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A Further Generalization of a Supercongruence Conjectured by He*

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Abstract In 2017, He conjectured that, for $p \equiv 3 \pmod 4$,

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac{1}{2})_k^3(\frac{1}{4})_k}{k!^4 4^k} \equiv 0 \pmod{p^4}.$$

He himself proved the modulus p^2 case, and later Liu proved the conjecture is true modulo p^3. This conjecture was finally confirmed by Wei in 2022. In this paper, motivated by Wei's work, through establishing its q-analogue, we give the following generalization of He's supercongruence: for any prime $p \equiv 3 \pmod 4$, and non-negative integer $s \leqslant (p-1)/6$,

$$\sum_{k=s}^{p-s-1}(6k+1)\frac{(\frac{1}{2})_{k-2s}(\frac{1}{2})_{k+2s}(\frac{1}{2})_k(\frac{1}{4})_k}{(k-s)!(k+s)!k!^2 4^k} \equiv 0 \pmod{p^4},$$

where $(x)_n = \Gamma(x+n)/\Gamma(x)$ denotes the Pochhammer symbol also for n not being a non-negative integer.

Keywords q-Supercongruences, p-adic Gamma function, Rahman's transformation, creative microscoping
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1 Introduction

For odd primes p, the p-adic Gamma function $\Gamma_p(x)$ can be defined as follows (see [15]): $\Gamma_p(0) = 1$,

$$\Gamma_p(n) = (-1)^n \prod_{\substack{1<j<n \\ p\nmid j}} j, \quad n = 1, 2, \ldots,$$

and for any p-adic integer x,

$$\Gamma_p(x) = \lim_{x_n \to x} \Gamma_p(x_n),$$

where x_n is any positive integer sequence that p-adically approaches x. For some basic properties of the p-adic Gamma function, we refer the reader to [14].

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In 2017, motivated by the work of [13, 14], He [7] built the following supercongruence:

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{(\frac{1}{2})_k^3(\frac{1}{4})_k}{k!^4 4^k} \equiv \begin{cases} (-1)^{(p+3)/4}p\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})^2 \pmod{p^2}, & \text{if } p\equiv 1 \pmod 4,\\ 0 \pmod{p^2}, & \text{if } p\equiv 3 \pmod 4. \end{cases} \tag{1.1}$$

Here and throughout the paper, $(x)_n=\Gamma(x+n)/\Gamma(x)$ is the Pochhammer symbol also for n not being a non-negative integer. It should be pointed out that the sign $(-1)^{(p+3)/4}$ was lost by He in his paper. In the same year, Liu [10] further proved that (1.1) is true modulo p^3. Both He and Liu utilized a ${}_7F_6$ summation in [3]. Note that we may compute the sum in (1.1) over k up to $p-1$, since the p-adic order of $(1/2)_k/k!$ is 1 for $(p+1)/2\leqslant k\leqslant p-1$. He [7] further conjectured that the following stronger version the second case of (1.1) should be true: for $p\equiv 3 \pmod 4$,

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac{1}{2})_k^3(\frac{1}{4})_k}{k!^4 4^k}\equiv 0 \pmod{p^4}. \tag{1.2}$$

During the past few years, q-analogues of supercongruences have been studied by a number of authors. See [5, 6, 8, 9, 11, 12, 16, 16, 18, 19, 20, 21]. In particular, employing the 'creative microscoping' method devised by the first author and Zudilin [6], Liu and Wang [11] established a q-analogue of (1.1) modulo p^3: for any positive odd integer n, modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{M}[6k+1]\frac{(q;q^2)_k^3(q;q^4)_k}{(q^2;q^2)_k(q^4;q^4)_k^3}q^{k^2+k} \equiv \begin{cases} \dfrac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}[n]q^{(1-n)/4}, & \text{if } n\equiv 1 \pmod 4,\\ 0, & \text{if } n\equiv 3 \pmod 4, \end{cases} \tag{1.3}$$

where $M=n-1$ or $(n-1)/2$. Here and in what follows, the *q-shifted factorial* is defined as

$$(a;q)_n=\begin{cases}(1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n=1,2,\dots,\\ 1, & \text{if } n=0,\\ \dfrac{1}{(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{n})}, & \text{if } n=-1,-2,\dots\end{cases}$$

For convenience, we will also use the abbreviated notation

$$(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$$

for $n=0,\pm1,\pm2,\dots$, or $n=\infty$. The *q-integer* is defined by $[n]=(1-q^n)/(1-q)$, and $\Phi_n(q)$ stands for the n-th *cyclotomic polynomial*, i.e.,

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

with ζ being an n-th primitive root of unity. Moreover, for any two rational functions $A(q)$ and $B(q)$, we say that $A(q)$ is congruent to $B(q)$ modulo an integer coefficient polynomial $P(q)$, denoted by $A(q) \equiv B(q) \pmod{P(q)}$, if $P(q)$ divides the numerator of the reduced form of $A(q)-B(q)$ in the polynomial ring $\mathbb{Z}[q]$.

