A q-CONGRUENCE IMPLYING THE BEUKERS–VAN HAMME CONGRUENCE

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ABSTRACT. By making use of Andrews' terminating q-analogue of Watson's formula and a double sum identity, we give a q-analogue of the following congruence: for any prime $p \equiv 1 \pmod 4$,

$$\sum_{k=0}^{(p-1)/2} \binom{(p-1)/2}{k} \binom{(p-1)/2+k}{k} \equiv \frac{1}{2^{(p-1)/2}} \binom{(p-1)/2}{(p-1)/4} \pmod{p^2}.$$

In view of the Chowla–Dwork–Evans congruence, our q-congruence may somewhat be regarded as a q-analogue of the Beukers–Van Hamme congruence:

$$\sum_{k=0}^{(p-1)/2} \binom{(p-1)/2}{k} \binom{(p-1)/2+k}{k} \equiv (-1)^{(p-1)/4} \left(2a - \frac{p}{2a}\right) \pmod{p^2},$$

where $p = a^2 + b^2$ with $a, b \in \mathbb{Z}$ and $a \equiv 1 \pmod 4$.

1. Introduction

The *Legendre polynomials* $P_n(x)$ can be defined as follows:

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \left(\frac{x-1}{2}\right)^k.$$

In different books on orthogonal polynomials, there are quite different definitions of Legendre polynomials (see [17] for a collection of such definitions). The numbers

$$c_n = P_n(3) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}$$

also known as the (central) Delannoy numbers (see [20]) play an important role in proving that $\log 2$ is irrational with measure of irrationality $4.622\ldots$ (see [1]). Carlitz [3] proved that the numbers c_n satisfy the following congruence: for any prime $p \equiv 1 \pmod 4$,

$$c_{(p-1)/2} \equiv (-1)^{(p-1)/4} \cdot 2a \pmod p, \tag{1.1}$$

1991 *Mathematics Subject Classification.* 05A10, 33D15, 11A07, 11B65.

Key words and phrases. Legendre polynomials; supercongruence; q-congruence; Andrews' terminating q-analogue of Watson's formula.

The second author was supported by the National Natural Science Foundation of China (grant 12171370).

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where $p=a^2+b^2$ with $a,b\in\mathbb{Z}$ and $a\equiv 1 \pmod 4$. Van Hamme [24] further established a stronger version of (1.1):

$$c_{(p-1)/2}\equiv(-1)^{(p-1)/4}(a+bi)\equiv(-1)^{(p-1)/4}\left(2a-\frac{p}{2a}\right) \pmod{p^2}, \tag{1.2}$$

where i is a p-adic integer such that $i^2=-1$. An Atkin and Swinnerton-Dyer type generalization of (1.2) was given by Coster and Van Hamme [5]. It is clear that (1.2) implies the following congruence:

$$c_{(p-1)/2}^2\equiv 4a^2-2p \pmod{p^2},$$

which was originally conjectured by Beukers.

For any odd prime p and $0\leqslant k\leqslant (p-1)/2$, one can easily check that

$$\begin{aligned}\binom{(p-1)/2}{k}\binom{(p-1)/2+k}{k}&=\prod_{j=1}^{k}\frac{((p+1)/2-j)((p-1)/2+j)}{j^2}\\&\equiv(-1)^k\prod_{j=1}^{k}\frac{(j-\frac12)^2}{j^2}=\frac{1}{16^k}\binom{2k}{k}^2(-1)^k \pmod{p^2},\end{aligned}$$

and so

$$c_{(p-1)/2}\equiv\sum_{k=0}^{(p-1)/2}\frac{1}{16^k}\binom{2k}{k}^2(-1)^k \pmod{p^2}.$$

Twenty-five years after Van Hamme's work [24], Z.-H. Sun [21] reproved the following result: for any prime $p\equiv 1 \pmod 4$,

$$\sum_{k=0}^{(p-1)/2}\frac{1}{16^k}\binom{2k}{k}^2(-1)^k\equiv(-1)^{(p-1)/4}\left(2a-\frac{p}{2a}\right) \pmod{p^2}, \tag{1.3}$$

where $p=a^2+b^2$ with $a,b\in\mathbb{Z}$ and $a\equiv 1 \pmod 4$. Note that the congruence (1.3) also appears as a conjecture in [22].

Recall that the *q-shifted factorials* are defined as $(a;q)_0=1$ and

$$(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1}) \text{ for } n=1,2,\dots,$$

and the *q-binomial coefficients* ${n\brack k}_q$ are defined by

$${n\brack k}={n\brack k}_q=\begin{cases}\dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{if } 0\leqslant k\leqslant n,\\ 0, & \text{otherwise.}\end{cases}$$

Moreover, the n-th *cyclotomic polynomial* $\Phi_n(q)$ may be given by

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity. It is well known that $\Phi_p(1)=p$ for any prime p.

