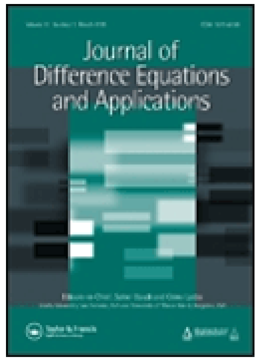

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A q-analogue of a curious supercongruence of Guillera and Zudilin

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Abstract. The following curious supercongruence of Guillera and Zudilin:

$$\sum_{k=1}^{p-1}(-1)^k\frac{(p+3k)2^k}{k}\binom{2k}{k}\equiv 8\left(1-2^{p-1}\right)\pmod{p^2}\quad\text{for any prime } p>2$$

plays an important part in their proof of a "divergent" Ramanujan-type supercongruence. In this paper, we give a q-analogue of this supercongruence by using the q-WZ method twice. Meanwhile, we confirm a q-analogue of the mentioned "divergent" Ramanujan-type supercongruence, which was early conjectured by the author.

Keywords: Wilf–Zeilberger method; q-WZ pair; q-binomial coefficients; cyclotomic polynomials

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1. Introduction

By using the Wilf–Zeilberger (abbr. WZ) method [13, 14], among other things, Guillera and Zudilin [2] proved the following supercongruence:

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{(\frac{1}{2})_k^3}{k!^3}(3k+1)(-1)^k2^{3k}\equiv(-1)^{\frac{p-1}{2}}p\pmod{p^3}\quad\text{for } p>2. \tag{1.1}$$

Here and throughout the paper, the letter p always denotes a prime, and $(a)_n=a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol.

In a previous paper, motivated by Guillera and Zudilin's work [2], the author [4, Theorem 1.3] utilized the q-WZ method [9, 15] to give the following q-analogue of (1.1):

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k[3k+1]\frac{(q;q^2)_k^3}{(q;q)_k^3}\equiv(-1)^{\frac{p-1}{2}}[p]q^{\frac{(p-1)^2}{4}}\pmod{[p]^3},$$

where the q-*shifted factorial* is defined by $(a;q)_0=1$ and $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n\geqslant 1$, while the q-*integer* is defined as $[n]=1+q+\cdots+q^{n-1}$ (see [1]). Note that two rational functions $A(q)$ and $B(q)$ are congruent modulo a polynomial $P(q)$ if the numerator of the reduced form of $A(q)-B(q)$ is divisible by $P(q)$ in the polynomial ring $\mathbb{Z}[q]$. For some other interesting q-congruences, we refer the reader to [3, 8, 10, 12].

Besides the WZ method, the following curious supercongruence

$$\sum_{k=1}^{p-1}(-1)^k\frac{(p+3k)2^k}{k}\binom{2k}{k}\equiv 8\left(1-2^{p-1}\right)\pmod{p^2}\quad\text{for } p>2 \tag{1.2}$$

plays an important role in Guillera and Zudilin's proof of (1.1).

Let $\Phi_n(q)$ be the n-th *cyclotomic polynomial*. Let ${M \brack N}$ be the *q-binomial coefficients* defined by

$$\begin{bmatrix}M\\N\end{bmatrix}=\begin{cases}\dfrac{(q;q)_M}{(q;q)_N(q;q)_{M-N}} & \text{if } 0\leqslant N\leqslant M.\\ 0 & \text{otherwise.}\end{cases}$$

In this paper, we shall give the following q-analogue of (1.2).