In 2022, Wei [20] confirmed He's conjecture (1.2) by giving the following q-supercongruence: for $n \equiv 3 \pmod 4$,

$$\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^2)_k^3(q;q^4)_k}{(q^2;q^2)_k(q^4;q^4)_k^3}q^{k^2+k} \equiv 0 \pmod{[n]\Phi_n(q)^3}, \tag{1.4}$$

which is also a generalization of the second case of (1.3) for $M=n-1$. It is easy to see that (1.2) follows by letting $n=p$ and $q\to 1$ in (1.4).

In this paper, we shall give the following generalization of (1.4) modulo $\Phi_n(q)^4$.

Theorem 1.1 Let n be a positive integer with $n\equiv 3 \pmod 4$. Let s be a non-negative integer with $s \leqslant (n-1)/6$. Then

$$\sum_{k=s}^{n-s-1}[6k+1]\frac{(q;q^2)_{k-2s}(q;q^2)_{k+2s}(q;q^2)_k(q;q^4)_k}{(q^2;q^2)_k(q^4;q^4)_{k-s}(q^4;q^4)_{k+s}(q^4;q^4)_k}q^{k^2+k} \equiv 0 \pmod{\Phi_n(q)^4}. \tag{1.5}$$

Note that the condition $s\leqslant (n-1)/6$ is necessary in Theorem 1.1. The q-congruence (1.5) does not hold for $(n-1)/6 < s \leqslant (n-1)/2$ in general.

Letting $n=p$ be a prime and $q\to 1$ in (1.5), we obtain the following supercongruence: for any prime $p\equiv 3 \pmod 4$, and non-negative integer $s\leqslant (p-1)/6$,

$$\sum_{k=s}^{p-s-1}(6k+1)\frac{(\frac12)_{k-2s}(\frac12)_{k+2s}(\frac12)_k(\frac14)_k}{(k-s)!(k+s)!k!^2 4^k} \equiv 0 \pmod{p^4}, \tag{1.6}$$

which clearly reduces to (1.2) when $s=0$.

Summation and transformation formulas for basic hypergeometric series play an important role in the study of q-supercongruences. See, for example, [6, 5]. Here we would like to mention Gasper and Rahman's quadratic summation (see [2, (3.8.12)]), which may be stated in the following form:

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{1-aq^{3k}}{1-a}\frac{(a,b,q/b;q)_k(d,f,a^2q/df;q^2)_k}{(aq/d,aq/f,df/a;q)_k(q^2,aq^2/b,abq;q^2)_k}q^k\\
&\quad+\frac{(aq,f/a,b,q/b;q)_\infty(d,aq^2/df,fq^2/d,df^2q/a^2;q^2)_\infty}{(a/f,fq/a,aq/d,df/a;q)_\infty(aq^2/b,abq,fq/ab,bf/a;q^2)_\infty}\\
&\quad\times {}_3\phi_2\left[\begin{matrix} f,bf/a,fq/ab\\ fq^2/d,df^2q/a^2\end{matrix};q^2,q^2\right]\\
&\quad=\frac{(aq,f/a;q)_\infty(aq^2/bd,abq/d,bdf/a,dfq/ab;q^2)_\infty}{(aq/d,df/a;q)_\infty(aq^2/b,abq,bf/a,fq/ab;q^2)_\infty},
\end{aligned}\tag{1.7}$$

where the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined by

$${}_{r+1}\phi_r\left[\begin{matrix} a_1,a_2,\dots,a_{r+1}\\ b_1,b_2,\dots,b_r\end{matrix};q,\,z\right]=\sum_{k=0}^{\infty}\frac{(a_1,a_2,\dots,a_{r+1};q)_k z^k}{(q,b_1,\dots,b_r;q)_k}.$$

Gasper and Rahman's summation (1.7) was utilized by Wei [20] to prove (1.4). It was also employed by the first author [4] to produce more q-supercongruences.