The objective of this note is to give the following q-congruence which implies the Beukers–Van Hamme congruence (1.2).

Theorem 1.1. *Let n be an integer with $n\equiv 1 \pmod 4$ and $n>1$. Then*

$$\sum_{k=0}^{(n-1)/2} \begin{bmatrix} (n-1)/2 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} (n-1)/2+k \\ k \end{bmatrix}_{q^2} q^{k^2+k-nk} \equiv q^{(n-1)/4}\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}} \pmod{\Phi_n(q)^2}. \tag{1.4}$$

Letting $n=p$ be a prime and taking $q\to 1$ in (1.4), we are led to

$$c_{(p-1)/2} \equiv \frac{1}{2^{(p-1)/2}}\binom{(p-1)/2}{(p-1)/4} \pmod{p^2}. \tag{1.5}$$

Note that Chowla, Dwork, and Evans [4] have proved the following result:

$$\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2}\left(2a-\frac{p}{2a}\right) \pmod{p^2}, \tag{1.6}$$

where $p\equiv 1 \pmod 4$ and $p=a^2+b^2$ with $a\equiv 1 \pmod 4$. Therefore,

$$\frac{1}{2^{(p-1)/2}}\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2^{(p+1)/2}}\left(2a-\frac{p}{2a}\right). \tag{1.7}$$

For $p\equiv 1 \pmod 4$, we have $2^{(p-1)/2}\equiv(-1)^{(p-1)/4} \pmod p$ and so

$$\frac{2^{p-1}+1}{2^{(p+1)/2}} \equiv (-1)^{(p-1)/4} \pmod{p^2}. \tag{1.8}$$

Combining (1.5), (1.7), and (1.8), we see that the q-congruence (1.4) implies the Beukers–Van Hamme congruence (1.2).

We have another weaker result as follows.

Theorem 1.2. *Let n be a positive integer with $n\equiv 3 \pmod 4$. Then*

$$\sum_{k=0}^{(n-1)/2} \begin{bmatrix} (n-1)/2 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} (n-1)/2+k \\ k \end{bmatrix}_{q^2} q^{k^2+k} \equiv 0 \pmod{\Phi_n(q)}. \tag{1.9}$$

Our proof of Theorems 1.1 and 1.2 is based on Andrews' terminating q-analogue of Watson's formula (see [2] or [6, (II.17)]):

$$ {}_4\phi_3\left[\begin{matrix} q^{-n},\ a^2q^{n+1},\ b,\ -b \\ aq,\ -aq,\ b^2 \end{matrix}; q, q\right] = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \dfrac{b^n(q,a^2q^2/b^2;q^2)_{n/2}}{(a^2q^2,\ b^2q;q^2)_{n/2}}, & \text{if } n \text{ is even,} \end{cases} \tag{1.10}$$

where the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined as

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_1,a_2,\dots,a_{r+1} \\ b_1,b_2,\dots,b_r \end{matrix}; q,\ z\right] = \sum_{k=0}^{\infty}\frac{(a_1,a_2,\dots,a_{r+1};q)_k z^k}{(q,b_1,\dots,b_r;q)_k},$$

and $(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$.

2. Two Lemmas

Besides Andrews' summation formula, we also need another two lemmas.

Lemma 2.1. *Let n be a positive odd integer. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^2}{(q^2;q^2)_k^2}(-q)^k \equiv (-1)^{(n-1)/2}q^{(1-n^2)/4}\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k(q^2;q^4)_k}{(q^2;q^2)_k^2}q^{2k}. \tag{2.1}$$

Proof. The first author [8, Theorem 4.2] has proved that, modulo $(1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{n-1}\frac{(aq;q^2)_k(q/a;q^2)_k}{(q^2;q^2)_k^2}x^k \equiv (-1)^{(n-1)/2}q^{(1-n^2)/4}\sum_{k=0}^{n-1}\frac{(aq;q^2)_k(q/a;q^2)_k}{(q^2;q^2)_k^2}q^{2k}(x;q^2)_k. \tag{2.2}$$

Note that $(1-q^n)^2$ has the factor $\Phi_n(q)^2$, which is coprime with the denominators on both sides of (2.2). Letting $a=1$ and $x=-q$ in (2.2), we arrive at (2.1). □