Theorem 1.1. *Let n be a positive odd integer. Then*

$$\sum_{k=1}^{n-1}(-1)^k\frac{[n+3k](-q^{n+1};q)_{k-1}}{[2k]}\begin{bmatrix}2k\\k\end{bmatrix}\equiv 2\left(q^{\binom{n}{2}}-(-q;q)_{n-1}\right)\pmod{\Phi_n(q)^2}. \tag{1.3}$$

It is clear that (1.2) follows from (1.3) by taking $n=p$ and $q=1$. Note that (1.2) is a combination of the following two congruences [2, Lemma 4]:

$$\sum_{k=1}^{p-1}(-1)^k\frac{2^k}{k}\binom{2k}{k}\equiv\frac{4(1-2^{p-1})}{p}\pmod{p}\quad\text{for } p>2, \tag{1.4}$$

$$3\sum_{k=1}^{p-1}(-1)^k2^k\binom{2k}{k}\equiv 4\left(1-2^{p-1}\right)\pmod{p^2}\quad\text{for } p>2, \tag{1.5}$$

and (1.4) follows from a result of Sun and Tauraso [11, Theorem 1.2]. However, we did not find any q-analogues of (1.4) and (1.5). From (1.3), we can only deduce the following partial q-analogue of (1.5):

$$\sum_{k=1}^{n-1}(-1)^k\frac{[3k]}{[2k]}\begin{bmatrix}2k\\k\end{bmatrix}(-q;q)_{k-1}\equiv 0\pmod{\Phi_n(q)}\quad\text{for odd } n.$$

We shall also prove the following q-analogue of (1.1), which was originally conjectured by the author [4, Conjecture 1,4].

Theorem 1.2. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{\frac{n-1}{2}}(-1)^k[3k+1]\frac{(q;q^2)_k^3}{(q;q)_k^3}\equiv(-1)^{\frac{n-1}{2}}[n]q^{\frac{(n-1)^2}{4}}\pmod{[n]\Phi_n(q)^2}. \tag{1.6}$$

It should be mentioned that the following similar q-congruence has already been obtained by the author [4, Theorem 1.3]: for odd n,

$$\sum_{k=0}^{n-1}(-1)^k[3k+1]\frac{(q;q^2)_k^3}{(q;q)_k^3} \equiv (-1)^{\frac{n-1}{2}}[n]q^{\frac{(n-1)^2}{4}} \pmod{[n]\Phi_n(q)^2}. \tag{1.7}$$

We will prove (1.3) using the q-WZ method. Our proof is based on (1.7), which was also established by the author [4] via the q-WZ method. The proof of (1.3) is a little complicated, and may be considered as a q-analogue of Guillera and Zudilin's proof of (1.7). As far as we know, no other q-supercongruences are proved in such a way (using the q-WZ method twice). It would be interesting if the reader can find a new proof of (1.3).

We will prove (1.6) using the root of unity technique arising in a joint paper by the author and Zudilin [6]. Our proof of (1.6) is again based on (1.7).

2. Two lemmas

We give two preliminary results, which will be used in our proof of Theorem 1.1.

Lemma 2.1. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{\frac{n-3}{2}}\frac{q^{2k}}{[2k+1](-q;q)_k^2}{2k \brack k} \equiv (-1)^{\frac{n+1}{2}}\frac{(-q;q)_{n-1}q^{-\frac{(n-1)(n-3)}{4}}-q^{\frac{(n-1)(n+3)}{4}}}{[n]} \pmod{\Phi_n(q)}. \tag{2.1}$$

Proof. Noticing that $(q;q^2)_k \equiv (q^{1-n};q^2)_k \pmod{\Phi_n(q)}$, we can write the left-hand side of (2.1) as

$$\sum_{k=0}^{\frac{n-3}{2}}\frac{q^{2k}(q;q^2)_k}{[2k+1](q^2;q^2)_k} \equiv \sum_{k=0}^{\frac{n-3}{2}}\frac{q^{2k}(q^{1-n};q^2)_k}{[2k+1](q^2;q^2)_k} \pmod{\Phi_n(q)}. \tag{2.2}$$

The latter is nothing but a terminating basic hypergeometric series with one term missing,

$$\begin{aligned}\sum_{k=0}^{\frac{n-3}{2}}\frac{(q^{1-n};q^2)_k(q;q^2)_k}{(q^2;q^2)_k(q^3;q^2)_k}q^{2k} &= -(-1)^{\frac{n-1}{2}}\frac{q^{-\frac{(n-1)(n-3)}{4}}}{[n]}+\sum_{k=0}^{\frac{n-1}{2}}\frac{(q^{1-n};q^2)_k(q;q^2)_k}{(q^2;q^2)_k(q^3;q^2)_k}q^{2k}\\ &= -(-1)^{\frac{n-1}{2}}\frac{q^{-\frac{(n-1)(n-3)}{4}}}{[n]}+\frac{(q^2;q^2)_{\frac{n-1}{2}}}{[n](q;q^2)_{\frac{n-1}{2}}}q^{\frac{n-1}{2}}\end{aligned} \tag{2.3}$$

by the q-Chu-Vandermonde summation formula (see [1, Appendix (II.6)]).

