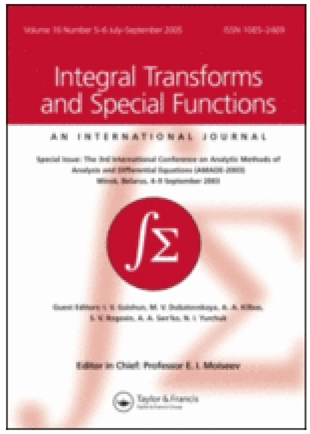

Some congruences related to a congruence of Van Hamme

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Some congruences related to a congruence of Van Hamme

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Abstract. We establish some supercongruences related to a supercongruence of Van Hamme, such as

$$\sum_{k=0}^{(p+1)/2}(-1)^k(4k-1)\frac{(-\frac{1}{2})_k^3}{k!^3}\equiv p(-1)^{(p+1)/2}+p^3(2-E_{p-3}) \pmod{p^4},$$

$$\sum_{k=0}^{(p+1)/2}(4k-1)^5\frac{(-\frac{1}{2})_k^4}{k!^4}\equiv 16p \pmod{p^4},$$

where p is an odd prime and E_{p-3} is the $(p-3)$-th Euler number. Our proof uses some congruences of Z.-W. Sun, the Wilf–Zeilberger method, Whipple's ${}_7F_6$ transformation, and the Mathematica package `Sigma` developed by Schneider. We also put forward two related conjectures.

Keywords: supercongruence; Euler numbers; gamma function; Whipple's ${}_7F_6$ transformation; WZ-pair.

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1. Introduction

In 1997, Van Hamme [32] conjectured that the Ramanujan-type formula for $1/\pi$:

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3}=\frac{2}{\pi},$$

due to Bauer [2], possesses a nice p-adic analogue:

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3}\equiv p(-1)^{(p-1)/2} \pmod{p^3}. \tag{1.1}$$

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Here and throughout the paper, p is an odd prime and $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochhammer symbol. The supercongruence (1.1) was first proved by Mortenson [24] in 2008 and reproved by Zudilin [34] in 2009. Motivated by Zudilin's work, in 2018, the first author [5] gave a q-analogue of (1.1) as follows:

$$\sum_{k=0}^{(p-1)/2} (-1)^k q^{k^2} [4k+1] \frac{(q;q^2)_k^3}{(q^2;q^2)_k^3} \equiv [p] q^{(p-1)^2/4} (-1)^{(p-1)/2} \pmod{[p]^3},$$

where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ and $[n] = [n]_q = 1 + q + \cdots + q^{n-1}$. For more supercongruences and q-supercongruences, we refer the reader to [4,6–21,27,30,31].

On the other hand, there is a similar supercongruence as follows:

$$\sum_{k=0}^{(p+1)/2} (-1)^k (4k-1) \frac{(-\frac{1}{2})_k^3}{k!^3} \equiv p(-1)^{(p+1)/2} \pmod{p^3}, \tag{1.2}$$

which is a special case of [9, Theorem 1.3] or [17, Theorem 4.9] (see also [16, Section 5]). Note that Sun [29] gave the following refinement of (1.1) modulo p^4:

$$\sum_{k=0}^{(p-1)/2} (-1)^k (4k+1) \frac{(\frac{1}{2})_k^3}{k!^3} \equiv p(-1)^{(p-1)/2} + p^3 E_{p-3} \pmod{p^4}, \tag{1.3}$$

where E_n is the n-th Euler number which may be defined by

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \frac{2e^x}{e^{2x}+1} \qquad \text{for} \quad |x| < \frac{\pi}{2}.$$

Inspired by Sun's result (1.3), we shall prove the following refinement of (1.2) modulo p^4.

Theorem 1.1. *We have*

$$\sum_{k=0}^{(p+1)/2} (-1)^k (4k-1) \frac{(-\frac{1}{2})_k^3}{k!^3} \equiv p(-1)^{(p+1)/2} + p^3(2 - E_{p-3}) \pmod{p^4}. \tag{1.4}$$

We shall also prove the following weaker supercongruence.

