Factors of a kind of alternating sums of products of q-binomial coefficients

Victor J. W. Guo

School of Mathematics, Hangzhou Normal University, Hangzhou 311121, People's Republic of China
jwguo@math.ecnu.edu.cn

Abstract. Let $[n]! = \prod_{i=1}^{n}(1+q+\cdots+q^{i-1})$ denote the q-factorials and let ${n \brack k} = [n]!/([k]![n-k]!)$ be the q-binomial coefficients, where $1/[k]!=0$ if k is a negative integer. Let $m_1,\dots,m_r,m_{r+1}=m_1$ and $n_1,\dots,n_s$, $n_{s+1}=n_1$ be positive integers with $r,s\geqslant 1$. We prove that the alternating sum

$$\frac{[m_1]![n_1]![m_r+n_s+1]!}{[m_1+m_r+1]![n_1+n_s]!}\sum_{k=-n_1}^{n_1}(-1)^k q^{ak^2+(2r-1){k \choose 2}}\prod_{i=1}^{r}{m_i+m_{i+1}+1 \brack m_i+k}\cdot\prod_{j=1}^{s}{n_j+n_{j+1} \brack n_j+k}$$

is a polynomial in q with non-negative integer coefficients for $0\leqslant a\leqslant s$. We also propose some related conjectures.

Keywords: q-binomial coefficients; q-Pfaff-Saalschütz sum; q-Chu-Vandermonde sum; q-Catalan numbers; q-Narayana numbers

AMS Subject Classifications: 05A10, 05A30, 33D15

1 Introduction

It is well known that, for $s=1$, 2, and 3, the alternating sum

$${2n \choose n}^{-1}\sum_{k=-n}^{n}(-1)^k{2n \choose n+k}^s \tag{1.1}$$

is equal to 0, 1, and ${3n \choose n}$, respectively. On the other hand, by using asymptotic techniques, de Bruijn [6] showed that there is no closed form for (1.1) for $s>3$. In 1998, Calkin [5] proved a surprising result: for all positive integers n and s, the expression (1.1) is always an integer. In 2007, the author, Jouhet, and Zeng [12] succeeded in giving the following generalization of (1.1): for all positive integers $n_1,\dots,n_s$, $n_{s+1}=n_1$, and $0\leqslant a\leqslant s-1$, the alternating sum

$${n_1+n_s \brack n_1}^{-1}\sum_{k=-n_1}^{n_1}(-1)^k q^{ak^2+{k \choose 2}}\prod_{i=1}^{s}{n_i+n_{i+1} \brack n_i+k}$$

is a polynomial in q with non-negative integer coefficients. Here and in what follows, the *q-shifted factorials* are defined by $(a;q)_0=1$ and $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n=1,2,\dots$, and the *q-binomial coefficients* are defined as

$$\begin{bmatrix}x\\k\end{bmatrix}=\begin{cases}\displaystyle\prod_{i=1}^{k}\frac{1-q^{x-i+1}}{1-q^i}, & \text{if } k\geqslant 0,\\ 0, & \text{if } k<0.\end{cases}$$

In particular, when $x=n$ is a non-negative integer, we have

$$\begin{bmatrix}n\\k\end{bmatrix}=\begin{cases}\dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{if } 0\leqslant k\leqslant n,\\ 0, & \text{otherwise.}\end{cases}$$

Moreover, the *q-integer* is defined as $[n]=(1-q^n)/(1-q)$ and the *q-factorial* is defined by $[n]!=[n][n-1]\cdots[1]$.

In this paper, we shall prove some similar results on another kind of alternating sums of products of q-binomial coefficients. Our approach makes use of two recurrence relations, which are similar to that in [12].

Here is a summary of our main results.

Theorem 1.1. *For $s\geqslant 3$ and all positive integers m, $n_1,\dots,n_s$, $n_{s+1}=n_1$, there holds*

$$\sum_{k=-n_1}^{n_1}(-1)^k q^{(s-1)k^2+\binom{k}{2}}\begin{bmatrix}2m+1\\m+k\end{bmatrix}\prod_{i=1}^{s}\begin{bmatrix}n_i+n_{i+1}\\n_i+k\end{bmatrix}$$
$$=\frac{[2m+1]![n_1+n_s]!}{[m]![n_1]![m+n_s+1]!}\sum_{\lambda}\begin{bmatrix}m+\lambda_{s-2}+n_s+1\\n_s\end{bmatrix}\prod_{i=1}^{s-2}q^{\lambda_i^2}\begin{bmatrix}\lambda_{i-1}\\\lambda_i\end{bmatrix}\begin{bmatrix}n_{i+1}+n_{i+2}\\n_{i+1}-\lambda_i\end{bmatrix},\qquad(1.2)$$

where $n_{s+1}=\lambda_0=n_1$ and the sum ranges over all sequences $\lambda=(\lambda_1,\dots,\lambda_{s-2})$ of non-negative integers such that $\lambda_0\geqslant\lambda_1\geqslant\cdots\geqslant\lambda_{s-2}$.

Theorem 1.2. *For $s\geqslant 3$ and all positive integers $m_1,m_2,n_1,\dots,n_s$, $n_{s+1}=n_1$, there holds*

$$\sum_{k=-n_1}^{n_1}(-1)^k q^{sk^2+\binom{k}{2}}\begin{bmatrix}m_1+m_2+1\\m_1+k\end{bmatrix}\begin{bmatrix}m_1+m_2+1\\m_2+k\end{bmatrix}\prod_{i=1}^{s}\begin{bmatrix}n_i+n_{i+1}\\n_i+k\end{bmatrix}$$
$$=\frac{[m_1]![n_1]![m_2+n_2+1]!}{[m_1+m_2+1]![n_1+n_2]!}\sum_{\lambda}q^{\lambda_{s-1}^2}\begin{bmatrix}\lambda_{s-2}\\\lambda_{s-1}\end{bmatrix}\begin{bmatrix}m_2+n_s+1\\n_s-\lambda_{s-1}\end{bmatrix}\begin{bmatrix}m_1+m_2+\lambda_{s-1}+1\\m_2\end{bmatrix}$$
$$\times\prod_{i=1}^{s-2}q^{\lambda_i^2}\begin{bmatrix}\lambda_{i-1}\\\lambda_i\end{bmatrix}\begin{bmatrix}n_{i+1}+n_{i+2}\\n_{i+1}-\lambda_i\end{bmatrix},\qquad(1.3)$$

where $n_{s+1}=\lambda_0=n_1$ and the sum ranges over all sequences $\lambda=(\lambda_1,\dots,\lambda_{s-1})$ of non-negative integers such that $\lambda_0\geqslant\lambda_1\geqslant\cdots\geqslant\lambda_{s-1}$.

Theorem 1.3. *Let $m_1, \ldots, m_r, m_{r+1} = m_1$ and $n_1, \ldots, n_s$, $n_{s+1} = n_1$ be positive integers with $r, s \geqslant 1$. Then the alternating sum*

$$\frac{[m_1]![n_1]![m_r+n_s+1]!}{[m_1+m_r+1]![n_1+n_s]!}\sum_{k=-n_1}^{n_1}(-1)^k q^{ak^2+(2r-1)\binom{k}{2}}\prod_{i=1}^{r}\begin{bmatrix} m_i+m_{i+1}+1 \\ m_i+k \end{bmatrix}\cdot\prod_{j=1}^{s}\begin{bmatrix} n_j+n_{j+1} \\ n_j+k \end{bmatrix} \tag{1.4}$$

is a polynomial in q with non-negative integer coefficients for $0 \leqslant a \leqslant s$.