For any positive odd integer n, a special case of a q-analogue of Morley's congruence [8, (1.5)], due to Pan, gives

$$\frac{(q;q^2)_{\frac{n-1}{2}}}{(q^2;q^2)_{\frac{n-1}{2}}} = \begin{bmatrix} n-1 \\ \frac{n-1}{2} \end{bmatrix}_{q^2} \frac{1}{(-q;q)_{n-1}} \equiv (-1)^{\frac{n-1}{2}} q^{\frac{1-n^2}{4}} (-q;q)_{n-1} \pmod{\Phi_n(q)^2}. \tag{2.4}$$

Combining (2.2)–(2.4) and using the following congruence (see [5, Lemma 3.2])

$$(-q;q)_{n-1} \equiv 1 \pmod{\Phi_n(q)}, \tag{2.5}$$

we are led to (2.1). □

Lemma 2.2. *Let $n > 1$ be a positive odd integer. Then, for $k = 1, 2, \ldots, n-1$, we have*

$$\frac{(q;q^2)_k (q^{n+2};q^2)_{k-1}^2}{(q;q)_k^2 (q^{n+1};q)_{k-1}} \equiv \begin{bmatrix} 2k \\ k \end{bmatrix} \frac{(-q^{n+1};q)_{k-1}}{1-q^{2k}} \pmod{\Phi_n(q)^2}. \tag{2.6}$$

Proof. For any integer j, we have

$$(1-q^{n+2j})^2 - (1-q^{2j})(1-q^{2n+2j}) = q^{2j}(1-q^n)^2 \equiv 0 \pmod{\Phi_n(q)^2},$$

and so

$$\begin{aligned}(q^{n+2};q^2)_{k-1}^2 &= \prod_{j=1}^{k-1}(1-q^{n+2j})^2 \\ &\equiv \prod_{j=1}^{k-1}(1-q^{2j})(1-q^{2n+2j}) = (q^2;q^2)_{k-1}(q^{2n+2};q^2)_{k-1} \pmod{\Phi_n(q)^2}. \end{aligned} \tag{2.7}$$

For $k = 1, 2, \ldots, n-1$, both $(q;q)_k$ and $(q^{n+1};q)_{k-1}$ are relatively prime to $\Phi_n(q)$. From (2.7) we deduce that

$$\frac{(q;q^2)_k (q^{n+2};q^2)_{k-1}^2}{(q;q)_k^2 (q^{n+1};q)_{k-1}} \equiv \frac{(q;q^2)_k (q^2;q^2)_{k-1} (-q^{n+1};q)_{k-1}}{(q;q)_k^2} = \begin{bmatrix} 2k \\ k \end{bmatrix} \frac{(-q^{n+1};q)_{k-1}}{1-q^{2k}} \pmod{\Phi_n(q)^2}.$$

□

3. Proof of Theorem 1.1

We introduce the following two rational functions in q:

$$F(n,k) = (-1)^n [3n+2k+1] \frac{(q;q^2)_n (q^{2k+1};q^2)_n^2 (q;q^2)_k}{(q;q)_n^2 (q^{2k+1};q)_n (q^2;q^2)_k},$$

$$G(n,k) = (-1)^n q^{n+2k-1} \frac{(q;q^2)_n (q^{2k+1};q^2)_{n-1}^2 (q;q^2)_k}{(1-q)(q;q)_{n-1}^2 (q^{2k+1};q)_{n-1} (q^2;q^2)_k},$$

where we use the convention that $1/(q^2;q^2)_m = 0$ for any negative integer m. We can check that $F(n,k)$ and $G(n,k)$ satisfy the relation

$$F(n,k-1)-F(n,k)=G(n+1,k)-G(n,k). \tag{3.1}$$

Namely, they form a q-WZ pair. Note that, for $q=1$, these functions are just the WZ pair given by Guillera and Zudilin [2] in their proof of (1.1).

