Further q-supercongruences from the q-Pfaff–Saalschütz identity

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Abstract. Recently, Wei, Liu, and Wang gave a q-analogue of a supercongruence of Long and Ramakrishna [Adv. Math. 290 (2016), 773–808] from the q-Pfaff–Saalschütz identity. In this note, we present a generalization of Wei–Liu–Wang's q-supercongruence with one more parameter by using the q-Pfaff–Saalschütz identity and the method of 'creative microscoping' introduced by the second author and Zudilin again.

Keywords: creative microscoping; p-adic Gamma function q-Pfaff–Saalschütz identity; q-supercongruences

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1 Introduction

Let p be an odd prime and $\mathbb{Z}_p$ the ring of all p-adic integers. Recall that Morita's p-adic Gamma function (see [12, Chapter 7]) is defined as follows:

$$\Gamma_p(0) = 1 \quad \text{and} \quad \Gamma_p(n) = (-1)^n \prod_{\substack{1 \leqslant k < n \\ p \nmid k}} k, \quad \text{for } n = 1, 2, \dots.$$

Notice that $\mathbb{N}$ is a dense subset of $\mathbb{Z}_p$ with the p-adic norm $|\cdot|_p$. For each $x \in \mathbb{Z}_p$, the p-adic Gamma function $\Gamma_p(x)$ is given by

$$\Gamma_p(x) = \lim_{\substack{n \in \mathbb{N} \\ |x-n|_p \to 0}} \Gamma_p(n).$$

In 2006, Long and Ramakrishna [11, Proposition 25] established the following supercongruence: for any odd prime p,

$$\sum_{k=0}^{p-1} \frac{(\frac{1}{3})_k^3}{k!^3} \equiv \begin{cases} \Gamma_p(\frac{1}{3})^6 \pmod{p^3}, & \text{if } p \equiv 1 \pmod 6, \\ -\frac{p^2}{3}\Gamma_p(\frac{1}{3})^6 \pmod{p^3}, & \text{if } p \equiv 5 \pmod 6, \end{cases} \tag{1.1}$$

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where $(x)_n = x(x+1)\cdots(x+n-1)$ is the rising factorial. Later the second author [3, Theorem 1.1] obtained a partial q-analogue of the second case of (1.1): for $n \equiv 2 \pmod 3$,

$$\sum_{k=0}^{n-1} \frac{(q;q^3)_k^3}{(q^3;q^3)_k^3} q^{3k} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.2}$$

Here and in what follows, the q-shifted factorial is defined as

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}) \quad \text{for } n = 1, 2, \dots,$$

and $\Phi_n(q)$ stands for the n-th cyclotomic polynomial in q:

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}} (q-\zeta^k),$$

where ζ is an n-th primitive root of unity. For convenience, we will also adopt the compact notation:

$$(x_1, x_2, \dots, x_m; q)_n = (x_1;q)_n (x_2;q)_n \cdots (x_m;q)_n.$$

Moreover, the basic hypergeometric series ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q,\ z\right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_k}{(q, b_1, b_2, \dots, b_r; q)_k} z^k.$$

Recently, using the q-Pfaff–Saalschütz identity (see [2, Appendix (II.12)]):

$${}_3\phi_2\left[\begin{matrix} q^{-n}, a, b \\ c, q^{1-n}ab/c \end{matrix}; q,\ q\right] = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}, \tag{1.3}$$

together with the method of 'creative microscoping' introduced by the second author and Zudilin [6] and the Chinese remainder theorem for coprime polynomials, Wei, Liu, and Wang [14] gave the following complete q-analogue of (1.1): for any positive integer n with $n \equiv 1 \pmod 3$, modulo $\Phi_n(q)^3$, we have

$$\sum_{k=0}^{(n-1)/3} \frac{(q;q^3)_k^3}{(q^3;q^3)_k^3} q^{3k} \equiv q^{(n-1)/3} \frac{(q^2;q^3)_{(n-1)/3}^2}{(q^3;q^3)_{(n-1)/3}^2} \left(1 + [n]^2 \sum_{k=1}^{(n-1)/3} \frac{q^{3k-1}}{[3k-1]^2}\right), \tag{1.4}$$

where $[n] = (1-q^n)/(1-q)$ is the q-integer, and for any positive integer n with $n \equiv 2 \pmod 3$, modulo $\Phi_n(q)^3$, we have

$$\sum_{k=0}^{(2n-1)/3} \frac{(q;q^3)_k^3}{(q^3;q^3)_k^3} q^{3k} \equiv q^{(2n-1)/3} \frac{(q^2;q^3)_{(2n-1)/3}^2}{(q^3;q^3)_{(2n-1)/3}^2} \left(1 - [2n]^2 \sum_{k=1}^{(2n-1)/3} \frac{q^{3k-1}}{[3k-1]^2}\right). \tag{1.5}$$

For more recent q-supercongruences, we refer the reader to [1, 4, 5, 7–10, 13, 15] and references therein.