Note that Theorem 1.3 may somewhat be regraded as a generalization of Theorem 1.1. In particular, when we take $r = 1$, $m_1 = m$, and $a = s-1$ in Theorem 1.3, the polynomial (1.4) is explicitly computed in Theorem 1.1. However, Theorem 1.2 is not a special case of Theorem 1.3.

The paper is organized as follows. We shall prove Theorems 1.1–1.3 in Sections 2-4, respectively. In Section 5, we propose a conjecture on the q-positivity of q-super Catalan numbers. In Section 6, we give some consequences of Theorem 1.3. Finally, we put forward some related conjectures in Section 7.

2 Proof of Theorem 1.1

We need two known formulas in basic hypergeometric series. One of them, already utilized by the author, Jouhet, and Zeng [12], is the q-Pfaff-Saalschütz sum [8, Appendix (II.12)] (see also [10, 15]):

$$\begin{bmatrix} n_1+n_2 \\ n_1+k \end{bmatrix}\begin{bmatrix} n_2+n_3 \\ n_2+k \end{bmatrix}\begin{bmatrix} n_3+n_1 \\ n_3+k \end{bmatrix} = \sum_{r=0}^{n_1-k}\frac{q^{k^2+2kr}(q;q)_{n_1+n_2+n_3-k-r}}{(q;q)_r(q;q)_{r+2k}(q;q)_{n_1-k-r}(q;q)_{n_2-k-r}(q;q)_{n_3-k-r}}, \tag{2.1}$$

where we have assumed that $1/(q;q)_n = 0$ if $n < 0$.

Recall that Jackson's terminating q-analogue of Dixon's sum can be written as

$$\sum_{k=-n_1}^{n_1}(-1)^k q^{(3k^2-k)/2}\begin{bmatrix} n_1+n_2 \\ n_1+k \end{bmatrix}\begin{bmatrix} n_2+n_3 \\ n_2+k \end{bmatrix}\begin{bmatrix} n_3+n_1 \\ n_3+k \end{bmatrix} = \frac{[n_1+n_2+n_3]!}{[n_1]![n_2]![n_3]!}. \tag{2.2}$$

The other formula we need is the following invariant of (2.2). Our proof is similar to that of (2.2) in [11].

Lemma 2.1. *For positive integers n_1, n_2, and n_3, there holds*

$$\sum_{k=-n_1}^{n_1}(-1)^k q^{(3k^2-k)/2}\begin{bmatrix} n_1+n_2 \\ n_1+k \end{bmatrix}\begin{bmatrix} n_2+n_3+1 \\ n_2+k \end{bmatrix}\begin{bmatrix} n_3+n_1 \\ n_3+k \end{bmatrix} = \frac{[n_1+n_2+n_3+1]!}{[n_1+n_3+1][n_1]![n_2]![n_3]!}. \tag{2.3}$$

Proof. Note that $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$ for $0 \leqslant k \leqslant n$. We can rewrite (2.3) as follows:

$$\begin{aligned} &[n_1+n_3+1]\sum_k (-1)^k q^{(3k^2-k)/2} \begin{bmatrix} n_1+n_2 \\ n_1+k \end{bmatrix} \begin{bmatrix} n_2+n_3+1 \\ n_2+k \end{bmatrix} \begin{bmatrix} n_3+n_1 \\ n_1-k \end{bmatrix} \\ &\quad = [n_1+n_2+n_3+1] \begin{bmatrix} n_1+n_2 \\ n_1 \end{bmatrix} \begin{bmatrix} n_1+n_2+n_3 \\ n_1+n_2 \end{bmatrix}. \end{aligned} \tag{2.4}$$

It is clear that both sides of (2.4) are polynomials in q^{n_3} of degree n_1+n_2+1. It suffices to check (2.4) for n_1+n_2+2 different values of n_3. Suppose that $n_1 \leqslant n_2$ (it is easy to see that the identity (2.3) is intrinsically symmetric in n_1 and n_2).

For $n_3 = 0$, we can verify that

$$[n_1+1]\left(\begin{bmatrix} n_1+n_2 \\ n_1 \end{bmatrix}[n_2+1] - q\begin{bmatrix} n_1+n_2 \\ n_1+1 \end{bmatrix}[n_1]\right) = [n_1+n_2+1]\begin{bmatrix} n_1+n_2 \\ n_1 \end{bmatrix},$$

and so (2.4) holds. For $n_3 = -m$ with $1 \leqslant m \leqslant n_1+n_2+1$ the right-hand side of (2.4) vanishes, while the left-hand side of (2.4) becomes

$$L = [n_1-m+1]\sum_k (-1)^k q^{(3k^2-k)/2} \begin{bmatrix} n_1+n_2 \\ n_1+k \end{bmatrix} \begin{bmatrix} n_2-m+1 \\ n_2+k \end{bmatrix} \begin{bmatrix} n_1-m \\ n_1-k \end{bmatrix}.$$

We now prove that $L = 0$ for $1 \leqslant m \leqslant n_1+n_2$ by considering the following four cases:

- If $m \in [1,\ n_1]$, then $\begin{bmatrix} n_1-m \\ n_1-k \end{bmatrix} = 0$ for $k \leqslant 0$ and $\begin{bmatrix} n_2-m+1 \\ n_2+k \end{bmatrix} = 0$ for $k \geqslant 1$.
- If $m = n_1+1$, then $[n_1-m+1] = 0$.
- If $m \in [n_1+2,\ n_2+1]$, then $\begin{bmatrix} n_2-m+1 \\ n_2+k \end{bmatrix} = 0$ for any k in the range $-n_1 \leqslant k \leqslant n_1$.
- If $m \in [n_2+2,\ n_1+n_2+1]$, since $\begin{bmatrix} -x \\ k \end{bmatrix} = (-1)^k q^{-kx-\binom{k}{2}} \begin{bmatrix} x+k-1 \\ k \end{bmatrix}$, we have

$$\begin{aligned} \frac{L}{[n_1-m+1]} &= \sum_k (-1)^{k+n_1+n_2} q^{\binom{n_2-k}{2}+A} \begin{bmatrix} n_1+n_2 \\ n_1+k \end{bmatrix} \begin{bmatrix} m+k-2 \\ m-n_2-2 \end{bmatrix} \begin{bmatrix} m-k-1 \\ m-n_1-1 \end{bmatrix} \\ &= \sum_k (-1)^{k+m+n_2-1} q^{\binom{n_2-k}{2}+B} \begin{bmatrix} n_1+n_2 \\ n_1+k \end{bmatrix} \begin{bmatrix} m+k-2 \\ m-n_2-2 \end{bmatrix} \begin{bmatrix} k-n_1-1 \\ m-n_1-1 \end{bmatrix}, \end{aligned} \tag{2.5}$$

  where $A = n_1(n_1+1)/2 + 2n_2 - (n_1+n_2)m + kn_2$ and $B = A + (m-k-1)(m-n_1-1) - \binom{m-n_1-1}{2}$.

