# A new extension of a "divergent" Ramanujan-type supercongruence

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**Abstract.** We give a new extension of a "divergent" Ramanujan-type supercongruence of Guillera and Zudilin by establishing a $q$-analogue of this result. Our proof makes use of the "creative microscoping" method, which was introduced by the second author and Zudilin in 2019. We also present a similar extension of the (L.2) supercongruence of Van Hamme in the modulus $p^2$ case.

*Keywords*: supercongruence; basic hypergeometric series; cyclotomic polynomials; creative microscoping.

*2020 Mathematics Subject Classifications*: 33D15; 11A07; 11B65.

# 1 Introduction

Employing the Wilf–Zeilberger (WZ) method, Guillera and Zudilin [5] established the following supercongruence:

$$\sum_{k=0}^{(p-1)/2}(-1)^k\frac{(\frac{1}{2})_k^3}{k!^3}(3k+1)2^{3k}\equiv p(-1)^{(p-1)/2}\pmod{p^3}\quad\text{for } p>2. \tag{1.1}$$

Here and in what follows, $p$ is a prime, and $(a)_b=\Gamma(a+b)/\Gamma(a)$ denotes the Pochhammer symbol also for $b$ not being a non-negative integer. In the spirit of [19], the supercongruence (1.1) corresponds to a "divergent" Ramanujan-type series for $1/\pi$:

$$\sum_{k=0}^{\infty}(-1)^k\frac{(\frac{1}{2})_k^3}{(1)_k^3}(3k+1)2^{3k}\ \text{“}=\text{”}\ \frac{1}{\pi} \tag{1.2}$$

(see [5, (47)]). Note that the left-hand side of (1.2) has to be understood as the analytic continuation of the corresponding hypergeometric series. Namely, the formula (1.2) can be written as

$$\frac{1}{2\pi i}\int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty}\frac{(\frac{1}{2})_s^3}{(1)_s^3}\Gamma(-s)(3s+1)2^{3s}ds=\frac{1}{\pi}.$$

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See [4] for more divergent hypergeometric formulas for $1/\pi$ and $1/\pi^2$.

Sun [14, Conjecture 5.1(ii)] formulated a refinement of (1.1) modulo $p^4$, which was later confirmed by Chen, Xie, and He [2]. A $q$-analogue of (1.1) was obtained by the second author [6] using the $q$-WZ method. A further generalization of this $q$-analogue was conjectured by the second author and Zudilin in [9, Conjecture 4.6] where two particular cases were settled by themselves. The second author and Schlosser [8, Theorem 6.1] finally confirmed this conjecture.

In this paper, we give the following new generalization of (1.1): for $p > 2$ and $0 \leqslant s \leqslant (p-1)/8$,

$$\sum_{k=2s}^{(p-1)/2+2s} (-1)^k \frac{(\frac{1}{2})_{k-s}(\frac{1}{2})_{k+s}(\frac{1}{2})_k}{(k-2s)!(k+2s)!k!}(3k+1)2^{3k} \equiv p(-1)^{(p-1)/2} \pmod{p^3}. \tag{1.3}$$

It is easy to see that the $s = 0$ case of (1.3) reduces to (1.1). We shall prove (1.3) by establishing the following $q$-congruence.

**Theorem 1.1.** *Let $n > 1$ be an odd integer and let $0 \leqslant s \leqslant (n-1)/8$. Then, modulo $\Phi_n(q)^3$,*

$$\sum_{k=2s}^{(n-1)/2+2s} (-1)^k [3k+1] \frac{(q;q^2)_{k-s}(q;q^2)_{k+s}(q;q^2)_k}{(q;q)_{k-2s}(q;q)_{k+2s}(q;q)_k} \equiv (-q)^{(n-1)^2/4}[n]. \tag{1.4}$$

Here and throughout the paper,

$$(a;q)_n = \begin{cases} (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n = 1, 2, \ldots, \\ 1, & \text{if } n = 0, \\ \dfrac{1}{(1-aq^{-1})(1-q^{-2})\cdots(1-aq^n)}, & \text{if } n = -1, -2, \ldots \end{cases}$$

is the *$q$-shifted factorial*, and $[n] = 1 + q + \cdots + q^{n-1}$ denotes the *$q$-integer*. For convenience, we will also adopt the compact $q$-notation for a product of $q$-shifted factorials: $(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$. Moreover, $\Phi_n(q)$ represents the $n$-th *cyclotomic polynomial* in $q$, i.e.,

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k \leqslant n \\ \gcd(k,n)=1}} (q - \zeta^k),$$

where $\zeta$ is an $n$-th primitive root of unity. It is not hard to see that (1.3) follows from (1.4) by taking $n = p$ and $q \to 1$.