In this note, using the same method of Wei–Liu–Wang, we shall prove the following generalizations of the q-supercongruences (1.4) and (1.5).

Theorem 1.1. *Let n and s be positive integers with $n\equiv 1 \pmod 3$ and $s\leqslant (n-1)/3$. Then, modulo $\Phi_n(q)^3$,*

$$\sum_{k=s}^{(n-1)/3}\frac{(q;q^3)_k^3q^{3k}}{(q^3;q^3)_{k-s}(q^3;q^3)_{k+s}(q^3;q^3)_k}$$
$$\equiv \frac{(q;q^3)_s^3(q^2,q^{2+3s};q^3)_{(n-1)/3-s}}{(q^3;q^3)_{(n-1)/3+s}(q^3;q^3)_{(n-1)/3}}q^A\left(1+[n]^2\sum_{k=1}^{(n-1)/3-s}\frac{q^{3k+3s-1}}{[3k+3s-1]^2}\right), \tag{1.6}$$

where $A=3s+(1+3s)((n-1)/3-s)$.

It is easy to see that the $s=0$ case of (1.6) reduces to (1.4). Letting $n=p$ be an odd prime and taking $q\to 1$ in (1.6), we obtain the following supercongruence: for any prime $p\equiv 1 \pmod 3$ and nonnegative integer $s\leqslant (p-1)/3$,

$$\sum_{k=s}^{(p-1)/3}\frac{(\frac{1}{3})_k^3}{(k-s)!(k+s)!k!}$$
$$\equiv \frac{(\frac{1}{3})_s^3(\frac{2}{3})_{(p-1)/3-s}(\frac{2}{3})_{(p-1)/3}}{(\frac{2}{3})_s(1)_{(p-1)/3+s}(1)_{(p-1)/3}}\left(1+\sum_{k=1}^{(p-1)/3-s}\frac{p^2}{(3k+3s-1)^2}\right) \pmod{p^3}.$$

Theorem 1.2. *Let n and s be positive integers with $n\equiv 2 \pmod 3$ and $s\leqslant (n-2)/3$. Then, modulo $\Phi_n(q)^3$,*

$$\sum_{k=s}^{(2n-1)/3}\frac{(q;q^3)_k^3q^{3k}}{(q^3;q^3)_{k-s}(q^3;q^3)_{k+s}(q^3;q^3)_k}$$
$$\equiv \frac{(q;q^3)_s^3(q^2,q^{2+3s};q^3)_{(2n-1)/3-s}}{(q^3;q^3)_{(2n-1)/3+s}(q^3;q^3)_{(2n-1)/3}}q^B\left(1-[2n]^2\sum_{k=1}^{(2n-1)/3-s}\frac{q^{3k+3s-1}}{[3k+3s-1]^2}\right), \tag{1.7}$$

where $B=3s+(1+3s)((2n-1)/3-s)$.

When $s=0$, the q-supercongruence (1.7) reduces to (1.5). Similarly as before, from (1.7) we can deduce the following supercongruence: for any prime $p\equiv 2 \pmod 3$ and nonnegative integer $s\leqslant (p-2)/3$,

$$\sum_{k=s}^{(2p-1)/3}\frac{(\frac{1}{3})_k^3}{(k-s)!(k+s)!k!}$$
$$\equiv \frac{(\frac{1}{3})_s^3(\frac{2}{3})_{(2p-1)/3-s}(\frac{2}{3})_{(2p-1)/3}}{(\frac{2}{3})_s(1)_{(2p-1)/3+s}(1)_{(2p-1)/3}}\left(1-\sum_{k=1}^{(2p-1)/3-s}\frac{4p^2}{(3k+3s-1)^2}\right) \pmod{p^3}.$$

2 Proof of Theorem 1.1

In order to prove Theorem 1.1, we first give two lemmas.