  Clearly, the expression $q^B \begin{bmatrix} m+k-2 \\ m-n_2-2 \end{bmatrix} \begin{bmatrix} k-n_1-1 \\ m-n_1-1 \end{bmatrix}$ is a polynomial in q^k of degree

$$n_2 - (m-n_1-1) + 2m - n_1 - n_2 - 3 = m-2 \leqslant n_1+n_2-1.$$

  The right-hand side of (2.5) vanishes if we can confirm that

$$\sum_k (-1)^k q^{\binom{n_2-k}{2}} \begin{bmatrix} n_1+n_2 \\ n_1+k \end{bmatrix} q^{jk} = 0, \quad \text{for } 0 \leqslant j \leqslant n_1+n_2-1. \tag{2.6}$$

Employing the q-binomial theorem

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} z^k = \prod_{k=0}^{n-1}(1-zq^k)$$

(see, for example, [2, p. 36]) with $z=q^{-j}$ and replacing k by $n-k$, we get

$$\sum_{k=0}^{n}(-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}+jk} = 0, \quad \text{for } 0 \leqslant j \leqslant n-1, \tag{2.7}$$

which is equivalent to (2.6) by putting $n=n_1+n_2$ and shifting k to n_1+k.

□

Let $S(m_1,\dots,m_r;n_1,\dots,n_s;a,q)$ denote the following alternating sum:

$$\frac{[m_1]![n_1]![m_r+n_s+1]!}{[m_1+m_r+1]![n_1+n_s]!}\sum_{k=-n_1}^{n_1}(-1)^k q^{ak^2+\binom{k}{2}} \prod_{i=1}^{r} \begin{bmatrix} m_i+m_{i+1}+1 \\ m_i+k \end{bmatrix} \cdot \prod_{j=1}^{s} \begin{bmatrix} n_j+n_{j+1} \\ n_j+k \end{bmatrix}$$

where $m_{r+1}=m_1$, $n_{s+1}=n_1$, and $m_1,\dots,m_r,n_1,\dots,n_s$ are allowed to be 0's. We need to establish the following recurrence relation.

Lemma 2.2. *Let $s \geqslant 3$. Then, for all non-negative integers $m,n_1,\dots,n_s$ and a,*

$$S(m;n_1,\dots,n_s;a,q) = \sum_{l=0}^{n_1} q^{l^2} \begin{bmatrix} n_1 \\ l \end{bmatrix} \begin{bmatrix} n_2+n_3 \\ n_2-l \end{bmatrix} S(m;l,n_3,\dots,n_s;a-1,q). \tag{2.8}$$

Proof. For any integer k and non-negative integers $m,a_1,\dots,a_l$, let

$$C(m;a_1,\dots,a_l;k) = \begin{bmatrix} 2m+1 \\ m+k \end{bmatrix} \prod_{i=1}^{l} \begin{bmatrix} a_i+a_{i+1} \\ a_i+k \end{bmatrix},$$

where $a_{l+1}=a_1$. Then

$$S(m;n_1,\dots,n_s;a,q) = \frac{(q;q)_m(q;q)_{n_1}(q;q)_{m+n_s+1}}{(q;q)_{2m+1}(q;q)_{n_1+n_s}} \sum_{k=-n_1}^{n_1}(-1)^k q^{ak^2+\binom{k}{2}} C(m;n_1,\dots,n_s;k). \tag{2.9}$$

Note that for $s \geqslant 3$ there holds

$$C(m;n_1,\dots,n_s;k) = \frac{(q;q)_{n_2+n_3}(q;q)_{n_s+n_1}}{(q;q)_{n_1+n_2}(q;q)_{n_s+n_3}} \begin{bmatrix} n_1+n_2 \\ n_1+k \end{bmatrix} \begin{bmatrix} n_1+n_2 \\ n_2+k \end{bmatrix} C(m;n_3,\dots,n_s;k),$$

and, taking the limits as $n_3 \to \infty$ in (2.1) leads to

$$\begin{bmatrix} n_1+n_2 \\ n_1+k \end{bmatrix} \begin{bmatrix} n_1+n_2 \\ n_2+k \end{bmatrix} = \sum_{r=0}^{n_1-k} \frac{q^{r^2+2kr}(q;q)_{n_1+n_2}}{(q;q)_r(q;q)_{r+2k}(q;q)_{n_1-k-r}(q;q)_{n_2-k-r}}. \tag{2.10}$$

Substituting the above two identities into (2.9), we obtain

$$S(m;n_1,\dots,n_s;a,q)=\frac{(q;q)_m(q;q)_{n_1}(q;q)_{n_2+n_3}(q;q)_{m+n_s+1}}{(q;q)_{2m+1}(q;q)_{n_s+n_3}}$$
$$\times\sum_{k=-n_1}^{n_1}\sum_{r=0}^{n_1-k}(-1)^k\frac{q^{(r+k)^2+(a-1)k^2+\binom{k}{2}}C(m;n_3,\dots,n_s;k)}{(q;q)_r(q;q)_{r+2k}(q;q)_{n_1-k-r}(q;q)_{n_2-k-r}}.$$

Letting $l=r+k$, then $-n_1\leqslant l\leqslant n_1$. However, if $l<0$, then at least one of the numbers $l+k$ and $l-k$ is negative for any k, which means that $1/((q;q)_{l-k}(q;q)_{l+k})=0$ by our assumption. Thus, exchanging the summation order, we obtain

$$S(m;n_1,\dots,n_s;a,q)$$
$$=\sum_{l=0}^{n_1}\frac{q^{l^2}(q;q)_m(q;q)_{n_1}(q;q)_{n_2+n_3}(q;q)_{m+n_s+1}}{(q;q)_{2m+1}(q;q)_{n_1-l}(q;q)_{n_2-l}(q;q)_{n_s+n_3}}\sum_{k=-l}^{l}\frac{(-1)^kq^{(a-1)k^2+\binom{k}{2}}C(m;n_3,\dots,n_s;k)}{(q;q)_{l-k}(q;q)_{l+k}}.$$

Finally, performing the following substitution

$$C(m;n_3,\dots,n_m;k)=\frac{(q;q)_{l-k}(q;q)_{l+k}(q;q)_{n_s+n_3}}{(q;q)_{n_3+l}(q;q)_{n_s+l}}C(m;l,n_3,\dots,n_s;k),$$

we complete the proof of (2.8). □

Proof of Theorem 1.1. It is not difficult to see that, the identity (2.3) has the following equivalent form:

$$\sum_{k=-n_1}^{n_1}(-1)^kq^{(3k^2-k)/2}\begin{bmatrix}2m+1\\m+k\end{bmatrix}\begin{bmatrix}n_1+n_2\\n_1+k\end{bmatrix}\begin{bmatrix}n_1+n_2\\n_2+k\end{bmatrix}$$
$$=\frac{[2m+1]![n_1+n_2]![m+n_1+n_2+1]!}{[m]![n_1]![n_2]![m+n_1+1]![m+n_2+1]!}. \tag{2.11}$$

It follows that

$$S(m;n_1,n_2;1)=\begin{bmatrix}m+n_1+n_2+1\\n_2\end{bmatrix}. \tag{2.12}$$

The identity (1.2) then follows by iterating the recurrence (2.8) $s-2$ times. □

3 Proof of Theorem 1.2

We have the following identity similar to (2.11):

$$\sum_{k=-n}^{n}(-1)^kq^{(3k^2-k)/2}\begin{bmatrix}m_1+m_2+1\\m_1+k\end{bmatrix}\begin{bmatrix}m_1+m_2+1\\m_2+k\end{bmatrix}\begin{bmatrix}2n\\n+k\end{bmatrix}$$
$$=\frac{[m_1+m_2+1]![2n]![m_1+m_2+n+1]!}{[m_1]![m_2]![n]![m_1+n+1]![m_2+n+1]!}. \tag{3.1}$$

On the basis of the above identity, we can establish the following result.