Let $m>1$ be a positive odd integer. Summing (3.1) over $n=0,1,\dots,m-1$, we obtain

$$\sum_{n=0}^{m-1}F(n,k-1)-\sum_{n=0}^{m-1}F(n,k)=G(m,k), \tag{3.2}$$

where we have used $G(0,k)=0$. Summing (3.2) further over $k=1,2,\dots,\frac{m-1}{2}$, we get

$$\sum_{n=0}^{m-1}F(n,0)=\sum_{n=0}^{m-1}F\left(n,\frac{m-1}{2}\right)+\sum_{k=1}^{\frac{m-1}{2}}G(m,k). \tag{3.3}$$

By (1.7) and noticing that $[m]$ is divisible by $\Phi_m(q)$ for $m>1$, we have

$$\sum_{n=0}^{m-1}F(n,0)=\sum_{n=0}^{m-1}(-1)^n[3n+1]\frac{(q;q^2)_n^3}{(q;q)_n^3}\equiv(-1)^{\frac{m-1}{2}}[m]q^{\frac{(m-1)^2}{4}} \pmod{\Phi_m(q)^3}. \tag{3.4}$$

Applying the relations $(q^{2k+1};q^2)_{m-1}=(q;q^2)_m(q^{2m+1};q^2)_{k-1}/(q;q^2)_k$ and $(q^{2k+1};q)_{m-1}=(1-q)(q;q)_{m-1}[m](q^{m+1};q)_{2k-1}/(q;q)_{2k}$, we get

$$\begin{aligned}G(m,k)&=\frac{(-1)^m q^{m+2k-1}(q^{2m+1};q^2)_{k-1}^2(q;q)_{2k}}{(1-q)^2(q;q^2)_k(q^{m+1};q)_{2k-1}(q^2;q^2)_k}\cdot\frac{(q;q^2)_m^3}{[m](q;q)_{m-1}^3}\\&=-\frac{q^{m+2k-1}(q^{2m+1};q^2)_{k-1}^2}{(1-q)^2(q^{m+1};q)_{2k-1}}\cdot\frac{(q;q^2)_m^3}{[m](q;q)_{m-1}^3}.\end{aligned}$$

Since $(q;q^2)_m$ has the factor $1-q^m$ and is therefore divisible by $\Phi_m(q)$, we see that

$$\frac{(q;q^2)_m^3}{[m](q;q)_{m-1}^3}\equiv 0 \pmod{\Phi_m(q)^2}.$$

Thus, using the relation $q^{2m}\equiv q^m\equiv 1 \pmod{\Phi_m(q)}$, we obtain

$$G(m,k)\equiv-\frac{q^{2k-1}(q;q^2)_{k-1}^2}{(1-q)^2(q;q)_{2k-1}}\cdot\frac{(q;q^2)_m^3}{[m](q;q)_{m-1}^3} \pmod{\Phi_m(q)^3},$$

where the congruence is valid for $k=1,2,\dots,\frac{m-1}{2}$. It follows that

$$\begin{aligned}\sum_{k=1}^{\frac{m-1}{2}}G(m,k)&\equiv-\frac{(q;q^2)_m^3}{[m](q;q)_{m-1}^3}\sum_{k=1}^{\frac{m-1}{2}}\frac{q^{2k-1}(q;q^2)_{k-1}^2}{(1-q)^2(q;q)_{2k-1}}\\&=-\frac{[m]^2}{(-q;q)_{m-1}^3}{2m-1\brack m-1}^3\sum_{k=1}^{\frac{m-1}{2}}\frac{q^{2k-1}}{[2k-1](-q;q)_{k-1}^2}{2k-2\brack k-1} \pmod{\Phi_m(q)^3}.\end{aligned}$$