We shall prove Theorem 1.1 by making use of the method of 'creative microscoping' and Gasper and Rahman's quadratic summation (1.7) again.

2 Proof of Theorem 1.1

We first give the following lemma, which has already appeared in [1]. In order to make the paper self-contained, we give the whole proof here.

Lemma 2.1 Let $n>1$ be an odd integer, and let s be a non-negative integer with $s\leqslant(n-1)/2$. Then

$$\sum_{k=s}^{n-s-1}\frac{1-q^{1+6k-n}}{1-q^{1-n}}\frac{(aq;q^2)_{k-2s}(q^{1-n};q^2)_{k+2s}(q/a;q^2)_k(q/b;q^4)_k}{(bq^2;q^2)_k(q^4;q^4)_{k-s}(q^{4-n}/a;q^4)_{k+s}(aq^{4-n};q^4)_k}(bq^{-n})^k q^{k^2+k}=0. \tag{2.1}$$

Proof. Clearly, the left-hand side of (2.1) may be written as

$$\begin{aligned}&\sum_{k=0}^{n-2s-1}\frac{(1-q^{1+6k+6s-n})(aq;q^2)_{k-s}(q^{1-n};q^2)_{k+3s}(q/a;q^2)_{k+s}(q/b;q^4)_{k+s}}{(1-q^{1-n})(bq^2;q^2)_{k+s}(q^4;q^4)_k(q^{4-n}/a;q^4)_{k+2s}(aq^{4-n};q^4)_{k+s}}(bq^{-n})^{k+s}q^{(k+s)^2+k+s}\\&=\frac{(aq;q^2)_{-s}(q^{1-n};q^2)_{3s}(q/a;q^2)_s(q/b;q^4)_s}{(bq^2;q^2)_s(q^{4-n}/a;q^4)_{2s}(aq^{4-n};q^4)_s}(bq^{-n})^s q^{s^2+s}\\&\quad\times\sum_{k=0}^{n-2s-1}\frac{(1-q^{1+6k+6s-n})(aq^{1-2s},q^{1+6s-n},q^{1+2s}/a;q^2)_k(q^{1+4s}/b;q^4)_k}{(1-q^{1-n})(bq^{2+2s};q^2)_k(q^4,q^{4+8s-n}/a,aq^{4+4s-n};q^4)_k}(bq^{-n})^k q^{k^2+2sk+k}\end{aligned} \tag{2.2}$$

If $s>(n-1)/6$, then $(q^{1-n};q^2)_{3s}=0$ or $(1-q^{1+6k+6s-n})(q^{1+6s-n};q^2)_k=0$, and therefore the right-hand side of (2.2) vanishes. We now suppose that $0\leqslant s\leqslant(n-1)/6$. Taking $d=q^{-2n}$ and then letting $n\to\infty$ in (1.7), we obtain

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{1-aq^{3k}}{1-a}\frac{(a,b,q/b;q)_k(f;q^2)_k}{(q^2,aq^2/b,abq;q^2)_k(aq/f;q)_k}\left(\frac{a}{f}\right)^k q^{\binom{k+1}{2}}\\&\quad=\frac{(aq,aq^2,aq^2/bf,abq/f;q^2)_\infty}{(aq/f,aq^2/f,aq^2/b,abq;q^2)_\infty}.\end{aligned} \tag{2.3}$$

Then, making the parameter replacements $q\mapsto q^2$, $a\mapsto q^{1+6s-n}$, $b\mapsto aq^{1-2s}$, and $f\mapsto q^{1+4s}/b$ in the above identity yields that

$$\sum_{k=0}^{(n-1)/6-s}\frac{(1-q^{1+6k+6s-n})(aq^{1-2s},q^{1+6s-n},q^{1+2s}/a;q^2)_k(q^{1+2s}/b;q^4)_k}{(1-q^{1+6s-n})(bq^{2+2s};q^2)_k(q^4,q^{4+8s-n}/a,aq^{4+4s-n};q^4)_k}(bq^{2s-n})^k q^{k^2+k}=0,$$

where we have utilized the fact that $(q^{1+6s-n};q^2)_k=0$ for $k>(n-1)/6-s$. This proves that the right-hand side of (2.2) vanishes. □

To prove Theorem 1.1, we also need another three lemmas.