Lemma 3.1. *Let m_1, m_2, n_1, n_2 be non-negative integers. Then*

$$S(m_1, m_2; n_1, n_2; 2, q) = \sum_{l=0}^{n_1} q^{l^2} \begin{bmatrix} n_1 \\ l \end{bmatrix} \begin{bmatrix} m_2 + n_2 + 1 \\ n_2 - l \end{bmatrix} \begin{bmatrix} m_1 + m_2 + l + 1 \\ m_2 \end{bmatrix}. \tag{3.2}$$

Proof. By definition, we have

$$\begin{aligned} &S(m_1, m_2; n_1, n_2; 2, q) \\ &\quad = \frac{[m_1]![n_1]![m_2 + n_2 + 1]!}{[m_1 + m_2 + 1]![n_1 + n_2]!} \\ &\quad\quad \times \sum_{k=-n_1}^{n_1} (-1)^k q^{2k^2 + \binom{k}{2}} \begin{bmatrix} m_1 + m_2 + 1 \\ m_1 + k \end{bmatrix} \begin{bmatrix} m_1 + m_2 + 1 \\ m_2 + k \end{bmatrix} \begin{bmatrix} n_1 + n_2 \\ n_1 + k \end{bmatrix} \begin{bmatrix} n_1 + n_2 \\ n_2 + k \end{bmatrix} \end{aligned}$$

Substituting (2.10) into the right-hand side of the above identity, we get

$$\begin{aligned} &S(m_1, m_2; n_1, n_2; 2, q) \\ &\quad = \frac{(q;q)_{m_1} (q;q)_{n_1} (q;q)_{m_2+n_2+1}}{(q;q)_{m_1+m_2+1}} \\ &\quad\quad \times \sum_{k=-n_1}^{n_1} \sum_{r=0}^{n_1-k} \frac{(-1)^k q^{(k+r)^2 + (3k^2-k)/2}}{(q;q)_r (q;q)_{r+2k} (q;q)_{n_1-k-r} (q;q)_{n_2-k-r}} \begin{bmatrix} m_1 + m_2 + 1 \\ m_1 + k \end{bmatrix} \begin{bmatrix} m_1 + m_2 + 1 \\ m_2 + k \end{bmatrix} \end{aligned}$$

Putting $l = r + k$, then $-n_1 \leqslant l \leqslant n_1$. However, if $l < 0$, then at least one of $l + k$ and $l - k$ is negative for any k, which indicates that $1/((q;q)_{l-k}(q;q)_{l+k}) = 0$. Therefore, exchanging the summation order, we can write the right-hand side as

$$\begin{aligned} &\sum_{l=0}^{n_1} \frac{q^{l^2} (q;q)_{m_1} (q;q)_{n_1} (q;q)_{m_2+n_2+1}}{(q;q)_{n_1-l} (q;q)_{n_2-l} (q;q)_{m_1+m_2+1} (q;q)_{2l}} \\ &\quad \times \sum_{k=-l}^{l} (-1)^k q^{(3k^2-k)/2} \begin{bmatrix} m_1 + m_2 + 1 \\ m_1 + k \end{bmatrix} \begin{bmatrix} m_1 + m_2 + 1 \\ m_2 + k \end{bmatrix} \begin{bmatrix} 2l \\ l + k \end{bmatrix} \\ &= \sum_{l=0}^{n_1} q^{l^2} \begin{bmatrix} n_1 \\ l \end{bmatrix} \begin{bmatrix} m_2 + n_2 + 1 \\ n_2 - l \end{bmatrix} \begin{bmatrix} m_1 + m_2 + l + 1 \\ m_2 \end{bmatrix}, \end{aligned}$$

where we have used the identity (3.1) with $n = l$. This completes the proof. □

Proof of Theorem 1.2. Like the proof of Lemma 2.2, we can prove that, for $s \geqslant 3$,

$$S(m_1, m_2; n_1, \ldots, n_s; a, q) = \sum_{l=0}^{n_1} q^{l^2} \begin{bmatrix} n_1 \\ l \end{bmatrix} \begin{bmatrix} n_2 + n_3 \\ n_2 - l \end{bmatrix} S(m_1, m_2; l, n_3, \ldots, n_s; a - 1, q). \tag{3.3}$$

The identity (1.3) then follows by repeatedly using the recurrence (3.3) $s - 2$ times. □

4 Proof of Theorem 1.3

Let $S_1(m_1, \ldots, m_r; n_1, \ldots, n_s; a, q)$ denote the expression (1.4). We need to establish the following recurrence relation.

Lemma 4.1. *Let $r \geqslant 3$. Then, for all non-negative integers $m_1, \ldots, m_r, n_1, \ldots, n_s$ and a,*

$$S_1(m_1, \ldots, m_r; n_1, \ldots, n_s; a, q) = \sum_{l=0}^{m_1} q^{l^2+l} \begin{bmatrix} m_1 \\ l \end{bmatrix} \begin{bmatrix} m_2 + m_3 + 1 \\ m_2 - l \end{bmatrix} S_1(l, m_3, \ldots, m_r; n_1, \ldots, n_s; a, q). \tag{4.1}$$

Proof. For any integer k and non-negative integers $a_1, \ldots, a_u, b_1, \ldots, b_v$, let

$$C(a_1, \ldots, a_u; b_1, \ldots, b_v; k) = \prod_{i=1}^{u} \begin{bmatrix} a_i + a_{i+1} + 1 \\ a_i + k \end{bmatrix} \cdot \prod_{j=1}^{v} \begin{bmatrix} b_j + b_{j+1} \\ b_j + k \end{bmatrix},$$

where $a_{u+1} = a_1$ and $b_{v+1} = b_1$. Since the k-th summand in the summation (1.4) without the prefactor is zero for

$$k > \min\{m_1 + 1, \ldots, m_r + 1, n_1, \ldots, n_s\}$$

or

$$k < \max\{-m_1, \ldots, -m_r, -n_1, \ldots, -n_s\},$$

we have

$$S_1(m_1, \ldots, m_r; n_1, \ldots, n_s; a, q) = \frac{(q;q)_{m_1}(q;q)_{n_1}(q;q)_{m_r+n_s+1}}{(q;q)_{m_1+m_r+1}(q;q)_{n_1+n_s}} \sum_{k=-m_1}^{m_1+1} (-1)^k q^{ak^2+(2r-1)\binom{k}{2}} C(m_1, \ldots, m_r; n_1, \ldots, n_s; k). \tag{4.2}$$

Note that for $r \geqslant 3$ there holds

$$C(m_1, \ldots, m_r; n_1, \ldots, n_s; k) = \frac{(q;q)_{m_2+m_3+1}(q;q)_{m_r+m_1+1}}{(q;q)_{m_1+m_2+1}(q;q)_{m_r+m_3+1}} \begin{bmatrix} m_1 + m_2 + 1 \\ m_1 + k \end{bmatrix} \begin{bmatrix} m_1 + m_2 + 1 \\ m_2 + k \end{bmatrix} C(m_3, \ldots, m_r; n_1, \ldots, n_s; k),$$