Lemma 2.2. *For any positive integer n, there holds*

$$\sum_{k=0}^{2n}\frac{(q^{-4n};q^2)_k(q^2;q^4)_k}{(q^2;q^2)_k^2}q^{2k}\sum_{i=1}^{k}\frac{q^{2i-1}}{1-q^{2i-1}} = -nq^{2n}\frac{(q^2;q^4)_n}{(q^4;q^4)_n}. \tag{2.3}$$

Proof. Let L_n and R_n denote the left-hand side and the right-hand side of (2.3), respectively. Using the symbolic summation package Sigma developed by Schneider [19], we can obtain the following recurrence satisfied by L_n:

$$\begin{aligned}
&-q^{10}(1-q^{4n+2})(1-q^{4n+4})(1-q^{4n+6})(1-q^{4n+9})L_n\\
&\quad+q^4(1-q^{4n+6})(1+2q^4-q^{4n+5}-4q^{4n+8}-2q^{4n+10}-2q^{4n+13}\\
&\qquad+2q^{8n+12}+2q^{8n+15}+4q^{8n+17}+q^{8n+20}-2q^{12n+21}+q^{12n+25})L_{n+1}\\
&\quad-q^2(1-q^{4n+8})(2+q^4-2q^{4n+5}-2q^{4n+8}-4q^{4n+10}-q^{4n+13}\\
&\qquad+q^{8n+12}+4q^{8n+15}+2q^{8n+17}+2q^{8n+20}-q^{12n+21}-2q^{12n+25})L_{n+2}\\
&\quad+(1-q^{4n+5})(1-q^{4n+8})(1-q^{4n+10})(1-q^{4n+12})L_{n+3}=0.
\end{aligned} \tag{2.4}$$

It is trivial to verify that $\{R_n\}_{n\geq 1}$ also satisfies the recurrence (2.4) and $L_n=R_n$ for $n=1,2,3$. This completes the proof of (2.3). □

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. It is clear that, for any $k\geqslant 0$ and indeterminates a,b,

$$(aq,bq;q^2)_k \equiv (q,abq;q^2)_k \pmod{(1-a)(1-b)}.$$

Putting $a=q^{-n}$ and $b=q^n$ in the above congruence and noticing that $1-q^{\pm n}$ contains the factor $\Phi_n(q)$, we get

$$(q^{1-n},q^{1+n};q^2)_k \equiv (q;q^2)_k^2 \pmod{\Phi_n(q)^2}.$$

Therefore, by the definition of q-binomial coefficients, the left-hand side of (1.4) can be written as

$$\sum_{k=0}^{(n-1)/2}\frac{(q^{1-n},q^{1+n};q^2)_k}{(q^2;q^2)_k^2}(-q)^k\equiv\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^2}{(q^2;q^2)_k^2}(-q)^k\pmod{\Phi_n(q)^2}.\tag{3.1}$$

Since $n\equiv 1\pmod 4$, by Lemma 2.1, the right-hand side modulo $\Phi_n(q)^2$ is congruent to

$$q^{(1-n^2)/4}\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k(q^2;q^4)_k}{(q^2;q^2)_k^2}q^{2k}.$$

To prove the theorem, it suffices to prove that

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k(q^2;q^4)_k}{(q^2;q^2)_k^2}q^{2k}\equiv q^{(n-1)(n+2)/4}\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}\pmod{\Phi_n(q)^2}.\tag{3.2}$$

Making the parameter substitutions $a\mapsto 0$, $b\mapsto q$, $q\mapsto q^2$, and $n\mapsto (n-1)/2$ in (1.10) gives

$$\sum_{k=0}^{(n-1)/2}\frac{(q^{1-n};q^2)_k(q^2;q^4)_k}{(q^2;q^2)_k^2}q^{2k}=q^{(n-1)/2}\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}.\tag{3.3}$$

Note that, for $0\leqslant k\leqslant (n-1)/2$,

$$\begin{aligned}(q^{1-n};q^2)_k&=\prod_{i=1}^{k}(1-q^{2i-1-n})\\&=\prod_{i=1}^{k}\big(1-q^{2i-1}-(1-q^n)q^{2i-1-n}\big)\\&\equiv\prod_{i=1}^{k}(1-q^{2i-1})\left(1-(1-q^n)\sum_{i=1}^{k}\frac{q^{2i-1}}{1-q^{2i-1}}\right)\\&=(q;q^2)_k\left(1-(1-q^n)\sum_{i=1}^{k}\frac{q^{2i-1}}{1-q^{2i-1}}\right)\pmod{\Phi_n(q)^2}.\end{aligned}\tag{3.4}$$