Theorem 1.2. *We have*

$$\sum_{k=0}^{(p+1)/2} (-1)^k (4k-1)^3 \frac{(-\frac{1}{2})_k^3}{k!^3} \equiv 3p(-1)^{(p-1)/2} \pmod{p^2}. \tag{1.5}$$

To the best of our knowledge, there are no q-congruences involving the Euler numbers E_{p-3} in the literature. For this reason, we believe that it is

difficult to find a q-analogue of (1.4). Recently, the first author [6, Theorem 5.1] gave a q-analogue of (1.5) as follows:

$$\sum_{k=0}^{(p+1)/2}(-1)^k q^{2k^2}[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^3}{(q^4;q^4)_k^3}\equiv\frac{2+q}{q^3}[p]_{q^2}q^{(p-1)^2/2}(-1)^{(p-1)/2}\pmod{[p]^3}.$$

The first author and Schlosser [15] proved the following supercongruence

$$\sum_{k=0}^{(p+1)/2}(4k-1)\frac{(-\frac{1}{2})_k^4}{k!^4}\equiv -5p^4\pmod{p^5}.$$

In this paper, using the same method in [20], we shall prove the following related result.

Theorem 1.3. *We have*

$$\sum_{k=0}^{(p+1)/2}(4k-1)^3\frac{(-\frac{1}{2})_k^4}{k!^4}\equiv 0\pmod{p^4},$$

$$\sum_{k=0}^{(p+1)/2}(4k-1)^5\frac{(-\frac{1}{2})_k^4}{k!^4}\equiv 16p\pmod{p^4},$$

$$\sum_{k=0}^{(p+1)/2}(4k-1)^7\frac{(-\frac{1}{2})_k^4}{k!^4}\equiv 80p\pmod{p^4}.$$

We can also give similar supercongruences for $\sum_{k=0}^{(p+1)/2}(4k-1)^m(-\frac{1}{2})_k^4/k!^4$ for some other odd integers $m\geqslant 9$. But the proofs will become more complicated and we omit these results here.

2. Proof of Theorem 1.1

We first give the following result due to Sun [28].

Lemma 2.1. *We have*

$$\sum_{k=1}^{(p-1)/2}\frac{4^k}{(2k-1)\binom{2k}{k}}\equiv E_{p-1}-1+(-1)^{(p-1)/2}\pmod{p}. \tag{2.1}$$

We also need the following congruence, which was given in the proof of [29, Theorem 1.1] implicitly.

Lemma 2.2. *For $1\leqslant k\leqslant (p-1)/2$, we have*

$$(-1)^{(p+1)/2+k}\frac{2(\frac{1}{2})_{(p+1)/2}^2(\frac{1}{2})_{(p-1)/2+k}}{(1)_{(p-1)/2}^2(1)_{(p+1)/2-k}(\frac{1}{2})_k^2}\equiv\frac{p^3 4^k}{2k(2k-1)\binom{2k}{k}}\pmod{p^4}. \tag{2.2}$$

Proof of Theorem 1.1. For all non-negative integers n and k, define the functions

$$F(n,k)=(-1)^{n+k}\frac{(4n-1)(-\frac{1}{2})_n^2(-\frac{1}{2})_{n+k}}{(1)_n^2(1)_{n-k}(-\frac{1}{2})_k^2},$$

$$G(n,k)=(-1)^{n+k}\frac{2(-\frac{1}{2})_n^2(-\frac{1}{2})_{n+k-1}}{(1)_{n-1}^2(1)_{n-k}(-\frac{1}{2})_k^2},$$

where we assume that $1/(1)_m=0$ for $m=-1,-2,\ldots$. The functions $F(n,k)$ and $G(n,k)$ form a Wilf–Zeilberger pair (WZ-pair). Namely, they satisfy the following relation

$$F(n,k-1)-F(n,k)=G(n+1,k)-G(n,k). \tag{2.3}$$

This WZ-pair is similar to one WZ-pair in [34] and can be found in the spirit of [3, 25]. Summing (2.3) over n from 0 to $(p+1)/2$, we obtain

$$\sum_{n=0}^{(p+1)/2}F(n,k-1)-\sum_{n=0}^{(p+1)/2}F(n,k)=G\left(\frac{p+3}{2},k\right)-G(0,k)=G\left(\frac{p+3}{2},k\right). \tag{2.4}$$

Summing (2.4) further over $k=1,2,\ldots,(p+1)/2$, we get

$$\sum_{n=0}^{(p+1)/2}F(n,0)=F\left(\frac{p+1}{2},\frac{p+1}{2}\right)+\sum_{k=1}^{(p+1)/2}G\left(\frac{p+3}{2},k\right), \tag{2.5}$$

where we have used $F(n,k)=0$ for $n<k$.