Note that a similar $q$-supercongruence also with the parameter $s$ was given by the second author [7]. For some other important work on $q$-supercongruences, we refer the reader to [1, 10–12, 16–18, 20].

The second author and Zudilin [9, Theorem 4.4] proved that, for any odd integer $n > 1$,

$$\sum_{k=0}^{(n-1)/2} (-1)^k [6k+1] \frac{(q;q^2)_k^3}{(q^4;q^4)_k^3} \equiv (-q)^{-(n-1)(n+5)/8}[n] \pmod{[n]\Phi_n(q)^2}, \tag{1.5}$$

which is a $q$-analogue of the (L.2) supercongruence of Van Hamme [15]:

$$\sum_{k=0}^{(p-1)/2} (-1)^k \frac{(\frac{1}{2})_k^3}{k!^3 2^{3k}} (6k+1) \equiv p\left(\frac{-2}{p}\right) \pmod{p^3} \quad \text{for } p > 2,$$

where $(\frac{\cdot}{p})$ is the Legendre symbol modulo $p$.

In this paper, we shall give the following generalization of (1.5) in the modulus $\Phi_n(q)^2$ case.

**Theorem 1.2.** *Let $n > 1$ be an odd integer and let $0 \leqslant s \leqslant (n-1)/4$. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=s}^{(n-1)/2+s} (-1)^k [6k+1] \frac{(q;q^2)_{k-2s}(q;q^2)_{k+2s}(q;q^2)_k}{(q^4;q^4)_{k-s}(q^4;q^4)_{k+s}(q^4;q^4)_k} \equiv (-q)^{-(n-1)(n+5)/8}[n]. \tag{1.6}$$

Note that the $q$-congruence (1.6) does not hold $\Phi_n(q)^3$ in general. Letting $n = p$ and $q \to 1$ in (1.6), we obtain the following supercongruence: for $p > 2$ and $0 \leqslant s \leqslant (p-1)/4$,

$$\sum_{k=s}^{(p-1)/2+s} (-1)^k \frac{(\frac{1}{2})_{k-2s}(\frac{1}{2})_{k+2s}(\frac{1}{2})_k}{(k-s)!(k+s)!k!2^{3k}} (6k+1) \equiv p(-1)^{(p-1)/2} \pmod{p^2},$$

which is an extension of (1.5) modulo $p^2$.

We shall prove Theorems 1.1 and 1.2 in Sections 2 and 3, respectively.

# 2 Proof of Theorem 1.1

Following Gasper and Rahman's monograph [3], the *basic hypergeometric series* ${}_{r+1}\phi_r$ with $r+1$ upper parameters $a_1, \ldots, a_{r+1}$, $r$ lower parameters $b_1, \ldots, b_r$, base $q$, and argument $z$ is defined as

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix}; q, z\right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_k z^k}{(q, b_1, \ldots, b_r; q)_k}.$$

Then Jackson's $q$-analogue of Dougall's ${}_7F_6$ summation formula (see [3, Appendix (II.22)]) can be written as

$${}_8\phi_7\left[\begin{matrix} a, & qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix}; q, q\right]$$

$$= \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n}, \tag{2.1}$$

where $a^2q = bcdeq^{-n}$, and a quadratic transformation formula of Rahman [3, (3.8.13)] can be stated as follows:

$$\sum_{k=0}^{\infty} \frac{(a;q^2)_k(1-aq^{3k})(d, aq/d; q^2)_k(b, c, aq/bc; q)_k}{(q;q)_k(1-a)(aq/d, d; q)_k(aq^2/b, aq^2/c, bcq; q^2)_k} q^k$$
$$= \frac{(aq^2, bq, cq, aq^2/bc; q^2)_\infty}{(q, aq^2/b, aq^2/c, bcq; q^2)_\infty} {}_3\phi_2\left[\begin{matrix} b, & c, & aq/bc \\ & dq, & aq^2/d \end{matrix}; q^2, q^2\right]. \tag{2.2}$$