Substituting (3.4) into the left-hand side of (3.3), we obtain

$$\begin{aligned}&\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k(q^2;q^4)_k}{(q^2;q^2)_k^2}q^{2k}\left(1-(1-q^n)\sum_{i=1}^{k}\frac{q^{2i-1}}{1-q^{2i-1}}\right)\\&\quad\equiv q^{(n-1)/2}\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}\pmod{\Phi_n(q)^2}.\end{aligned}\tag{3.5}$$

Now, letting $n \mapsto (n-1)/4$ in (2.3) and using $q^{-n} \equiv 1 \pmod{\Phi_n(q)}$, we arrive at

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k (q^2;q^4)_k}{(q^2;q^2)_k^2} q^{2k} \sum_{i=1}^{k} \frac{q^{2i-1}}{1-q^{2i-1}} \equiv \frac{1-n}{4} q^{(n-1)/2} \frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}} \pmod{\Phi_n(q)}. \tag{3.6}$$

Combining (3.5) and (3.6) yields that

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k (q^2;q^4)_k}{(q^2;q^2)_k^2} q^{2k} \\
&\quad \equiv q^{(n-1)/2} \frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}} + \frac{1-n}{4} q^{(n-1)/2} (1-q^n) \frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}} \\
&\quad = q^{(n-1)/2} \left(1 + \frac{1-n}{4}(1-q^n)\right) \frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}} \pmod{\Phi_n(q)^2}.
\end{aligned} \tag{3.7}$$

Since for all integers s,

$$\begin{aligned}
q^{sn} &= 1 - (1 - q^{sn}) \\
&= 1 - (1-q^n)(1 + q^n + q^{2n} + \cdots + q^{(s-1)n}) \\
&\equiv 1 - s(1-q^n) \pmod{\Phi_n(q)^2},
\end{aligned}$$

we have

$$q^{(n-1)(n+2)/4} = q^{(n-1)n/4 + (n-1)/2} \equiv q^{(n-1)/2} \left(1 + \frac{1-n}{4}(1-q^n)\right) \pmod{\Phi_n(q)^2}. \tag{3.8}$$

The proof of (3.2) then follows from (3.7) and (3.8). □

Proof of Theorem 1.2. Since $n \equiv 3 \pmod 4$, similarly as the proof of Theorem 1.1, we know that the left-hand side of (1.9) is congruent to

$$\begin{aligned}
\sum_{k=0}^{(n-1)/2} \frac{(q^{1-n}, q^{1+n}; q^2)_k}{(q^2;q^2)_k^2} (-q)^k &\equiv \sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} (-q)^k \\
&\equiv -q^{(1-n^2)/4} \sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k (q^2;q^4)_k}{(q^2;q^2)_k^2} q^{2k} \pmod{\Phi_n(q)}.
\end{aligned} \tag{3.9}$$

Performing the parameter substitutions $a \mapsto 0$, $b \mapsto q$, $q \mapsto q^2$, and $n \mapsto (n-1)/2$ in (1.10) produces

$$\sum_{k=0}^{(n-1)/2} \frac{(q^{1-n};q^2)_k (q^2;q^4)_k}{(q^2;q^2)_k^2} q^{2k} = 0.$$

In light of $q^{-n} \equiv 1 \pmod{\Phi_n(q)}$, we immediately conclude that the right-hand side of (3.9) is congruent to 0 modulo $\Phi_n(q)$. This completes the proof. □

4. Concluding remarks

In this paper, we mainly give a q-analogue of the congruence (1.5), which, in view of the Chowla–Dwork–Evans congruence (1.6), implies the Beukers–Van Hamme congruence (1.2). It remains a challenging problem to find a q-analogue of the Chowla–Dwork–Evans congruence (1.6) (or its weaker form (1.1)).

There are also some other congruences related to (1.2). For example, the (H.2) supercongruence of Van Hamme [25] can be stated as follows: for any odd prime p,

$$\sum_{k=0}^{(p-1)/2}\frac{1}{64^k}\binom{2k}{k}^3\equiv\begin{cases}-\Gamma_p(\tfrac14)^4 \pmod{p^2} & \text{if } p\equiv 1 \pmod 4,\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\end{cases}\tag{4.1}$$

where $\Gamma_p(x)$ denotes the p-adic Gamma function. Recently, the first author and Zudilin [12, Theorem 2] proved that, for positive odd integers n, modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k}\equiv\begin{cases}\dfrac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}q^{(n-1)/2} & \text{if } n\equiv 1 \pmod 4,\\ 0 & \text{if } n\equiv 3 \pmod 4,\end{cases}$$

which is a q-analogue of the following congruence:

$$\sum_{k=0}^{(p-1)/2}\frac{1}{64^k}\binom{2k}{k}^3\equiv\begin{cases}\frac{1}{2^{p-1}}\binom{(p-1)/2}{(p-1)/4}^2 \pmod{p^2} & \text{if } p\equiv 1 \pmod 4,\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}\tag{4.2}$$