Lemma 2.2 Let n be a positive integer with $n\equiv 3\pmod 4$, and let s be a non-negative integer with $s\leqslant(n-1)/6$. Then, modulo $1-aq^n$,

$$\sum_{k=s}^{n-s-1}[6k+1]\frac{(aq;q^2)_{k-2s}(q;q^2)_{k+2s}(q/a;q^2)_k(q/b;q^4)_k}{(bq^2;q^2)_k(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k}b^k q^{k^2+k}$$

$$
\begin{aligned}
&= [6s+1]\frac{(aq;q^2)_{-s}(q;q^2)_{3s}(q/a;q^2)_s(q/b;q^4)_s}{(bq^2;q^2)_s(q^4/a;q^4)_{2s}(aq^4;q^4)_s} b^s q^{s^2+s} \\
&\quad\times \begin{cases} \dfrac{(q^{3+6s},bq^{3-n};q^4)_{(n+1+2s)/4}}{(bq^{2+2s},q^{4+4s-n};q^4)_{(n+1+2s)/4}}, & \text{if } s \text{ is even,} \\ \dfrac{(q^{5+6s},bq^{3-n};q^4)_{(n-1+2s)/4}}{(bq^{4+2s},q^{4+4s-n};q^4)_{(n-1+2s)/4}}, & \text{otherwise.} \end{cases} \qquad (2.4)
\end{aligned}
$$

Proof. For $a=q^{-n}$, the left-hand side of (2.4) can be written as

$$
\begin{aligned}
&\sum_{k=s}^{n-s-1} [6k+1]\frac{(q^{1-n};q^2)_{k-2s}(q;q^2)_{k+2s}(q^{1+n};q^2)_k(q/b;q^4)_k}{(bq^2;q^2)_k(q^4;q^4)_{k-s}(q^{4+n};q^4)_{k+s}(q^{4-n};q^4)_k} b^k q^{k^2+k} \\
&\quad= \sum_{k=0}^{n-2s-1} [6k+6s+1]\frac{(q^{1-n};q^2)_{k-s}(q;q^2)_{k+3s}(q^{1+n};q^2)_{k+s}(q/b;q^4)_{k+s}}{(bq^2;q^2)_{k+s}(q^4;q^4)_k(q^{4+n};q^4)_{k+2s}(q^{4-n};q^4)_{k+s}} b^{k+s} q^{(k+s)^2+k+s} \\
&\quad= \frac{(q^{1-n};q^2)_{-s}(q;q^2)_{3s}(q^{1+n};q^2)_s(q/b;q^4)_s}{(bq^2;q^2)_s(q^{4+n};q^4)_{2s}(q^{4-n};q^4)_s} b^s q^{s^2+s} \\
&\qquad\times \sum_{k=0}^{n-2s-1} [6k+6s+1]\frac{(q^{1-2s-n},q^{1+6s},q^{1+2s+n};q^2)_k(q^{1+4s}/b;q^4)_k}{(bq^{2+2s};q^2)_k(q^4,q^{4+8s+n},q^{4+4s-n};q^4)_k} b^k q^{k^2+2sk+k} \qquad (2.5)
\end{aligned}
$$

Making the parameter substitutions $q\mapsto q^2$, $a\mapsto q^{1+6s}$, $b\mapsto q^{1-2s-n}$, and $f\mapsto q^{1+4s}/b$ in (2.3), we are led to

$$
\begin{aligned}
&\sum_{k=0}^{n-2s-1} \frac{[6k+6s+1](q^{1-2s-n},q^{1+6s},q^{1+2s+n};q^2)_k(q^{1+4s}/b;q^4)_k}{[6s+1](bq^{2+2s};q^2)_k(q^4,q^{4+8s+n},q^{4+4s-n};q^4)_k} b^k q^{k^2+2sk+k} \\
&\quad= \frac{(q^{3+6s},q^{5+6s},bq^{3+4s+n},bq^{3-n};q^4)_\infty}{(bq^{2+2s},bq^{4+2s},q^{4+8s+n},q^{4+4s-n};q^4)_\infty},
\end{aligned}
$$

where we have used $(q^{1-2s-n};q^2)_k=0$ for $k>(n-1)/2+s$ and $n-2s-1\geqslant(n-1)/2+s$. Substituting the above identity into (2.5) and making some simplifications, we see that both sides of (2.4) are equal for $a=q^{-n}$. Namely, the q-congruence (2.4) holds. □

We only use Lemmas 2.3 and 2.4 under the condition $s\leqslant(n-1)/6$. However, to make the results more general, we give them in the following forms.