Lemma 2.1. *Let n and s be positive integers with $n \equiv 1 \pmod 3$ and $s \leqslant (n-1)/3$. Then, modulo $(1-aq^n)(a-q^n)$,*

$$\sum_{k=s}^{(n-1)/3} \frac{(aq, q/a, q/b; q^3)_k q^{3k}}{(q^3;q^3)_{k-s}(q^3;q^3)_{k+s}(q^3/b;q^3)_k} \equiv \frac{(aq, q/a, q/b; q^3)_s (q^2, bq^{2+3s}; q^3)_{(n-1)/3-s} q^{3s}}{(q^3;q^3)_{(n-1)/3+s}(q^3/b;q^3)_{(n-1)/3}} \left(\frac{q^{1+3s}}{b}\right)^{(n-1)/3-s}. \tag{2.1}$$

Proof. When $a = q^{-n}$ or $a = q^n$, the left-hand side of (2.1) is equal to

$$\begin{aligned} &\sum_{k=s}^{(n-1)/3} \frac{(q^{1-n}, q^{1+n}, q/b; q^3)_k q^{3k}}{(q^3;q^3)_{k-s}(q^3;q^3)_{k+s}(q^3/b;q^3)_k} \\ &= \sum_{k=0}^{(n-1)/3-s} \frac{(q^{1-n}, q^{1+n}, q/b; q^3)_{k+s} q^{3k+3s}}{(q^3;q^3)_k(q^3;q^3)_{k+2s}(q^3/b;q^3)_{k+s}} \\ &= \frac{(q^{1-n}, q^{1+n}, q/b; q^3)_s q^{3s}}{(q^3;q^3)_{2s}(q^3/b;q^3)_s}\, {}_3\phi_2\left[\begin{matrix} q^{1-n+3s}, q^{1+n+3s}, q^{1+3s}/b \\ q^{3+6s}, q^{3+3s}/b \end{matrix}; q^3, q^3\right]. \end{aligned} \tag{2.2}$$

In view of the q-Pfaff–Saalschütz identity (1.3), the right-hand side of (2.2) can be simplified as

$$\begin{aligned} &\frac{(q^{1-n}, q^{1+n}, q/b; q^3)_s q^{3s}}{(q^3;q^3)_{2s}(q^3/b;q^3)_s} \cdot \frac{(q^{2+3s-n}, bq^{2+3s}; q^3)_{(n-1)/3-s}}{(q^{3+6s}, bq^{1-n}; q^3)_{(n-1)/3-s}} \\ &= \frac{(q^{1-n}, q^{1+n}, q/b; q^3)_s (q^2, bq^{2+3s}; q^3)_{(n-1)/3-s} q^{3s}}{(q^3;q^3)_{(n-1)/3+s}(q^3/b;q^3)_{(n-1)/3}} \left(\frac{q^{1+3s}}{b}\right)^{(n-1)/3-s}, \end{aligned}$$

which is equal to the right-hand side of (2.1) for $a = q^{-n}$ or $a = q^n$. Namely, the q-congruence (2.1) holds modulo $1-aq^n$ and $a-q^n$. Since $1-aq^n$ and $a-q^n$ are coprime polynomials in q, we complete the proof. □

Lemma 2.2. *Let n and s be positive integers with $n \equiv 1 \pmod 3$ and $s \leqslant (n-1)/3$. Then, modulo $1-bq^n$,*

$$\sum_{k=s}^{(n-1)/3} \frac{(aq, q/a, q/b; q^3)_k q^{3k}}{(q^3;q^3)_{k-s}(q^3;q^3)_{k+s}(q^3/b;q^3)_k} \equiv \frac{(aq, q/a, q/b; q^3)_s (aq^{2+3s}, q^{2+3s}/a; q^3)_{(n-1)/3-s} q^{3s}}{(q^3;q^3)_{(n-1)/3+s}(q^3/b;q^3)_s(q;q^3)_{(n-1)/3-s}}. \tag{2.3}$$

Proof. When $b = q^n$, the left-hand side of (2.3) is equal to

$$
\begin{aligned}
&\sum_{k=s}^{(n-1)/3} \frac{(aq, q/a, q^{1-n}; q^3)_k q^{3k}}{(q^3; q^3)_{k-s}(q^3; q^3)_{k+s}(q^{3-n}; q^3)_k} \\
&= \sum_{k=0}^{(n-1)/3-s} \frac{(aq, q/a, q^{1-n}; q^3)_{k+s} q^{3k+3s}}{(q^3; q^3)_k (q^3; q^3)_{k+2s}(q^{3-n}; q^3)_{k+s}} \\
&= \frac{(aq, q/a, q^{1-n}; q^3)_s q^{3s}}{(q^3; q^3)_{2s}(q^{3-n}; q^3)_s}\, {}_3\phi_2\left[\begin{matrix} aq^{1+3s}, q^{1+3s}/a, q^{1-n+3s} \\ q^{3+6s}, q^{3-n+3s} \end{matrix}; q^3, q^3\right]. \qquad (2.4)
\end{aligned}
$$