We also need the following quadratic transformation formula due to Gasper and Rahman (see [3, (3.8.14)]):

$$\sum_{k=0}^{n} \frac{1-acq^{3k}}{1-ac} \frac{(a, b, cq/b; q)_k(f, a^2c^2q^{2n+1}/f, q^{-2n}; q^2)_k}{(cq^2, acq^2/b, abq; q^2)_k(acq/f, f/acq^{2n}, acq^{2n+1}; q)_k} q^k$$
$$= \frac{(acq;q)_{2n}(ac^2q^2/bf, abq/f; q^2)_n}{(acq/f;q)_{2n}(abq, ac^2q^2/b; q^2)_n}$$
$$\times {}_{10}W_9(ac^2/b; f, ac/b, c, cq/b, cq^2/b, a^2c^2q^{2n+1}/f, q^{-2n}; q^2, q^2), \tag{2.3}$$

where

$${}_{r+3}W_{r+2}(a_0; a_1, a_2, ..., a_r; q, z) = \sum_{k=0}^{\infty} \frac{(1-a_0q^{2k})}{(1-a_0)} \frac{(a_0;q)_k(a_1;q)_k \cdots (a_r;q)_k}{(q;q)_k(qa_0/a_1;q)_k \cdots (qa_0/a_r;q)_k} z^k.$$

**Lemma 2.1.** *Let $n > 1$ be an odd integer. Let $a$ be an indeterminate and let $s \geqslant 0$. Then*

$$\sum_{k=2s}^{(n-1)/2+2s} (-1)^k \frac{(1-q^{3k+1-n})(aq;q^2)_{k-s}(q^{1-n};q^2)_{k+s}(q^{1-n}/a;q^2)_k}{(1-q^{1-n})(q;q)_{k-2s}(q^{1-n}/a;q)_{k+2s}(aq;q)_k} = 0. \tag{2.4}$$

*Proof.* It is easy to see that the left-hand side of (2.4) can be written as

$$\sum_{k=0}^{(n-1)/2} (-1)^{k+2s} \frac{(1-q^{3k+6s+1-n})(aq;q^2)_{k+s}(q^{1-n};q^2)_{k+3s}(q^{1-n}/a;q^2)_{k+2s}}{(1-q^{1-n})(q;q)_k(q^{1-n}/a;q)_{k+4s}(aq;q)_{k+2s}}$$
$$= \frac{(aq;q^2)_s(q^{1-n};q^2)_{3s}(q^{1-n}/a;q^2)_{2s}}{(q^{1-n}/a;q)_{4s}(aq;q)_{2s}}$$
$$\times \sum_{k=0}^{(n-1)/2} (-1)^k \frac{(1-q^{3k+6s+1-n})(aq^{2s+1}, q^{6s+1-n}, q^{4s+1-n}/a; q^2)_k}{(1-q^{1-n})(q, q^{4s+1-n}/a, aq^{2s+1}; q)_k}. \tag{2.5}$$

If $s \geqslant (n-1)/6$, then either $(q^{1-n};q^2)_{3s} = 0$ or $(1-q^{3k+6s+1-n})(q^{6s+1-n};q^2)_k = 0$, and so the right-hand side of (2.5) vanishes. If $s < (n-1)/6$, then taking $a = q^{6s+1-n}$, $d = aq^{2s+1}$, $b \mapsto q/b$, $c \mapsto q/c$ in (2.2), we get

$$\sum_{k=0}^{\infty} \frac{1-q^{3k+6s+1-n}}{1-q^{6s+1-n}} \frac{(q^{6s+1-n}, aq^{2s+1}, q^{4s+1-n}/a; q^2)_k(q/b, q/c, q^{6s-n}bc; q)_k}{(q^{4s+1-n}/a, aq^{2s+1}, q; q)_k(q^{6s+2-n}b, q^{6s+2-n}c, q^3/bc; q^2)_k} q^k$$

$$= \frac{(q^{6s+3-n}, q^2/b, q^2/c, q^{6s+1-n}bc; q^2)_\infty}{(q, q^{6s+2-n}b, q^{6s+2-n}c, q^3/bc; q^2)_\infty} {}_3\phi_2\left[\begin{matrix} q/b, & q/c, & q^{6s-n}bc \\ & aq^{2s+2}, & q^{4s+2-n}/a \end{matrix}; q^2, q^2\right]. \tag{2.6}$$

The left-hand side of (2.6) terminates at $k = (n-1)/2 - 3s$, while the right-hand side of (2.6) is equal to 0, because the numerator of the fraction before the ${}_3\phi_2$ series contains the factor $(q^{6s+3-n}; q^2)_\infty$. Thus, letting $b, c \to 0$ in (2.6), we immediately conclude that the right-hand side of (2.5) vanishes. □

We now give a parametric version of Theorem 1.1.