By the q-Chu–Vandermonde sum [8, Appendix (II.7)], we have

$$\begin{bmatrix} m_1 + m_2 + 1 \\ m_1 + k \end{bmatrix} = \sum_{t=0}^{m_1-k+1} \begin{bmatrix} m_1 - k + 1 \\ t \end{bmatrix} \begin{bmatrix} m_2 + k \\ t + 2k - 1 \end{bmatrix} q^{t^2+2kt-t},$$

which can be written as

$$\begin{bmatrix} m_1 + m_2 + 1 \\ m_1 + k \end{bmatrix} \begin{bmatrix} m_1 + m_2 + 1 \\ m_2 + k \end{bmatrix} = \sum_{t=0}^{m_1-k+1} \frac{q^{t^2+2kt-t}(q;q)_{m_1+m_2+1}}{(q;q)_t(q;q)_{t+2k-1}(q;q)_{m_1-k-t+1}(q;q)_{m_2-k-t+1}}.$$

Plugging the above two identities into (4.2), we get

$$
\begin{aligned}
&S_1(m_1,\dots,m_r;n_1,\dots,n_s;a,q)\\
&\quad=\frac{(q;q)_{m_1}(q;q)_{n_1}(q;q)_{m_2+m_3+1}(q;q)_{m_r+n_s+1}}{(q;q)_{m_r+m_3+1}(q;q)_{n_1+n_s}}\\
&\qquad\times\sum_{k=-m_1}^{m_1+1}\sum_{t=0}^{m_1-k+1}(-1)^k\frac{q^{(t+k)^2-t+(a-1)k^2+(2r-1)\binom{k}{2}}C(m_3,\dots,m_r;n_1,\dots,n_s;k)}{(q;q)_t(q;q)_{t+2k-1}(q;q)_{m_1-k-t+1}(q;q)_{m_2-k-t+1}}.
\end{aligned}
$$

Letting $l=t+k-1$, then $-m_1\leqslant l\leqslant m_1+1$. However, if $l<0$, then at least one of $l+k$ and $l-k+1$ is negative for any integer k, which means that $1/((q;q)_{l-k}(q;q)_{l+k-1})=0$ by our assumption. Thus, exchanging the order of summation, we have

$$
\begin{aligned}
&S_1(m_1,\dots,m_r;n_1,\dots,n_s;a,q)\\
&\quad=\sum_{l=0}^{m_1}\frac{q^{l^2+l}(q;q)_{m_1}(q;q)_{n_1}(q;q)_{m_2+m_3+1}(q;q)_{m_r+n_s+1}}{(q;q)_{m_r+m_3+1}(q;q)_{m_1-l}(q;q)_{m_2-l}(q;q)_{n_1+n_s}}\\
&\qquad\times\sum_{k=-l}^{l+1}\frac{(-1)^kq^{ak^2+(2r-3)\binom{k}{2}}C(m_3,\dots,m_r;n_1,\dots,n_s;k)}{(q;q)_{l-k+1}(q;q)_{l+k}}.
\end{aligned}
$$

Finally, performing the following substitution

$$
\begin{aligned}
&C(m_3,\dots,m_r;n_1,\dots,n_s;k)\\
&\quad=\frac{(q;q)_{l-k+1}(q;q)_{l+k}(q;q)_{m_r+m_3+1}}{(q;q)_{m_3+l+1}(q;q)_{m_r+l+1}}C(l,m_3,\dots,m_r;n_1,\dots,n_s;k),
\end{aligned}
$$

we complete the proof of (2.8). □

Proof of Theorem 1.3 for $r=1$. We proceed by induction on $s\geqslant 1$. Letting $n_2\to\infty$ in (2.11), we get

$$
S_1(m_1;n_1;1,q)=\frac{[m_1]![n_1]![m_1+n_1+1]!}{[2m_1+1]![2n_1]!}\sum_{k=-n_1}^{n_1}(-1)^kq^{(3k^2-k)/2}\begin{bmatrix}2m_1+1\\ m_1+k\end{bmatrix}\begin{bmatrix}2n_1\\ n_1+k\end{bmatrix}=1, \tag{4.3}
$$

and so

$$
S_1(m_1;n_1;0,q)=S_1(m_1;n_1;1,q^{-1})q^{m_1n_1+n_1}=q^{m_1n_1+n_1}.
$$

Consider the $s=2$ case. Similarly as before, we have

$$
\begin{aligned}
&S_1(m_1;n_1,n_2;2,q)\\
&\quad=\sum_{l=0}^{n_1}\frac{q^{l^2}(q;q)_{m_1}(q;q)_{n_1}(q;q)_{m_1+n_2+1}}{(q;q)_{n_1-l}(q;q)_{n_2-l}(q;q)_{2m_1+1}(q;q)_{2l}}\sum_{k=-l}^{l}(-1)^kq^{(3k^2-k)/2}\begin{bmatrix}2m_1+1\\ m_1+k\end{bmatrix}\begin{bmatrix}2l\\ l+k\end{bmatrix}\\
&\quad=\sum_{l=0}^{n_1}q^{l^2}\begin{bmatrix}n_1\\ l\end{bmatrix}\begin{bmatrix}m_1+n_2+1\\ n_2-l\end{bmatrix},
\end{aligned}
$$

where we have used the identity (4.3) with $n_1 = l$ in the last step. Consequently,

$$S_1(m_1; n_1, n_2; 0, q) = S_1(m_1; n_1, n_2; 2, q^{-1})q^{(m_1+n_1+1)n_2}$$

is also a polynomial in q with non-negative integer coefficients. This together with (2.12), with S replaced by S_1, proves the theorem for $s = 2$.

We now assume that the alternating sum $S_1(m_1; n_1, \ldots, n_{s-1}; a, q)$ is a polynomial in q with non-negative integer coefficients for some $s \geqslant 3$ and all a in the range $0 \leqslant a \leqslant s-1$. Then, in light of the recurrence (2.8), so is $S_1(m_1; n_1, \ldots, n_s; a, q)$ for $1 \leqslant a \leqslant s$. In what follows we shall prove that $S(m_1; n_1, \ldots, n_s; 0, q)$ also has the required property. Since the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ is a polynomial in q of degree $k(n-k)$ (see [2, p. 33]) and $\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)}$, it is easy to see that

$$S_1(m_1; n_1, \ldots, n_s; 0, q) = S_1(m_1; n_1, \ldots, n_s; s, q^{-1})q^{n_1n_2+n_2n_3+\cdots+n_{s-1}n_s+m_1n_s+n_s}$$

and the degree of the polynomial $S_1(m_1; n_1, \ldots, n_s; s, q)$ is at most $n_1n_2 + n_2n_3 + \cdots + n_{s-1}n_s + m_1n_s + n_s$. Hence, $S_1(m_1; n_1, \ldots, n_s; 0, q)$ is also a polynomial in q with non-negative integer coefficients. This completes the inductive step, and by the principle of mathematical induction, we conclude that Theorem 1.3 is true for all $s \geqslant 1$. □