By (1.3), the right-hand side of (2.4) can be simplified as

$$
\frac{(aq, q/a, q^{1-n}; q^3)_s q^{3s}}{(q^3; q^3)_{2s}(q^{3-n}; q^3)_s} \cdot \frac{(aq^{2+3s}, q^{2+3s}/a; q^3)_{(n-1)/3-s}}{(q^{3+6s}; q^3)_{(n-1)/3-s}(q; q^3)_{(n-1)/3-s}},
$$

which is just the right-hand side of (2.3) for $b = q^n$. Namely, the q-congruence (2.3) holds. □

Proof of Theorem 1.1. It is obvious that the polynomials $(1 - aq^n)(a - q^n)$ and $b - q^n$ are coprime polynomials in q. Noticing the q-congruences

$$
\begin{aligned}
\frac{(b - q^n)(ab - 1 - a^2 + aq^n)}{(a - b)(1 - ab)} &\equiv 1 \pmod{(1 - aq^n)(a - q^n)}, \\
\frac{(1 - aq^n)(a - q^n)}{(a - b)(1 - ab)} &\equiv 1 \pmod{b - q^n},
\end{aligned}
$$

and applying the Chinese remainder theorem for coprime polynomials, we are led to the following q-congruence: modulo $(1 - aq^n)(a - q^n)(1 - bq^n)$,

$$
\begin{aligned}
&\sum_{k=s}^{(n-1)/3} \frac{(aq, q/a, q/b; q^3)_k q^{3k}}{(q^3; q^3)_{k-s}(q^3; q^3)_{k+s}(q^3/b; q^3)_k} \\
&\equiv \frac{(b - q^n)(ab - 1 - a^2 + aq^n)}{(a - b)(1 - ab)} \\
&\quad \times \frac{(aq, q/a, q/b; q^3)_s (q^2, bq^{2+3s}; q^3)_{(n-1)/3-s} q^{3s}}{(q^3; q^3)_{(n-1)/3+s}(q^3/b; q^3)_{(n-1)/3}} \left(\frac{q^{1+3s}}{b}\right)^{(n-1)/3-s} \\
&\quad + \frac{(1 - aq^n)(a - q^n)}{(a - b)(1 - ab)} \frac{(aq, q/a, q/b; q^3)_s (aq^{2+3s}, q^{2+3s}/a; q^3)_{(n-1)/3-s} q^{3s}}{(q^3; q^3)_{(n-1)/3+s}(q^3/b; q^3)_s (q; q^3)_{(n-1)/3-s}}. \qquad (2.5)
\end{aligned}
$$

Taking $b = 1$ in (2.5) and using the identity

$$
(1 - q^n)(1 + a^2 - a - aq^n) = (1 - a)^2 + (1 - aq^n)(a - q^n),
$$

we arrive at the following q-congruence: modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,

$$\begin{aligned}
&\sum_{k=s}^{(n-1)/3} \frac{(aq,q/a,q;q^3)_k q^{3k}}{(q^3;q^3)_{k-s}(q^3;q^3)_{k+s}(q^3;q^3)_k} \\
&\quad\equiv \frac{(aq,q/a,q;q^3)_s(q^2,q^{2+3s};q^3)_{(n-1)/3-s}q^{3s+(1+3s)((n-1)/3-s)}}{(q^3;q^3)_{(n-1)/3+s}(q^3;q^3)_{(n-1)/3}} \\
&\qquad+\frac{(1-aq^n)(a-q^n)(aq,q/a,q;q^3)_s q^{3s}}{(1-a)^2(q^3;q^3)_{(n-1)/3+s}} \\
&\qquad\times\left(\frac{(q^{2+3s},q^{2+3s};q^3)_{(n-1)/3-s}}{(q^3;q^3)_s(q;q^3)_{(n-1)/3-s}}-\frac{(aq^{2+3s},q^{2+3s}/a;q^3)_{(n-1)/3-s}}{(q^3;q^3)_s(q;q^3)_{(n-1)/3-s}}\right),
\end{aligned} \tag{2.6}$$