**Theorem 2.2.** *Let $n > 1$ be an odd integer and let $0 \leqslant s \leqslant (n-1)/8$. Then, modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,*

$$\sum_{k=2s}^{(n-1)/2+2s} (-1)^k [3k+1] \frac{(aq; q^2)_{k-s}(q; q^2)_{k+s}(q/a; q^2)_k}{(q; q)_{k-2s}(q/a; q)_{k+2s}(aq; q)_k} \equiv (-q)^{(n-1)^2/4}[n]. \tag{2.7}$$

*Proof.* Since $q^n \equiv 1 \pmod{\Phi_n(q)}$, it follows from (2.4) that

$$\sum_{k=2s}^{(n-1)/2+2s} (-1)^k [3k+1] \frac{(aq; q^2)_{k-s}(q; q^2)_{k+s}(q/a; q^2)_k}{(q; q)_{k-2s}(q/a; q)_{k+2s}(aq; q)_k} \equiv 0 \pmod{\Phi_n(q)}. \tag{2.8}$$

Namely, the $q$-congruence (2.7) is true modulo $\Phi_n(q)$.

For $a = q^{-n}$, the left-hand side of (2.7) can be written as

$$\begin{aligned} &\sum_{k=2s}^{(n-1)/2+2s} (-1)^k [3k+1] \frac{(q^{1-n}; q^2)_{k-s}(q; q^2)_{k+s}(q^{1+n}; q^2)_k}{(q; q)_{k-2s}(q^{1+n}; q)_{k+2s}(q^{1-n}; q)_k} \\ &= \sum_{k=0}^{(n-1)/2-s} (-1)^{k+2s} [3k+6s+1] \frac{(q^{1-n}; q^2)_{k+s}(q; q^2)_{k+3s}(q^{1+n}; q^2)_{k+2s}}{(q; q)_k (q^{1+n}; q)_{k+4s}(q^{1-n}; q)_{k+2s}} \\ &= \frac{(q^{1-n}; q^2)_s (q; q^2)_{3s}(q^{1+n}; q^2)_{2s}}{(q^{1+n}; q)_{4s}(q^{1-n}; q)_{2s}} \\ &\quad \times \sum_{k=0}^{(n-1)/2-s} (-1)^k [3k+6s+1] \frac{(q^{2s+1-n}, q^{6s+1}, q^{4s+1+n}; q^2)_k}{(q, q^{4s+1+n}, q^{2s+1-n}; q)_k}. \end{aligned} \tag{2.9}$$

Applying (2.3) in which the parameters $a$, $b$, $c$, $f$, and $n$ are replaced by $q^{6s+1}/b$, $q/c$, $b$, $q^{6s+1}$, and $(n-1)/2 - s$, respectively, we have

$$\begin{aligned} &\sum_{k=0}^{(n-1)/2-s} \frac{(1-q^{3k+6s+1})(q^{6s+1}/b, q/c, bc; q)_k (q^{6s+1}, q^{4s+1+n}, q^{2s+1-n}; q^2)_k}{(1-q^{6s+1})(bq^2, cq^{6s+2}, q^{6s+3}/bc; q^2)_k (q, q^{2s+1-n}, q^{4s+1+n}; q)_k} q^k \\ &= \frac{(q^{6s+2}; q)_{n-2s-1}(bcq, q^2/bc; q^2)_{(n-1)/2-s}}{(q; q)_{n-2s-1}(q^{6s+3}/bc, bcq^{6s+2}; q^2)_{(n-1)/2-s}} \end{aligned}$$

$$
\begin{aligned}
&\times {}_{10}W_9(bcq^{6s};q^{6s+1},b,cq^{6s},bc,bcq,q^{4s+1+n},q^{1+2s-n};q^2,q^2)\\
&=\frac{(q^{6s+2};q)_{n-2s-1}(bcq,q^2/bc;q^2)_{(n-1)/2-s}}{(q;q)_{n-2s-1}(q^{6s+3}/bc,bcq^{6s+2};q^2)_{(n-1)/2-s}}\\
&\times {}_8W_7(bcq^{6s};b,cq^{6s},bc,q^{4s+1+n},q^{1+2s-n};q^2,q^2)\\
&=\frac{(q^{6s+2};q)_{n-2s-1}(bcq,q^2/bc;q^2)_{(n-1)/2-s}(bcq^{6s+2},q^2,q^{6s+2}/b,q^2/c;q^2)_{(n-1)/2-s}}{(q;q)_{n-2s-1}(q^{6s+3}/bc,bcq^{6s+2};q^2)_{(n-1)/2-s}(cq^{6s+2},q^{6s+2},bq^2,q^2/bc;q^2)_{(n-1)/2-s}}\\
&=\frac{(q^{6s+2};q)_{n-2s-1}(bcq;q^2)_{(n-1)/2-s}(q^{6s+2}/b,q^2/c,q^2;q^2)_{(n-1)/2-s}}{(q;q)_{n-2s-1}(q^{6s+3}/bc;q^2)_{(n-1)/2-s}(cq^{6s+2},bq^2,q^{6s+2};q^2)_{(n-1)/2-s}},
\end{aligned}
$$