Proof of Theorem 1.3 for $r = 2$. We proceed by induction on $s \geqslant 1$ again. Like the proof of Lemma 4.1, for $0 \leqslant a \leqslant 1$, we have

$$\begin{aligned}
&S_1(m_1, m_2; n_1; a, q) \\
&\quad = \frac{[m_1]![n_1]![m_2+n_1+1]!}{[m_1+m_2+1]![2n_1]!} \sum_{k=-m_1}^{m_1+1} (-1)^k q^{ak^2+3\binom{k}{2}} \begin{bmatrix} m_1+m_2+1 \\ m_1+k \end{bmatrix} \begin{bmatrix} m_1+m_2+1 \\ m_2+k \end{bmatrix} \begin{bmatrix} 2n_1 \\ n_1+k \end{bmatrix} \\
&\quad = \sum_{l=0}^{m_1} \frac{q^{l^2+l}(q;q)_{m_1}(q;q)_{n_1}(q;q)_{m_2+n_1+1}}{(q;q)_{m_1-l}(q;q)_{m_2-l}(q;q)_{2n_1}(q;q)_{2l+1}} \sum_{k=-l}^{l+1} (-1)^k q^{ak^2+\binom{k}{2}} \begin{bmatrix} 2l+1 \\ l+k \end{bmatrix} \begin{bmatrix} 2n_1 \\ n_1+k \end{bmatrix} \\
&\quad = \sum_{l=0}^{m_1} q^{l^2+l} \begin{bmatrix} m_1 \\ l \end{bmatrix} \begin{bmatrix} m_2+n_1+1 \\ m_2-l \end{bmatrix} S_1(l; n_1; a, q).
\end{aligned}$$

Similarly as the proof of Theorem 1.2, we get

$$\begin{aligned}
&S_1(m_1, m_2; n_1, n_2; a, q) \\
&\quad = \sum_{l=0}^{n_1} \frac{q^{l^2}(q;q)_{m_1}(q;q)_{n_1}(q;q)_{m_2+n_2+1}}{(q;q)_{n_1-l}(q;q)_{n_2-l}(q;q)_{m_1+m_2+1}(q;q)_{2l}} \\
&\qquad \times \sum_{k=-l}^{l} (-1)^k q^{(a-1)k^2+3\binom{k}{2}} \begin{bmatrix} m_1+m_2+1 \\ m_1+k \end{bmatrix} \begin{bmatrix} m_1+m_2+1 \\ m_2+k \end{bmatrix} \begin{bmatrix} 2l \\ l+k \end{bmatrix} \\
&\quad = \sum_{l=0}^{n_1} q^{l^2} \begin{bmatrix} n_1 \\ l \end{bmatrix} \begin{bmatrix} m_2+n_2+1 \\ n_2-l \end{bmatrix} S_1(m_1, m_2; l; a-1, q)
\end{aligned}$$

for $1 \leqslant a \leqslant 2$. Moreover,

$$
\begin{aligned}
&S_1(m_1, m_2; n_1, n_2; 0, q) \\
&\quad = \sum_{l=0}^{m_1} \frac{q^{l^2+l}(q;q)_{m_1}(q;q)_{n_1}(q;q)_{m_2+n_1+1}}{(q;q)_{m_1-l}(q;q)_{m_2-l}(q;q)_{2n_1}(q;q)_{2l+1}} \sum_{k=-l}^{l+1} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2l+1 \\ l+k \end{bmatrix} \begin{bmatrix} n_1+n_2 \\ n_1+k \end{bmatrix} \begin{bmatrix} n_1+n_2 \\ n_2+k \end{bmatrix} \\
&\quad = \sum_{l=0}^{m_1} q^{l^2+l} \begin{bmatrix} m_1 \\ l \end{bmatrix} \begin{bmatrix} m_2+n_1+1 \\ m_2-l \end{bmatrix} S_1(l; n_1, n_2; 0, q).
\end{aligned}
$$

We now assume that the alternating sum $S_1(m_1, m_2; n_1, \dots, n_{s-1}; a, q)$ is a polynomial in q with non-negative integer coefficients for some $s \geqslant 3$ and all a satisfying $0 \leqslant a \leqslant s-1$. Then, so is $S_1(m_1, m_2; n_1, \dots, n_s; a, q)$ for $1 \leqslant a \leqslant s$, since the recurrence (3.3) is also valid when we replace S by S_1. We now consider $S_1(m_1, m_2; n_1, \dots, n_s; 0, q)$. Along the lines of the proof of Lemma 4.1, we can prove that

$$
\begin{aligned}
&S_1(m_1, m_2; n_1, \dots, n_s; 0, q) \\
&\quad = \sum_{l=0}^{m_1} \frac{q^{l^2+l}(q;q)_{m_1}(q;q)_{n_1}(q;q)_{m_2+n_1+1}}{(q;q)_{m_1-l}(q;q)_{m_2-l}(q;q)_{2n_1}(q;q)_{2l+1}} \sum_{k=-l}^{l+1} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2l+1 \\ l+k \end{bmatrix} \prod_{j=1}^{s} \begin{bmatrix} n_j+n_{j+1} \\ n_j+k \end{bmatrix} \\
&\quad = \sum_{l=0}^{m_1} q^{l^2+l} \begin{bmatrix} m_1 \\ l \end{bmatrix} \begin{bmatrix} m_2+n_1+1 \\ m_2-l \end{bmatrix} S_1(l; n_1, \dots, n_s; 0, q).
\end{aligned}
$$

is also a polynomial in q with non-negative integer coefficients. This completes the inductive step, and we deduce that Theorem 1.3 is true for all $s \geqslant 1$. □

Proof of Theorem 1.3 for $r \geqslant 3$. We proceed by induction on r. We have proved that Theorem 1.3 is true for $r = 1, 2$. Assume that the alternating sum $S_1(m_1, \dots, m_{r-1}; n_1, \dots, n_s; a, q)$ is a polynomial in q with non-negative integer coefficients for some $r \geqslant 3$. Then, by the recurrence (4.1), we immediately conclude that so is $S_1(m_1, \dots, m_r; n_1, \dots, n_s; a, q)$. □

5 The q-positivity of q-super Catalan numbers

Warnaar and Zudilin [13] have proved that the *q-super Catalan numbers*

$$A_{m,n}(q) = \frac{[2m]![2n]!}{[m+n]![m]![n]!}$$

are polynomials in q with non-negative integer coefficients for all $m, n \geqslant 0$. It is reasonable to call

$$C_{m,n}(q) = \frac{[2m+1]![2n]!}{[m+n+1]![m]![n]!}$$

the q-super Catalan numbers too, since $C_{0,n}(q) = \frac{1}{[n+1]}\begin{bmatrix} 2n \\ n \end{bmatrix}$ are the so-called q-Catalan numbers, and are polynomials with non-negative integer coefficients (see [7]). Recently, Xia [14, Lemma 3.1] showed that $C_{m,n}(1)$ is always an integer. Motivated by Warnaar–Zudilin's and Xia's work, we shall propose the following conjecture.