where we have utilized the q-congruence

$$\begin{aligned}
q^{(1+3s)((n-1)/3-s)}\frac{(q^2;q^3)_{(n-1)/3-s}}{(q^3;q^3)_{(n-1)/3}} &= \frac{q^{(1+3s)((n-1)/3-s)}(q^2;q^3)_{(n-1)/3-s}}{(q^3;q^3)_s(q^{3+3s};q^3)_{(n-1)/3-s}} \\
&= \frac{(q^{2-n+3s};q^3)_{(n-1)/3-s}}{(q^3;q^3)_s(q^{1-n};q^3)_{(n-1)/3-s}} \\
&\equiv \frac{(q^{2+3s};q^3)_{(n-1)/3-s}}{(q^3;q^3)_s(q;q^3)_{(n-1)/3-s}} \pmod{\Phi_n(q)}
\end{aligned} \tag{2.7}$$

in the brackets.

By the L'Hôspital rule, we get

$$\begin{aligned}
&\lim_{a\to1}\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\big((q^{2+3s},q^{2+3s};q^3)_{(n-1)/3-s}-(aq^{2+3s},q^{2+3s}/a;q^3)_{(n-1)/3-s}\big) \\
&\quad=[n]^2(q^{2+3s};q^3)^2_{(n-1)/3-s}\sum_{k=1}^{(n-1)/3-s}\frac{q^{3k+3s-1}}{[3k+3s-1]^2}.
\end{aligned}$$

Letting $a\to1$ in (3.1) and applying the above limit, we obtain the following result: modulo $\Phi_n(q)^3$,

$$\begin{aligned}
&\sum_{k=s}^{(n-1)/3}\frac{(q;q^3)_k^3 q^{3k}}{(q^3;q^3)_{k-s}(q^3;q^3)_{k+s}(q^3;q^3)_k} \\
&\quad\equiv\frac{(q;q^3)_s^3(q^2,q^{2+3s};q^3)_{(n-1)/3-s}}{(q^3;q^3)_{(n-1)/3+s}(q^3;q^3)_{(n-1)/3}}q^{3s+(1+3s)((n-1)/3-s)} \\
&\qquad+\frac{[n]^2(q;q^3)_s^3(q^{2+3s};q^3)^2_{(n-1)/3-s}q^{3s}}{(q^3;q^3)_{(n-1)/3+s}(q^3;q^3)_s(q;q^3)_{(n-1)/3-s}}\sum_{k=1}^{(n-1)/3-s}\frac{q^{3k+3s-1}}{[3k+3s-1]^2}.
\end{aligned} \tag{2.8}$$

Substituting (2.7) into (2.8) and noticing that the denominator of $\sum_{k=1}^{(n-1)/3-s}\frac{q^{3k+3s-1}}{[3k+3s-1]^2}$ is coprime with $\Phi_n(q)$, we complete the proof of the theorem. □

3 Proof of Theorem 1.2

Note that $n \equiv 2 \pmod 3$ and $s \leqslant (n-2)/3$. Similarly to the proof of Theorem 1.1, we can prove that, modulo $(1-aq^{2n})(a-q^{2n})(1-bq^{2n})$,

$$\sum_{k=s}^{(2n-1)/3} \frac{(aq,q/a,q/b;q^3)_k q^{3k}}{(q^3;q^3)_{k-s}(q^3;q^3)_{k+s}(q^3/b;q^3)_k}$$
$$\equiv \frac{(b-q^{2n})(ab-1-a^2+aq^{2n})}{(a-b)(1-ab)}$$
$$\times \frac{(aq,q/a,q/b;q^3)_s(q^2,bq^{2+3s};q^3)_{(2n-1)/3-s}q^{3s}}{(q^3;q^3)_{(2n-1)/3+s}(q^3/b;q^3)_{(2n-1)/3}}\left(\frac{q^{1+3s}}{b}\right)^{(2n-1)/3-s}$$
$$+\frac{(1-aq^{2n})(a-q^{2n})}{(a-b)(1-ab)}\frac{(aq,q/a,q/b;q^3)_s(aq^{2+3s},q^{2+3s}/a;q^3)_{(2n-1)/3-s}q^{3s}}{(q^3;q^3)_{(2n-1)/3+s}(q^3/b;q^3)_s(q;q^3)_{(2n-1)/3-s}}. \qquad (3.1)$$