where we have used the summation formula (2.1). Letting $b,c\to 0$ in the above identity leads to

$$
\begin{aligned}
&\sum_{k=0}^{(n-1)/2-s}(-1)^k\frac{[3k+6s+1](q^{2s+1-n},q^{6s+1},q^{4s+1+n};q^2)_k}{[6s+1](q,q^{4s+1+n},q^{2s+1-n};q)_k}\\
&\quad=(-1)^{(n-1)/2-s}\frac{(q^{6s+3};q^2)_{(n-1)/2-s}}{(q;q^2)_{(n-1)/2-s}}q^{(n-1-2s)^2/4}.
\end{aligned}\tag{2.10}
$$

Substituting (2.10) into (2.9) and making some simplifications, we get

$$
\sum_{k=2s}^{(n-1)/2+2s}(-1)^k[3k+1]\frac{(q^{1-n};q^2)_{k-s}(q;q^2)_{k+s}(q^{1+n};q^2)_k}{(q;q)_{k-2s}(q^{1+n};q)_{k+2s}(q^{1-n};q)_k}=(-q)^{(n-1)^2/4}[n].
$$

This indicates that the $q$-congruence (2.7) holds modulo $1-aq^n$.

Similarly, for $a=q^n$, since $0\leqslant s\leqslant (n-1)/8$, the left-hand side of (2.7) is well-defined and can be written as

$$
\begin{aligned}
&\sum_{k=2s}^{(n-1)/2+2s}(-1)^k[3k+1]\frac{(q^{1+n};q^2)_{k-s}(q;q^2)_{k+s}(q^{1-n};q^2)_k}{(q;q)_{k-2s}(q^{1-n};q)_{k+2s}(q^{1+n};q)_k}\\
&=\sum_{k=0}^{(n-1)/2}(-1)^{k+2s}[3k+6s+1]\frac{(q^{1+n};q^2)_{k+s}(q;q^2)_{k+3s}(q^{1-n};q^2)_{k+2s}}{(q;q)_k(q^{1-n};q)_{k+4s}(q^{1+n};q)_{k+2s}}\\
&=\frac{(q^{1+n};q^2)_s(q;q^2)_{3s}(q^{1-n};q^2)_{2s}}{(q^{1-n};q)_{4s}(q^{1+n};q)_{2s}}\\
&\quad\times\sum_{k=0}^{(n-1)/2-2s}(-1)^k[3k+6s+1]\frac{(q^{2s+1+n},q^{6s+1},q^{4s+1-n};q^2)_k}{(q,q^{4s+1-n},q^{2s+1+n};q)_k},
\end{aligned}\tag{2.11}
$$

Replacing $a$, $b$, $c$, $f$, and $n$ in (2.3) by $q^{6s+1}/b$, $q/c$, $b$, $q^{6s+1}$, and $(n-1)/2-2s$, respectively, we get

$$
\sum_{k=0}^{(n-1)/2-2s}\frac{(1-q^{3k+6s+1})(q^{6s+1}/b,q/c,bc;q)_k(q^{6s+1},q^{4s+1-n},q^{2s+1+n};q^2)_k}{(1-q^{6s+1})(bq^2,cq^{6s+2},q^{6s+3}/bc;q^2)_k(q,q^{2s+1+n},q^{4s+1-n};q)_k}q^k
$$

$$
\begin{aligned}
&= \frac{(q^{6s+2};q)_{n-4s-1}(bcq,q^2/bc;q^2)_{(n-1)/2-2s}}{(q;q)_{n-4s-1}(q^{6s+3}/bc,bcq^{6s+2};q^2)_{(n-1)/2-2s}} \\
&\quad \times {}_{10}W_9(bcq^{6s};q^{6s+1},b,cq^{6s},bc,bcq,q^{2s+1+n},q^{1+4s-n};q^2,q^2) \\
&= \frac{(q^{6s+2};q)_{n-4s-1}(bcq,q^2/bc;q^2)_{(n-1)/2-2s}}{(q;q)_{n-4s-1}(q^{6s+3}/bc,q^{6s+2}bc;q^2)_{(n-1)/2-2s}} \\
&\quad \times {}_{8}W_7(bcq^{6s};b,cq^{6s},bc,q^{2s+1+n},q^{1+4s-n};q^2,q^2) \\
&= \frac{(q^{6s+3};q^2)_{(n-1)/2-2s}(bcq;q^2)_{(n-1)/2-2s}(q^{6s+2}/b,q^2/c;q^2)_{(n-1)/2-2s}}{(q;q^2)_{(n-1)/2-2s}(q^{6s+3}/bc;q^2)_{(n-1)/2-2s}(q^{6s+2}c,q^2b;q^2)_{(n-1)/2-2s}}.
\end{aligned}
$$