Conjecture 5.1. *The q-super Catalan numbers $C_{m,n}(q)$ are polynomials in q with non-negative integer coefficients.*

The fact that $C_{m,n}(q)$ are polynomials in q immediately follows from (4.3). Warnaar and Zudilin [13] also proved that, for $m \geqslant n$, the ratios

$$B_{m,n}(q) = \frac{[2m]![n]!}{[m]![2n]![m-n]!}$$

are polynomials in q with non-negative integer coefficients. This means that Conjecture 5.1 is true for $n = 3m+1$, since $C_{m,3m+1}(q) = B_{3m+1,2m+1}(q)$. Moreover, Warnaar and Zudilin [13, Conjeture 1] have made the following interesting conjecture: If $\mathbf{a} = (a_1, \dots, a_r)$ and $\mathbf{b} = (b_1, \dots, b_s)$ are tuples of positive integers subject to the condition

$$\sum_{i=1}^{r} \lfloor a_i x \rfloor - \sum_{j=1}^{s} \lfloor b_j x \rfloor \geqslant 0 \quad \text{for} \quad x \geqslant 0, \tag{5.1}$$

where $\lfloor x \rfloor$ denotes the integer part of x, then the polynomial

$$D(\mathbf{a}, \mathbf{b}; q) = \frac{[a_1]! \cdots [a_r]!}{[b_1]! \cdots [b_s]!}$$

has non-negative integer coefficients. After looking at the 52 choices for $\mathbf{a}$ and $\mathbf{b}$ listed in [3, Table 2], we found that, besides $C_{m,3m+1}(q)$, the q-positivity of $C_{1,6}(q)$, $C_{4,2}(q)$, $C_{1,7}(q)$ and $C_{1,10}(q)$ was already predicted by the above conjecture of Warnaar and Zudilin. However, in general, the tuples $\mathbf{a} = (2m+1, 2n)$ and $\mathbf{b} = (m, n, m+n+1)$ do not satisfy the condition (5.1). For instance, when $x = 4/7$,

$$\lfloor 5x \rfloor + \lfloor 8x \rfloor - \lfloor 2x \rfloor - \lfloor 4x \rfloor - \lfloor 7x \rfloor = -1.$$

This illustrates that Conjecture 5.1 is not a special case of Warnaar and Zudilin's conjecture.

Let n and h be positive integers. Using the q-Chu–Vandermonde sum [8, Appendix (II.7)] twice in the following form

$$\begin{bmatrix} a+b \\ r \end{bmatrix} = \sum_{j=0}^{r} q^{k(b-r+k)} \begin{bmatrix} a \\ j \end{bmatrix} \begin{bmatrix} b \\ r-j \end{bmatrix}$$

leads to

$$\begin{aligned} \begin{bmatrix} 2n+2h \\ h-1 \end{bmatrix} &= \sum_{j=0}^{h-1} q^{j(n+j+1)} \begin{bmatrix} n+h \\ j \end{bmatrix} \begin{bmatrix} n+h \\ h-j-1 \end{bmatrix} \\ &= \sum_{j=0}^{h-1} q^{j(n+j+1)} \begin{bmatrix} n+h \\ j \end{bmatrix} \sum_{k=0}^{j} q^{k(n+k+1)} \begin{bmatrix} j \\ k \end{bmatrix} \begin{bmatrix} n+h-j \\ h-j-k-1 \end{bmatrix}. \end{aligned}$$

Multiplying both sides by $[2n]![h]!/([n]![n+h]!)$ yields the identity

$$\frac{[h][2n+2h]![2n]!}{[n+h]![n]![2n+h+1]!} = \sum_{k=0}^{\lfloor (h-1)/2 \rfloor} C_{k,n}(q) \sum_{j=k}^{h-k-1} q^{k(n+k+1)+j(n+j+1)} \begin{bmatrix} h \\ 2k+1 \end{bmatrix} \begin{bmatrix} h-2k-1 \\ j-k \end{bmatrix}.$$

Since the left-hand side can be written as

$$C_{n+h,n}(q) - q^h A_{n+h,n}(q),$$

we obtain a recurrence for $C_{n+h,n}(q)$. It is clear that this recurrence support Conjecture 5.1. However, we did not find the corresponding recurrence for $C_{n,n+h}(q)$ and therefore cannot confirm Conjecture 5.1 by induction.

6 Some consequences of Theorem 1.3

Letting $m_1 = \cdots = m_r = m$ and $n_1 = \cdots = n_s = n$ in Theorem 1.3, we obtain the following result.

Corollary 6.1. *Let m, n, r, and s be positive integers. Then the alternating sum*

$$\frac{1}{C_{m,n}(q)} \sum_{k=-m}^{m+1} (-1)^k q^{ak^2+(2r-1)\binom{k}{2}} \begin{bmatrix} 2m+1 \\ m+k \end{bmatrix}^r \begin{bmatrix} 2n \\ n+k \end{bmatrix}^s$$

is a polynomial in q with non-negative integer coefficients for $0 \leqslant a \leqslant s$.

Letting $m_1 = \cdots = m_r = n$, $n_{2i-1} = n+1$ and $n_{2i} = n$ for $1 \leqslant i \leqslant s$ in Theorem 1.3, we obtain

Corollary 6.2. *Let n, r, and s be positive integers. Then the alternating sum*

$$\begin{bmatrix} 2n+1 \\ n \end{bmatrix}^{-1} \sum_{k=-n}^{n} (-1)^k q^{ak^2+(2r-1)\binom{k}{2}} \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}^{r+s} \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}^s$$

is a polynomial in q with non-negative integer coefficients for $0 \leqslant a \leqslant 2s$.

Recall that the q-Narayana numbers can be defined as

$$N_q(n,k) = \frac{1}{[n]} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ k-1 \end{bmatrix}.$$

From [4] we know that the q-Narayana numbers $N_q(n,k)$ are polynomials in q with non-negative integer coefficients. This implies that $\begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ k-1 \end{bmatrix}$ is divisible by $[n]$. Since

$$\begin{bmatrix} 2n+1 \\ n \end{bmatrix} = \frac{[2n+1]}{[n]} \begin{bmatrix} 2n \\ n-1 \end{bmatrix}, \quad \text{and} \quad \gcd\left(\begin{bmatrix} 2n \\ n-1 \end{bmatrix}, [2n+1] \right) = 1,$$

Corollary 6.2 leads to the following result.

Corollary 6.3. *Let n, r, and s be positive integers. Then the alternating sum*

$$\begin{bmatrix}2n+1\\ n\end{bmatrix}^{-1}\frac{1}{[2n+1]^{s-1}}\sum_{k=-n}^{n}(-1)^k q^{ak^2+(2r-1)\binom{k}{2}}\begin{bmatrix}2n+1\\ n+k\end{bmatrix}^{r+s}\begin{bmatrix}2n+1\\ n+k+1\end{bmatrix}^{s} \tag{6.1}$$

is a polynomial in q for $0 \leqslant a \leqslant 2s$.

Letting $n_1 = \cdots = n_s = n$, $m_{2i-1} = n$ and $m_{2i} = n-1$ for $1 \leqslant i \leqslant r$ in Theorem 1.3, we get

Corollary 6.4. *Let n, r, and s be positive integers. Then the alternating sum*

$$\begin{bmatrix}2n\\ n\end{bmatrix}^{-1}\sum_{k=1-n}^{n}(-1)^k q^{ak^2+(4r-1)\binom{k}{2}}\begin{bmatrix}2n\\ n+k\end{bmatrix}^{r+s}\begin{bmatrix}2n\\ n+k-1\end{bmatrix}^{r}$$

is a polynomial in q with non-negative integer coefficients for $0 \leqslant a \leqslant s$.

Similarly, since

$$\begin{bmatrix}2n\\ n\end{bmatrix} = \frac{[2n]}{[n]}\begin{bmatrix}2n-1\\ n\end{bmatrix}, \quad \text{and} \quad \gcd\left(\begin{bmatrix}2n-1\\ n-1\end{bmatrix}, [2n]\right) = 1,$$

Corollary 6.3 implies the following result.