Then take $b=1$ in (3.1) to obtain the following result: modulo $\Phi_n(q)(1-aq^{2n})(a-q^{2n})$,

$$\sum_{k=s}^{(2n-1)/3} \frac{(aq,q/a,q;q^3)_k q^{3k}}{(q^3;q^3)_{k-s}(q^3;q^3)_{k+s}(q^3;q^3)_k}$$
$$\equiv \frac{(aq,q/a,q;q^3)_s(q^2,q^{2+3s};q^3)_{(2n-1)/3-s}q^{3s+(1+3s)((2n-1)/3-s)}}{(q^3;q^3)_{(2n-1)/3+s}(q^3;q^3)_{(2n-1)/3}}$$
$$+\frac{(1-aq^{2n})(a-q^{2n})(aq,q/a,q;q^3)_s q^{3s}}{(1-a)^2(q^3;q^3)_{(2n-1)/3+s}}$$
$$\times\left(\frac{(q^{2+3s},q^{2+3s};q^3)_{(2n-1)/3-s}}{(q^3;q^3)_s(q;q^3)_{(2n-1)/3-s}}-\frac{(aq^{2+3s},q^{2+3s}/a;q^3)_{(2n-1)/3-s}}{(q^3;q^3)_s(q;q^3)_{(2n-1)/3-s}}\right), \qquad (3.2)$$

where we have applied the q-congruence

$$q^{(1+3s)((2n-1)/3-s)}\frac{(q^2;q^3)_{(2n-1)/3-s}}{(q^3;q^3)_{(2n-1)/3}} \equiv \frac{(q^{2+3s};q^3)_{(2n-1)/3-s}}{(q^3;q^3)_s(q;q^3)_{(2n-1)/3-s}}(\equiv 0) \pmod{\Phi_n(q)}$$

in the brackets.

Letting $a\to 1$ in (3.2) and applying the L'Hôspital rule, we arrive at the following q-supercongruence: modulo $\Phi_n(q)^3$,

$$\sum_{k=s}^{(2n-1)/3} \frac{(q;q^3)_k^3 q^{3k}}{(q^3;q^3)_{k-s}(q^3;q^3)_{k+s}(q^3;q^3)_k}$$
$$\equiv \frac{(q;q^3)_s^3(q^2,q^{2+3s};q^3)_{(2n-1)/3-s}}{(q^3;q^3)_{(2n-1)/3+s}(q^3;q^3)_{(2n-1)/3}}q^{3s+(1+3s)((2n-1)/3-s)}$$
$$+\frac{[2n]^2(q;q^3)_s^3(q^{2+3s};q^3)_{(2n-1)/3-s}^2q^{3s}}{(q^3;q^3)_{(2n-1)/3+s}(q^3;q^3)_s(q;q^3)_{(2n-1)/3-s}}\sum_{k=1}^{(2n-1)/3-s}\frac{q^{3k+3s-1}}{[3k+3s-1]^2}. \qquad (3.3)$$

Moreover, it is routine to verify the congruence:

$$
\begin{aligned}
&(q^3;q^3)_s(q;q^3)_{(2n-1)/3-s} \\
&\quad = (q^3;q^3)_s(1-q)(1-q^4)\cdots(1-q^{2n-3-3s}) \\
&\quad \equiv (q^3;q^3)_s(1-q^{1-2n})(1-q^{4-2n})\cdots(1-q^{-3-3s}) \\
&\quad = (-1)^{(2n-1)/3-s}q^{-(2n+2+3s)((2n-1)/3-s)/2}(q^3;q^3)_{(2n-1)/3} \pmod{\Phi_n(q)}, \qquad (3.4)
\end{aligned}
$$

and

$$
\frac{(q^{2+3s};q^3)_{(2n-1)/3-s}}{(q^2;q^3)_{(2n-1)/3-s}} = \frac{(q^{2n+1-3s};q^3)_s}{(q^2;q^3)_s} \equiv \frac{(q^{1-3s};q^3)_s}{(q^2;q^3)_s} = (-1)^s q^{-s(3s+1)/2} \pmod{\Phi_n(q)}. \qquad (3.5)
$$

Substituting (3.4) and (3.5) into (3.3) and noticing that $(-1)^{(2n-1)/3} = -1$, we finish the proof of Theorem 1.2.

4 Declarations

Conflicts of interest: No potential conflict of interest was reported by the author.

Availability of data and material: Not applicable.

Code availability: Not applicable.

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