Letting $b,c\to 0$ in the above identity implies that

$$
\begin{aligned}
&\sum_{k=0}^{(n-1)/2-2s}(-1)^k\frac{[3k+6s+1](q^{2s+1+n},q^{6s+1},q^{4s+1-n};q^2)_k}{[6s+1](q,q^{4s+1-n},q^{2s+1+n};q)_k} \\
&\qquad = (-1)^{(n-1)/2-2s}\frac{(q^{6s+3};q^2)_{(n-1)/2-2s}}{(q;q^2)_{(n-1)/2-2s}}q^{(n-1-4s)^2/4}. \qquad (2.12)
\end{aligned}
$$

Substituting (2.12) into (2.11), we are led to

$$
\sum_{k=2s}^{(n-1)/2+2s}(-1)^k[3k+1]\frac{(q^{1+n};q^2)_{k-s}(q;q^2)_{k+s}(q^{1-n};q^2)_k}{(q;q)_{k-2s}(q^{1-n};q)_{k+2s}(q^{1+n};q)_k}=(-q)^{(n-1)^2/4}[n].
$$

This means that the $q$-congruence (2.7) holds modulo $a-q^n$.

Since $\Phi_n(q)$, $1-aq^n$, and $a-q^n$ are pairwise relatively prime polynomials in $q$, we complete the proof of (2.7). □

*Proof of Theorem 1.1.* When $a=1$ the denominators on both sides of (2.7) are coprime with $\Phi_n(q)$, and the polynomial $(1-aq^n)(a-q^n)=(1-q^n)^2$ contains the factor $\Phi_n(q)^2$. Hence, letting $a=1$ in (2.7), we obtain the desired $q$-supercongruence (1.4). □

# 3 Proof of Theorem 1.2

We need the following summation formula of Rahman [13, (4.6)]:

$$
\sum_{k=0}^{\infty}\frac{(1-aq^{3k})(a,d,q/d;q)_k(b;q^2)_k}{(1-a)(q^2,aq^2/d,adq;q^2)_k(aq/b;q)_k}\frac{a^kq^{\binom{k+1}{2}}}{b^k}=\frac{(aq,aq^2,adq/b,aq^2/bd;q^2)_\infty}{(aq/b,aq^2/b,aq^2/d,adq;q^2)_\infty}, \qquad (3.1)
$$

which was already utilized by the second author and Zudilin [9] to prove (1.5).

Like before, we first give the following $q$-series identity.

**Lemma 3.1.** *Let $n>1$ be an odd integer. Let $a$ be an indeterminate and let $s\geqslant 0$. Then*

$$
\sum_{k=s}^{(n-1)/2+s}(-1)^k\frac{(1-q^{6k+1-n})(aq;q^2)_{k-2s}(q^{1-n};q^2)_{k+2s}(q/a;q^2)_k}{(1-q^{1-n})(q^4;q^4)_{k-s}(q^{4-n}/a;q^4)_{k+s}(aq^{4-n};q^4)_k}q^{3k^2-nk}=0. \qquad (3.2)
$$

*Proof.* It is easy to see that the left-hand side of (3.2) can be written as

$$\sum_{k=0}^{(n-1)/2}(-1)^{k+s}\frac{(1-q^{6k+6s+1-n})(aq;q^2)_{k-s}(q^{1-n};q^2)_{k+3s}(q/a;q^2)_{k+s}}{(1-q^{1-n})(q^4;q^4)_k(q^{4-n}/a;q^4)_{k+2s}(aq^{4-n};q^4)_{k+s}}q^{3(k+s)^2-n(k+s)}$$
$$=(-1)^s\frac{(q/a;q^2)_s(aq;q^2)_{-s}(q^{1-n};q^2)_{3s}}{(q^{4-n}/a;q^4)_{2s}(aq^{4-n};q^4)_s}q^{3s^2-ns}$$
$$\times\sum_{k=0}^{(n-1)/2}(-1)^k\frac{(1-q^{6k+6s+1-n})(q^{6s+1-n},aq^{1-2s},q^{2s+1}/a;q^2)_k}{(1-q^{1-n})(q^4,q^{8s+4-n}/a,aq^{4s+4-n};q^4)_k}q^{3k^2+6ks-nk}. \tag{3.3}$$