Lemma 2.3 Let n be a positive integer with $n\equiv 3 \pmod 4$, and let s be a non-negative integer with $s\leqslant(n-1)/2$. Then, modulo $a-q^n$,

$$
\begin{aligned}
&\sum_{k=s}^{n-s-1} [6k+1]\frac{(aq;q^2)_{k-2s}(q;q^2)_{k+2s}(q/a;q^2)_k(q/b;q^4)_k}{(bq^2;q^2)_k(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k} b^k q^{k^2+k} \\
&\quad= [6s+1]\frac{(aq;q^2)_{-s}(q;q^2)_{3s}(q/a;q^2)_s(q/b;q^4)_s}{(bq^2;q^2)_s(q^4/a;q^4)_{2s}(aq^4;q^4)_s} b^s q^{s^2+s} \\
&\quad\times \begin{cases} \dfrac{(q^{3+6s},bq^{3+4s-n};q^4)_{(n+1-2s)/4}}{(bq^{2+2s},q^{4+8s-n};q^4)_{(n+1-2s)/4}}, & \text{if } s \text{ is even,} \\ \dfrac{(q^{5+6s},bq^{3+4s-n};q^4)_{(n-1-2s)/4}}{(bq^{4+2s},q^{4+8s-n};q^4)_{(n-1-2s)/4}}, & \text{otherwise.} \end{cases} \qquad (2.6)
\end{aligned}
$$

Proof. For $a=q^n$, the left-hand side of (2.6) can be written as

$$
\begin{aligned}
&\sum_{k=s}^{n-s-1}[6k+1]\frac{(q^{1+n};q^2)_{k-2s}(q;q^2)_{k+2s}(q^{1-n};q^2)_k(q/b;q^4)_k}{(bq^2;q^2)_k(q^4;q^4)_{k-s}(q^{4-n};q^4)_{k+s}(q^{4+n};q^4)_k}b^kq^{k^2+k}\\
&\quad=\frac{(q^{1+n};q^2)_{-s}(q;q^2)_{3s}(q^{1-n};q^2)_s(q/b;q^4)_s}{(bq^2;q^2)_s(q^{4-n};q^4)_{2s}(q^{4+n};q^4)_s}b^sq^{s^2+s}\\
&\quad\times\sum_{k=0}^{n-2s-1}[6k+6s+1]\frac{(q^{1-2s+n},q^{1+6s},q^{1+2s-n};q^2)_k(q^{1+4s}/b;q^4)_k}{(bq^{2+2s};q^2)_k(q^4,q^{4+8s-n},q^{4+4s+n};q^4)_k}b^kq^{k^2+2sk+k} \qquad (2.7)
\end{aligned}
$$

Making the parameter substitutions $q\mapsto q^2$, $a\mapsto q^{1+6s}$, $b\mapsto q^{1-2s+n}$, and $f\mapsto q^{1+4s}/b$ in (2.3), we arrive at

$$
\begin{aligned}
&\sum_{k=0}^{n-2s-1}\frac{[6k+6s+1](q^{1-2s+n},q^{1+6s},q^{1+2s-n};q^2)_k(q^{1+4s}/b;q^4)_k}{[6s+1](bq^{2+2s};q^2)_k(q^4,q^{4+8s-n},q^{4+4s+n};q^4)_k}b^kq^{k^2+2sk+k}\\
&\quad=\frac{(q^{3+6s},q^{5+6s},bq^{3+4s-n},bq^{3+n};q^4)_\infty}{(bq^{2+2s},bq^{4+2s},q^{4+8s-n},q^{4+4s+n};q^4)_\infty}.
\end{aligned}
$$

where we have used $(q^{1+2s-n};q^2)_k=0$ for $k>(n-1)/2-s$. Plugging the above identity into (2.7), we conclude that both sides of (2.6) are equal for $a=q^n$. That is, the q-congruence (2.6) holds. □