Noticing that $(-q;q)_{m-1}$ is relatively prime to $[m]$, we can use ${2m-1 \brack m-1} \equiv 1 \pmod{\Phi_m(q)}$, (2.1), and (2.5) to obtain

$$\sum_{k=1}^{\frac{m-1}{2}} G(m,k) \equiv (-1)^{\frac{m-1}{2}}[m]\left((-q;q)_{m-1}q^{-\frac{(m-1)(m-3)}{4}} - q^{\frac{(m-1)(m+3)}{4}}\right)q \pmod{\Phi_m(q)^3}. \tag{3.5}$$

On the other hand, by (2.4) we have

$$F\left(0,\frac{m-1}{2}\right) = [m]\frac{(q;q^2)_{\frac{m-1}{2}}}{(q^2;q^2)_{\frac{m-1}{2}}} \equiv (-1)^{\frac{m-1}{2}}q^{\frac{1-m^2}{4}}[m](-q;q)_{m-1} \pmod{\Phi_m(q)^3}. \tag{3.6}$$

Moreover, by (2.4) and (2.6), for $1 \leqslant n \leqslant m-1$,

$$\begin{aligned}
F\left(n,\frac{m-1}{2}\right) &= (-1)^n[3n+m]\frac{(q;q^2)_n(q^m;q^2)_n^2(q;q^2)_{\frac{m-1}{2}}}{(q;q)_n^2(q^m;q)_n(q^2;q^2)_{\frac{m-1}{2}}} \\
&= (-1)^n[3n+m]\frac{(1-q)(q;q^2)_n(q^{m+2};q^2)_{n-1}^2}{(q;q)_n^2(q^{m+1};q)_{n-1}}\cdot[m]\frac{(q;q^2)_{\frac{m-1}{2}}}{(q^2;q^2)_{\frac{m-1}{2}}} \\
&\equiv (-1)^n[3n+m]\frac{(-q^{m+1};q)_{n-1}}{[2n]}{2n \brack n} \\
&\quad \times (-1)^{\frac{m-1}{2}}q^{\frac{1-m^2}{4}}[m](-q;q)_{m-1} \pmod{\Phi_m(q)^3}.
\end{aligned} \tag{3.7}$$

Substituting (3.4)–(3.7) into (3.3), we are led to

$$\begin{aligned}
&(-1)^{\frac{m-1}{2}}[m]q^{\frac{(m-1)^2}{4}} \\
&\equiv (-1)^{\frac{m-1}{2}}q^{\frac{1-m^2}{4}}[m](-q;q)_{m-1} \\
&\quad + (-1)^{\frac{m-1}{2}}q^{\frac{1-m^2}{4}}[m](-q;q)_{m-1}\sum_{n=1}^{m-1}(-1)^n[3n+m]\frac{(-q^{m+1};q)_{n-1}}{[2n]}{2n \brack n} \\
&\quad + (-1)^{\frac{m-1}{2}}[m]\left((-q;q)_{m-1}q^{-\frac{(m-1)(m-3)}{4}} - q^{\frac{(m-1)(m+3)}{4}}\right)q \pmod{\Phi_m(q)^3}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{n=1}^{m-1}(-1)^n[3n+m]\frac{(-q^{m+1};q)_{n-1}}{[2n]}{2n \brack n} \\
&\equiv \frac{q^{\frac{(m-1)^2}{4}} - q^{\frac{1-m^2}{4}}(-q;q)_{m-1} + q^{\frac{(m-1)(m+3)}{4}+1} - (-q;q)_{m-1}q^{1-\frac{(m-1)(m-3)}{4}}}{q^{\frac{1-m^2}{4}}(-q;q)_{m-1}} \\
&= \frac{(1+q^m)(q^{\binom{m}{2}} - (-q;q)_{m-1})}{(-q;q)_{m-1}} \pmod{\Phi_m(q)^2}.
\end{aligned} \tag{3.8}$$