In view of [24, Theorem 3]:

$$\frac{1}{2^{(p-1)/2}}\binom{(p-1)/2}{(p-1)/4}=\binom{-1/2}{(p-1)/4}\equiv-\frac{\Gamma_p(\frac14)^2}{\Gamma_p(\frac12)} \pmod{p^2},$$

and $\Gamma_p(\frac12)^2=-1$ for $p\equiv 1 \pmod 4$. Thus, we see that the congruence (4.2) is equivalent to (4.1). In light of (1.6), the congruence (4.2) can also be written as

$$\sum_{k=0}^{(p-1)/2}\frac{1}{64^k}\binom{2k}{k}^3\equiv\begin{cases}2(a^2-b^2) \pmod{p^2} & \text{if } p=a^2+b^2 \text{ with } a \text{ odd},\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

The first author and Zudilin [13] also gave another q-analogue of (4.2) as follows:

$$\begin{aligned}&\sum_{k=0}^{(n-1)/2}\frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)(q^4;q^4)_k^3}q^k\\ &\equiv\begin{cases}\dfrac{(1+q^n)(q^2;q^4)_{(n-1)/4}^2}{(1+q)(q^4;q^4)_{(n-1)/4}^2} \pmod{\Phi_n(q)^2} & \text{if } n\equiv 1 \pmod 4,\\ 0 \pmod{\Phi_n(q)^2} & \text{if } n\equiv 3 \pmod 4.\end{cases}\end{aligned}$$

(A typo has been corrected here.)

Moreover, Van Hamme [25, (A.2)] made the following conjecture:

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5}\equiv\begin{cases}-\dfrac{p}{\Gamma_p(\frac{3}{4})^4}\pmod{p^3} & \text{if } p\equiv 1 \pmod 4,\\ 0\pmod{p^3} & \text{if } p\equiv 3\pmod 4.\end{cases}\tag{4.3}$$

This supercongruence was first proved by McCarthy and Osburn [18]. Swisher [23] then showed that (4.3) also holds modulo p^5 for any prime $p\equiv 1 \pmod 4$ and $p>5$. The second author [15] extended the second part of (4.3) to the modulus p^4 case. Wang and Yue [26] and the first author [9] built the following q-congruence: for odd n, modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]\frac{(q;q^2)_k^4(q^2;q^4)_k}{(q^2;q^2)_k^4(q^4;q^4)_k}q^k\equiv\begin{cases}\dfrac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2}[n] & \text{if } n\equiv 1\pmod 4,\\ 0 & \text{if } n\equiv 3\pmod 4,\end{cases}$$

which may somewhat be deemed a q-analogue of (4.3).

The last congruence we want to mention is a result due to He [14]: for any odd prime p, modulo p^2,

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{(\frac{1}{2})_k^3(\frac{1}{4})_k}{k!^4 4^k}\equiv\begin{cases}(-1)^{(p+3)/4}p\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})^2 & \text{if } p\equiv 1\pmod 4,\\ 0 & \text{if } p\equiv 3\pmod 4.\end{cases}\tag{4.4}$$

Liu and Wang [16] established the following q-congruence: for positive odd integers n, modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q;q^2)_k^3(q;q^4)_k q^{k^2+k}}{(q^2;q^2)_k(q^4;q^4)_k^3}\equiv\begin{cases}\dfrac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}[n]q^{(1-n)/4} & \text{if } n\equiv 1\pmod 4,\\ 0 & \text{if } n\equiv 3\pmod 4,\end{cases}$$

which again may somewhat be considered as a q-analogue of (4.4) modulo p^3.

Therefore, if one can find a q-analogue of Beukers–Van Hamme congruence (1.2), then we will obtain more q-congruences, which are full q-analogues of some classical congruences in the literature. However, this work seems rather difficult!

In 2019, the first author and Zudilin [11] devised a method, called "creative microscoping", to prove a number of q-supercongruences. Although this method is very useful in many cases (see, for example, [7, 9, 10, 12, 13, 16, 26, 27]), we do not know how to use it to prove Theorem 1.1 in this paper. Thus, our proof of Theorem 1.1 is a little complicated. Moreover, we only give a computer proof of Lemma 2.2. It would be very interesting if one can find out a human proof.

Declaration of competing interest. There is no competing interest.

Data availability. No data was used for the research described in the article.

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