If $s\geqslant(n-1)/6$, then either $(q^{1-n};q^2)_{3s}=0$ or $(1-q^{6k+6s+1-n})(q^{6s+1-n};q^2)_k=0$, and so the right-hand side of (3.3) vanishes. If $s<(n-1)/6$, then performing the parameter substitutions $q\mapsto q^2, a=q^{6s+1-n}, d=aq^{1-2s}$ in (3.1), we obtain

$$\sum_{k=0}^{\infty}\frac{(1-q^{6k+6s+1-n})(q^{6s+1-n},aq^{1-2s},q^{2s+1}/a;q^2)_k(b;q^4)_k}{(1-q^{6s+1-n})(q^4,q^{8s+4-n}/a,aq^{4s+4-n};q^4)_k(q^{6s+3-n}/b;q^2)_k}\frac{q^{(6s+1-n)k+(k^2+k)}}{b^k}$$
$$=\frac{(q^{6s+3-n},q^{6s+5-n},aq^{4s+4-n}/b,q^{8s+4-n}/ab;q^4)_\infty}{(q^{6s+3-n}/b,q^{6s+5-n}/b,q^{8s+4-n}/a,aq^{4s+4-n};q^4)_\infty}. \tag{3.4}$$

The left-hand side of (3.4) terminates at $k=(n-1)/2-3s$, while the right-hand side of (3.4) vanishes because the numerator has the factor $(q^{6s+3-n},q^{6s+5-n};q^4)_\infty$. Letting $b\to\infty$ in (3.4), we see that the summation on the right-hand side of (3.3) vanishes. □

We have the following parametric generalization of Theorem 1.2.

**Theorem 3.2.** *Let $n>1$ be an odd integer and let $0\leqslant s\leqslant(n-1)/4$. Then, modulo $\Phi_n(q)(a-q^n)$,*

$$\sum_{k=s}^{(n-1)/2+s}(-1)^k[6k+1]\frac{(aq;q^2)_{k-2s}(q;q^2)_{k+2s}(q/a;q^2)_k}{(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k}\equiv(-q)^{-(n-1)(n+5)/8}[n]. \tag{3.5}$$

*Proof.* Replacing $a$ and $q$ by $a^{-1}$ and $q^{-1}$, respectively, the $q$-congruence (3.5) is equivalent to

$$\sum_{k=s}^{(n-1)/2+s}(-1)^k[6k+1]\frac{(aq;q^2)_{k-2s}(q;q^2)_{k+2s}(q/a;q^2)_k}{(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k}q^{3k^2}$$
$$\equiv(-q)^{(n-1)(n-3)/8+4s^2}a^{-s}[n]\pmod{\Phi_n(q)(a-q^n)}. \tag{3.6}$$

In view of $q^n\equiv1\pmod{\Phi_n(q)}$, from (3.2) we deduce that

$$\sum_{k=s}^{(n-1)/2+s}(-1)^k[6k+1]\frac{(aq;q^2)_{k-2s}(q;q^2)_{k+2s}(q/a;q^2)_k}{(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k}q^{3k^2}\equiv0\pmod{\Phi_n(q)}.$$

Namely, the $q$-congruence (3.5) holds modulo $\Phi_n(q)$.

For $a=q^n$, the right-hand side of (3.6) can be written as

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2}(-1)^{k+s}[6k+6s+1]\frac{(q^{1+n};q^2)_{k-s}(q;q^2)_{k+3s}(q^{1-n};q^2)_{k+s}}{(q^4;q^4)_k(q^{4-n};q^4)_{k+2s}(q^{4+n};q^4)_{k+s}}q^{3(k+s)^2}\\
&\quad=(-1)^s q^{3s^2}\frac{(q^{1+n};q^2)_{-s}(q;q^2)_{3s}(q^{1-n};q^2)_s}{(q^{4-n};q^4)_{2s}(q^{4+n};q^4)_s}\\
&\qquad\times\sum_{k=0}^{(n-1)/2}(-1)^k[6k+6s+1]\frac{(q^{1+n-2s},q^{6s+1},q^{2s+1-n};q^2)_k}{(q^4,q^{8s+4-n},q^{4s+4+n};q^4)_k}q^{3k^2+6ks}. \qquad (3.7)
\end{aligned}$$

Setting $q\mapsto q^2, a=q^{6s+1}, d=q^{n+1-2s}$ and $b\to\infty$ in (3.1), we get