Lemma 2.4 Let n be a positive integer with $n\equiv 3 \pmod 4$, and let s be a non-negative integer with $s\leqslant (n-3)/4$. Then, modulo $b-q^{3n}$,

$$
\begin{aligned}
&\sum_{k=s}^{n-s-1}[6k+1]\frac{(aq;q^2)_{k-2s}(q;q^2)_{k+2s}(q/a;q^2)_k(q/b;q^4)_k}{(bq^2;q^2)_k(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k}b^kq^{k^2+k}\\
&\quad=[3n+2s]\frac{(aq;q^2)_{-s}(q;q^2)_{(3n-1)/2+s}(q/a;q^2)_s(q/b;q^4)_s}{(bq^2;q^2)_s(q^4/a;q^4)_{(3n-1)/4+s}(aq^4;q^4)_{(3n-1)/4}}b^sq^{s^2+s} \qquad (2.8)
\end{aligned}
$$

Proof. For $b=q^{3n}$, the left-hand side of (2.8) is equal to

$$
\begin{aligned}
&\sum_{k=s}^{n-s-1}[6k+1]\frac{(aq;q^2)_{k-2s}(q;q^2)_{k+2s}(q/a;q^2)_k(q^{1-3n};q^4)_k}{(q^{2+3n};q^2)_k(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k}q^{k^2+3nk+k}\\
&\quad=\frac{(aq;q^2)_{-s}(q;q^2)_{3s}(q/a;q^2)_s(q^{1-3n};q^4)_s}{(q^{2+3n};q^2)_s(q^4/a;q^4)_{2s}(aq^4;q^4)_s}q^{s^2+3ns+s}\\
&\quad\times\sum_{k=0}^{n-2s-1}[6k+6s+1]\frac{(aq^{1-2s},q^{1+6s},q^{1+2s}/a;q^2)_k(q^{1+4s-3n};q^4)_k}{(q^{2+2s+3n};q^2)_k(q^4,q^{4+8s}/a,aq^{4+4s};q^4)_k}q^{k^2+2sk+3nk+k} \qquad (2.9)
\end{aligned}
$$

Letting $q\mapsto q^2$, $a\mapsto q^{1+6s}$, $b\mapsto aq^{1-2s}$, and $f\mapsto q^{1+4s-3n}$ in (2.3), and noticing $(q^{1+4s-3n};q^4)_k=0$ for $k>(3n-1)/4-s$ and $n-2s-1\geqslant (3n-1)/4-s$. we see that the right-hand side of (2.9) is equal to

$$
\begin{aligned}
&[6s+1]\frac{(aq;q^2)_{-s}(q;q^2)_{3s}(q/a;q^2)_s(q^{1-3n};q^4)_s}{(q^{2+3n};q^2)_s(q^4/a;q^4)_{2s}(aq^4;q^4)_s}q^{s^2+3ns+s}\\
&\quad\times\frac{(q^{3+6s},q^{5+6s},q^{3+4s+3n}/a,aq^{3+3n};q^4)_\infty}{(q^{2+2s+3n},q^{4+2s+3n},q^{4+8s}/a,aq^{4+4s};q^4)_\infty},
\end{aligned}
$$

which is just the $b=q^{3n}$ case of right-hand side of (2.8). This proves the q-congruence. □

Proof of Theorem 1.1. Since $q^n \equiv 1 \pmod{\Phi_n(q)}$, we deduce from (2.1) that

$$\sum_{k=s}^{n-s-1}[6k+1]\frac{(aq;q^2)_{k-2s}(q;q^2)_{k+2s}(q/a;q^2)_k(q/b;q^4)_k}{(bq^2;q^2)_k(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k}b^kq^{k^2+k} \equiv 0 \pmod{\Phi_n(q)}. \tag{2.10}$$

It is clear that $\Phi_n(q)$, $1-aq^n$, $a-q^n$, and $b-q^{3n}$ are pairwise coprime polynomials in q. Meanwhile, the right-hand sides of (2.4), (2.6), and (2.8) are all congruent to 0 modulo $\Phi_n(q)$. Therefore, employing the following congruences:

$$\begin{aligned}
\frac{(a-q^n)(b-q^{3n})}{(1-a^2)(1-a^3b)}a^4 &\equiv 1 \pmod{1-aq^n},\\
\frac{(1-aq^n)(b-q^{3n})}{(1-a^2)(b-a^3)} &\equiv 1 \pmod{a-q^n},\\
\frac{(1-a^3q^{3n})(a^3-q^{3n})}{(a^3-b)(1-a^3b)} &\equiv 1 \pmod{b-q^{3n}},
\end{aligned}$$

and the Chinese remainder theorem for coprime polynomials, we conclude from (2.4), (2.6), (2.8), and (2.10) that, for even s with $0\leqslant s\leqslant (n-1)/6$, modulo $\Phi_n(q)(1-aq^n)(a-q^n)(b-q^{3n})$,