Corollary 6.5. *Let m, n, r and s be positive integers. Then the alternating sum*

$$\begin{bmatrix}2n-1\\ n\end{bmatrix}^{-1}\frac{1}{[2n]^{r}}\sum_{k=1-n}^{n}(-1)^k q^{ak^2+(4r-1)\binom{k}{2}}\begin{bmatrix}2n\\ n+k\end{bmatrix}^{r+s}\begin{bmatrix}2n\\ n+k-1\end{bmatrix}^{r} \tag{6.2}$$

is a polynomial in q for $0 \leqslant a \leqslant s$.

Letting $m_{2i-1} = m$ and $m_{2i} = m-1$ for $1 \leqslant i \leqslant r$, and letting $n_{2j-1} = n$ and $n_{2i} = n+1$ for $1 \leqslant j \leqslant s$ in Theorem 1.3, we arrive at

Corollary 6.6. *Let m, n, r, and s be positive integers. Then the alternating sum*

$$\frac{1}{C_{n,m}(q)}\sum_{k=1-m}^{m}(-1)^k q^{ak^2+(4r-1)\binom{k}{2}}\begin{bmatrix}2m\\ m+k\end{bmatrix}^{r}\begin{bmatrix}2m\\ m+k-1\end{bmatrix}^{r}\begin{bmatrix}2n+1\\ n+k\end{bmatrix}^{s}\begin{bmatrix}2n+1\\ n+k+1\end{bmatrix}^{s}$$

is a polynomial in q with non-negative integer coefficients for $0 \leqslant a \leqslant 2s$.

As before, we have the following conclusion from Corollary 6.6, though it is difficult to give an explicit formula for $\gcd(C_{n,m}(q), [2m]^r[2n+1]^s)$ for general m and n.

Corollary 6.7. *Let m, n, r, and s be positive integers. Then the alternating sum*

$$\frac{\gcd(C_{n,m}(q), [2m]^r[2n+1]^s)}{C_{n,m}(q)[2m]^r[2n+1]^s} \times \sum_{k=1-m}^{m}(-1)^k q^{ak^2+(4r-1)\binom{k}{2}}\begin{bmatrix}2m\\ m+k\end{bmatrix}^{r}\begin{bmatrix}2m\\ m+k-1\end{bmatrix}^{r}\begin{bmatrix}2n+1\\ n+k\end{bmatrix}^{s}\begin{bmatrix}2n+1\\ n+k+1\end{bmatrix}^{s} \tag{6.3}$$

is a polynomial in q for $0 \leqslant a \leqslant 2s$.

7 Some conjectures

On the basis of numerical calculations, we would like to present the following conjecture, which is a generalization of Theorem 1.3.

Conjecture 7.1. *Let $m_1, \dots, m_r, m_{r+1} = m_1$ and $n_1, \dots, n_s$, $n_{s+1} = n_1$ be positive integers with $r, s \geqslant 1$. Then the alternating sum*

$$\frac{[m_1]![n_1]![m_r+n_s+1]!}{[m_1+m_r+1]![n_1+n_s]!} \sum_{k=-n_1}^{n_1} (-1)^k q^{ak^2+(2b-1)\binom{k}{2}} \prod_{i=1}^{r} \begin{bmatrix} m_i+m_{i+1}+1 \\ m_i+k \end{bmatrix} \cdot \prod_{j=1}^{s} \begin{bmatrix} n_j+n_{j+1} \\ n_j+k \end{bmatrix}$$

is a polynomial in q with non-negative integer coefficients for $0 \leqslant a \leqslant s$ and $1 \leqslant b \leqslant r$.

We also have the following similar conjecture on alternating sums.

Conjecture 7.2. *Let $m_1, \dots, m_r, m_{r+1} = m_1$ and $n_1, \dots, n_s$, $n_{s+1} = n_1$ be positive integers with $r, s \geqslant 1$. Then the alternating sum*

$$\frac{[m_1]![n_1]![m_r+n_s+1]!}{[m_1+m_r]![n_1+n_s]!} \sum_{k=-n_1}^{n_1} (-1)^k q^{ak^2+\binom{k}{2}} \\ \times \prod_{i=1}^{r} \frac{1}{[m_i+m_{i+1}+1]} \begin{bmatrix} m_i+m_{i+1}+1 \\ m_i+k \end{bmatrix} \begin{bmatrix} m_i+m_{i+1}+1 \\ m_i+k+1 \end{bmatrix} \cdot \prod_{j=1}^{s} \begin{bmatrix} n_j+n_{j+1} \\ n_j+k \end{bmatrix}$$

is a polynomial in q with non-negative integer coefficients for $0 \leqslant a \leqslant 2r+s-1$.

Note that perhaps Andrews' multiseries transformation [1] can be utilized to prove the above two conjectures for some special cases. See [9] for such an example.

Finally, we believe that the Corollaries 6.3, 6.5, and 6.7 have the following stronger versions.

Conjecture 7.3. *The polynomials* (6.1)–(6.3) *have non-negative integer coefficients.*

Data Availability Statements. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] G.E. Andrews, Problems and prospects for basic hypergeometric functions, in: Theory and Application for Basic Hypergeometric Functions, R.A. Askey, ed., Math. Res. Center, Univ. Wisconsin, Publ. No. 35, Academic Press, New York, 1975, pp. 191–224.

[2] G.E. Andrews, The Theory of Partitions, Cambridge University Press, Cambridge, 1998.

[3] J.W. Bober, Factorial ratios, hypergeometric series, and a family of step functions, J. Lond. Math. Soc. (2) 79 (2009), 422–444.

[4] Brändén, q-Narayana numbers and the flag h-vector of $J(2\times n)$, Discrete Math. 281 (2004), 67–81.

[5] N.J. Calkin, Factors of sums of powers of binomial coefficients, Acta Arith. 86 (1998), 17–26.

[6] N.G. de Bruijn, Asymptotic Methods in Analysis, Dover, New York, 1981.

[7] J. Fürlinger and J. Hofbauer, q-Catalan numbers, J. Combin. Theory, Ser. A 2 (1985), 248–264.

[8] G. Gasper and M. Rahman, Basic Hypergeometric Series, Second Edition, Encyclopedia of Mathematics and Its Applications, Vol. 96, Cambridge University Press, Cambridge, 2004.

[9] V.J.W. Guo and X. Lian, On a generalization of a congruence related to q-Narayana numbers, J. Algebraic Combin. 55 (2022), 1299–1305.

[10] V.J.W. Guo and J. Zeng, A combinatorial proof of a symmetric q-Pfaff-Saalschütz identity, Electron. J. Combin. 12 (2005), #N2.

[11] V.J.W. Guo and J. Zeng, A short proof of the q-Dixon identity, Discrete Math. 296 (2005), 259–261.

[12] V.J.W. Guo, F. Jouhet, and J. Zeng, Factors of alternating sums of products of binomial and q-binomial coefficients, Acta Arith. 127 (2007), 17–31.

[13] S.O. Warnaar and W. Zudilin, A q-rious positivity, Aequat. Math. 81 (2011), 177–183.

[14] W. Xia, Proof of a conjecture of Sun and its extension by Guo, Ramanujan J. 62 (2023), 617–631.

[15] D. Zeilberger, A q-Foata proof of the q-Saalschütz identity, European J. Combin. 8 (1987), 461–463.