Since $q^m \equiv 1 \pmod{\Phi_m(q)}$ and $q^{\binom{m}{2}} - (-q;q)_{m-1} \equiv 1 - (-q;q)_{m-1} \equiv 0 \pmod{\Phi_m(q)}$, we see that the right-hand side of (3.8) can be simplified as $2(q^{\binom{m}{2}} - (-q;q)_{m-1})$. Finally, for $m = 1$, each side of (3.8) is equal to 0 and the congruence still holds. □

Remark. The author [4] proved (1.7) by the following q-WZ pair:

$$F(n,k) = (-1)^n [3n-2k+1] \begin{bmatrix} 2n-2k \\ n \end{bmatrix} \frac{(q;q^2)_n (q;q^2)_{n-k}}{(q;q)_n (q^2;q^2)_{n-k}},$$

$$G(n,k) = (-1)^{n+1} [n] \begin{bmatrix} 2n-2k \\ n-1 \end{bmatrix} \frac{(q;q^2)_n (q;q^2)_{n-k} q^{n+1-2k}}{(q;q)_n (q^2;q^2)_{n-k}},$$

which is a q-analogue of the WZ pair given by He in his proof of [7, Theorem 1.1].

4. Proof of Theorem 1.2

When $n = 1$, both sides of (1.6) are equal to 1. For $n > 1$, let $\zeta \neq 1$ be an n-th root of unity (not necessarily primitive). In other words, ζ is a primitive root of unity of odd degree $d \mid n$. Let $c_q(k)$ stand for the k-th term on the left-hand side of (1.6), i.e.,

$$c_q(k) = (-1)^k [3k+1] \frac{(q;q^2)_k^3}{(q;q)_k^3} = (-1)^k \frac{[3k+1]}{(-q;q)_k^3} \begin{bmatrix} 2k \\ k \end{bmatrix}^3.$$

Since $(q;q^2)_k \equiv 0 \pmod{\Phi_n(q)}$ for $\frac{n+1}{2} \leqslant k \leqslant n-1$, from (1.7) we deduce that

$$\sum_{k=0}^{\frac{n-1}{2}} (-1)^k [3k+1] \frac{(q;q^2)_k^3}{(q;q)_k^3} \equiv \sum_{k=0}^{n-1} (-1)^k [3k+1] \frac{(q;q^2)_k^3}{(q;q)_k^3} \equiv (-1)^{\frac{n-1}{2}} [n] q^{\frac{(n-1)^2}{4}} \pmod{\Phi_n(q)^3}. \tag{4.1}$$

These two congruences with $n = d$ imply that

$$\sum_{k=0}^{\frac{d-1}{2}} c_\zeta(k) = \sum_{k=0}^{d-1} c_\zeta(k) = 0.$$

Noticing that

$$\frac{c_\zeta(\ell d + k)}{c_\zeta(\ell d)} = \lim_{q \to \zeta} \frac{c_q(\ell d + k)}{c_q(\ell d)} = c_\zeta(k),$$

we have

$$\sum_{k=0}^{\frac{n-1}{2}} c_\zeta(k) = \sum_{\ell=0}^{\frac{1}{2}(\frac{n}{d}-3)} c_\zeta(\ell d) \sum_{k=0}^{d-1} c_\zeta(k) + \sum_{k=0}^{\frac{d-1}{2}} c_\zeta\left(\frac{n-d}{2} + k\right) = 0,$$

which means that the sum $\sum_{k=0}^{\frac{n-1}{2}} c_q(k)$ is divisible by the cyclotomic polynomial $\Phi_d(q)$. Since this is true for any divisor $d > 1$ of n, we conclude that this sum is divisible by

$$\prod_{d \mid n,\, d>1} \Phi_d(q) = [n].$$

Namely, we have the following congruence:

$$\sum_{k=0}^{\frac{n-1}{2}}(-1)^k[3k+1]\frac{(q;q^2)_k^3}{(q;q)_k^3} \equiv (-1)^{\frac{n-1}{2}}[n]q^{\frac{(n-1)^2}{4}} \pmod{[n]}. \tag{4.2}$$

Combining (4.1) and (4.2), we complete the proof of (1.6). □

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