$$\begin{aligned}
&\sum_{k=s}^{n-s-1}[6k+1]\frac{(aq;q^2)_{k-2s}(q;q^2)_{k+2s}(q/a;q^2)_k(q/b;q^4)_k}{(bq^2;q^2)_k(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k}b^kq^{k^2+k}\\
&\quad\equiv [6s+1]\frac{(aq;q^2)_{-s}(q;q^2)_{3s}(q/a;q^2)_s(q/b;q^4)_s}{(bq^2;q^2)_s(q^4/a;q^4)_{2s}(aq^4;q^4)_s}b^sq^{s^2+s}\\
&\qquad\times\frac{(q^{3+6s},bq^{3-n};q^4)_{(n+1+2s)/4}}{(bq^{2+2s},q^{4+4s-n};q^4)_{(n+1+2s)/4}}\frac{(a-q^n)(b-q^{3n})}{(1-a^2)(1-a^3b)}a^4\\
&\qquad+[6s+1]\frac{(aq;q^2)_{-s}(q;q^2)_{3s}(q/a;q^2)_s(q/b;q^4)_s}{(bq^2;q^2)_s(q^4/a;q^4)_{2s}(aq^4;q^4)_s}b^sq^{s^2+s}\\
&\qquad\times\frac{(q^{3+6s},bq^{3+4s-n};q^4)_{(n+1-2s)/4}}{(bq^{2+2s},q^{4+8s-n};q^4)_{(n+1-2s)/4}}\frac{(1-aq^n)(b-q^{3n})}{(1-a^2)(b-a^3)}\\
&\qquad+[3n+2s]\frac{(aq;q^2)_{-s}(q;q^2)_{(3n-1)/2+s}(q/a;q^2)_s(q/b;q^4)_s}{(bq^2;q^2)_s(q^4/a;q^4)_{(3n-1)/4+s}(aq^4;q^4)_{(3n-1)/4}}b^sq^{s^2+s}\\
&\qquad\times\frac{(1-a^3q^{3n})(a^3-q^{3n})}{(a^3-b)(1-a^3b)}.
\end{aligned}\tag{2.11}$$

We now consider the $b=1$ case of (2.11). It is easy to see that all the factors involving b in the denominators of (2.11) are coprime with $\Phi_n(q)$, and $b-q^{3n}=1-q^{3n}$ contains the factor $\Phi_n(q)$. Moreover, $(bq^{3-n};q^4)_{(n+1+2s)/4}=(q^{3-n};q^4)_{(n+1+2s)/4}=0$ and $(bq^{3+4s-n};q^4)_{(n+1-2s)/4}=(q^{3+4s-n};q^4)_{(n+1-2s)/4}=0$. Thus, letting $b=1$ in (2.11), we conclude that, for even s with $0\leqslant s\leqslant (n-1)/6$, modulo $\Phi_n(q)^2(1-aq^n)(a-q^n)$,

$$\begin{aligned}
&\sum_{k=s}^{n-s-1}[6k+1]\frac{(aq;q^2)_{k-2s}(q;q^2)_{k+2s}(q/a;q^2)_k(q;q^4)_k}{(q^2;q^2)_k(q^4;q^4)_{k-s}(q^4/a;q^4)_{k+s}(aq^4;q^4)_k}q^{k^2+k}\\
&\quad\equiv -[3n+2s]\frac{(aq;q^2)_{-s}(q;q^2)_{(3n-1)/2+s}(q/a;q^2)_s(q;q^4)_s}{(q^2;q^2)_s(q^4/a;q^4)_{(3n-1)/4+s}(aq^4;q^4)_{(3n-1)/4}}q^{s^2+s}\frac{(1-a^3q^{3n})(a^3-q^{3n})}{(1-a^3)^2}\\
&\quad\equiv 0,
\end{aligned}$$

since $(q;q^2)_{(3n-1)/2+s}$ contains the factor $(1-q^n)(1-q^{3n})$ for $s>0$. Finally, taking $a=1$ in the above q-congruence, we immediately get (1.5) for even s. Similarly, we can prove (1.5) in the case where s is odd. $\square$

3 An open problem

Numerical evaluation implies that the following stronger version of Theorem 1.1 should be true.

Conjecture 3.1 The q-supercongruence (1.5) holds modulo $[n]\Phi_n(q)^3$.

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