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2-s}(-1)^k\frac{1-q^{6k+6s+1}}{1-q^{6s+1}}\frac{(q^{1+n-2s};q^2)_k(q^{6s+1};q^2)_k(q^{2s+1-n};q^2)_k}{(q^4;q^4)_k(q^{8s+4-n};q^4)_k(q^{4s+4+n};q^4)_k}q^{3k^2+6ks}\\
&=\frac{(q^{6s+3},q^{6s+5};q^4)_\infty}{(q^{8s+4-n},q^{4s+4+n};q^4)_\infty}\\
&=\begin{cases}\dfrac{(q^{6s+5};q^4)_{(n-1-2s)/4}}{(q^{8s+4-n};q^4)_{(n-1-2s)/4}}, & \text{if } n\equiv 2s+1 \pmod 4,\\[2ex] \dfrac{(q^{6s+3};q^4)_{(n+1-2s)/4}}{(q^{8s+4-n};q^4)_{(n+1-2s)/4}}, & \text{if } n\equiv 2s+3 \pmod 4.\end{cases}
\end{aligned}$$

Plugging the above identity into (3.7) and making some simplifications, we obtain

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2}(-1)^{k+s}[6k+6s+1]\frac{(q^{1+n};q^2)_{k-s}(q;q^2)_{k+3s}(q^{1-n};q^2)_{k+s}}{(q^4;q^4)_k(q^{4-n};q^4)_{k+2s}(q^{4+n};q^4)_{k+s}}q^{3(k+s)^2}\\
&\quad=(-q)^{(n-1)(n-3)/8+4s^2}q^{-sn}[n].
\end{aligned}$$

This proves that (3.5) holds modulo $a-q^n$. Since $\Phi_n(q)$ and $a-q^n$ are coprime polynomials in $q$, we complete the proof of (3.5). □

*Proof of Theorem 1.2.* When $a=1$ the denominators on the two sides of (3.5) are coprime with $\Phi_n(q)$, and the polynomial $a-q^n=1-q^n$ has the factor $\Phi_n(q)$. Therefore, letting $a=1$ in (2.7), we are led to (1.6). □

# 4 Concluding remarks

Guillera and Zudilin [5] also proved the following "divergent" Ramanujan-type supercongruence:

$$\sum_{k=0}^{(p-1)/2}\frac{(\frac12)_k^3}{k!^3}(3k+1)2^{2k}\equiv p \pmod{p^3} \quad \text{for } p>2. \qquad (4.1)$$

However, we did not find any interesting generalization of (4.1) like (1.3), nor in the modulus $p^2$ case. Perhaps an interested reader may make some progress on this problem.

We believe that the following stronger version of Theorem 1.1 should be true.

**Conjecture 4.1.** *The $q$-supercongruence* (1.4) *holds modulo* $[n]\Phi_n(q)^2$.

Note that the $s=0$ case of Conjecture 4.1 was already proved by the second author [6]. A natural way to prove Conjecture 4.1 is to establish the following $q$-congruence: for any odd integer $n>1$ and non-negative integer $s$,

$$\sum_{k=2s}^{N+2s}(-1)^k[3k+1]\frac{(q;q^2)_{k-s}(q;q^2)_{k+s}(q;q^2)_k}{(q;q)_{k-2s}(q;q)_{k+2s}(q;q)_k}\equiv 0 \pmod{\Phi_n(q)},$$

where $N=(n-1)/2$ or $n-1$, and then use the technique of roots of unity in [9]. However, we cannot deduce this $q$-congruence for $N=(n-1)/2$ from (2.8) by taking $a=1$ directly, since the denominators of the left-hand side of (2.8) might contain the factor $\Phi_n(q)$ when $a=1$.

We also conjecture that Theorem 1.2 has the following generalization.

**Conjecture 4.2.** *The $q$-supercongruence* (1.6) *holds modulo* $[n]\Phi_n(q)$.

Similarly, to prove Conjecture 4.2, we need to build the $q$-congruence: for any odd integer $n>1$ and non-negative integer $s$,

$$\sum_{k=2s}^{N+2s}(-1)^k[3k+1]\frac{(q;q^2)_{k-s}(q;q^2)_{k+s}(q;q^2)_k}{(q;q)_{k-2s}(q;q)_{k+2s}(q;q)_k}\equiv 0 \pmod{\Phi_n(q)},$$

where $N=(n-1)/2$ or $n-1$. But it seems difficult to accomplish this. We hope that an interested reader can settle these two conjectures.

# 5 Statements and Declarations

**Competing interests:** The authors have no relevant financial or non-financial interests to disclose.

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