It is easy to see that

$$\begin{aligned}F\left(\frac{p+1}{2},\frac{p+1}{2}\right)&=\frac{(2p+1)(-\frac{1}{2})_{p+1}}{(1)_{(p+1)/2}^2}=-\frac{2p(2p+1)}{4^p(p+1)^2}\binom{2p}{p}\binom{p-1}{(p-1)/2}\\&\equiv(-1)^{(p+1)/2}\frac{p(2p+1)}{(p+1)^2}\equiv(-1)^{(p-1)/2}(p^3-p)\pmod{p^4},\end{aligned} \tag{2.6}$$

where we have used Wolstenholme's congruence [33]:

$$\binom{2p}{p}\equiv 2 \pmod{p^3}\quad\text{for } p>3,$$

and Morley's congruence [23]:

$$\binom{p-1}{(p-1)/2}\equiv(-1)^{(p-1)/2}4^{p-1}\pmod{p^3}\quad\text{for } p>3.$$

Moreover, we have

$$\begin{aligned}\sum_{k=1}^{(p+1)/2} G\left(\frac{p+3}{2},k\right) &= G\left(\frac{p+3}{2},1\right)+\sum_{k=1}^{(p-1)/2} G\left(\frac{p+3}{2},k+1\right)\\ &= (-1)^{(p-1)/2}\frac{p^3}{(p+1)^3 2^{3(p-1)}}\binom{p-1}{(p-1)/2}^3\\ &\quad+\sum_{k=1}^{(p-1)/2}(-1)^{(p+5)/2+k}\frac{2(-\frac{1}{2})^2_{(p+3)/2}(-\frac{1}{2})_{(p+3)/2+k}}{(1)^2_{(p+1)/2}(1)_{(p+1)/2-k}(-\frac{1}{2})^2_{k+1}}\\ &\equiv p^3-\sum_{k=1}^{(p-1)/2}(-1)^{(p+1)/2+k}\frac{4(\frac{1}{2})^2_{(p+1)/2}(\frac{1}{2})_{(p+1)/2+k}(\frac{p}{2}+k)}{(1)^2_{(p-1)/2}(1)_{(p+1)/2-k}(\frac{1}{2})^2_k(p+1)^2}\pmod{p^4}.\end{aligned}$$

By (2.2), modulo p^4 we may write the right-hand side of the above congruence as

$$p^3-\frac{p^3}{(p+1)^2}\sum_{k=1}^{(p-1)/2}\frac{4^k}{(2k-1)\binom{2k}{k}}.$$

Therefore, by (2.1), we obtain

$$\sum_{k=1}^{(p+1)/2} G\left(\frac{p+3}{2},k\right)\equiv p^3-p^3(E_{p-3}-1+(-1)^{(p-1)/2})\pmod{p^4}. \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5), we arrive at (1.4). □

3. Proof of Theorem 1.2

Recall that the *gamma function* $\Gamma(z)$, for any complex number z with the real part positive, may be defined by

$$\Gamma(z)=\int_0^\infty x^{z-1}e^{-x}dx,$$

and can be uniquely analytically extended to a meromorphic function defined for all complex numbers z, except for non-positive integers. It is worthwhile to mention that the gamma function has the property $\Gamma(z+1)=z\Gamma(z)$.

We need the following hypergeometric identity, which is a specialization of Whipple's ${}_7F_6$ transformation (see [1, p. 28]):

$$\begin{aligned}&{}_6F_5\left[\begin{matrix}a, & 1+\frac{1}{2}a, & b, & c, & d, & e\\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e\end{matrix};-1\right]\\ &=\frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)}{}_3F_2\left[\begin{matrix}1+a-b-c, & d, & e\\ & 1+a-b, & 1+a-c\end{matrix};1\right],\end{aligned} \tag{3.1}$$

where

$$ {}_{r+1}F_r\left[\begin{matrix} a_0, & a_1, & \ldots, & a_r \\ & b_1, & \ldots, & b_r \end{matrix};z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_r)_k}{k!(b_1)_k \cdots (b_r)_k} z^k. $$

Motivated by McCarthy and Osburn [22] and Mortenson [24], we take the following choice of variables in (3.1). Letting $a = -\frac{1}{2}$, $b = c = \frac{3}{4}$, $d = \frac{-1-p}{2}$, and $e = \frac{-1+p}{2}$, we conclude immediately that

$$ \begin{aligned} &{}_6F_5\left[\begin{matrix} -\frac{1}{2}, & \frac{3}{4}, & \frac{3}{4}, & \frac{3}{4}, & \frac{-1-p}{2}, & \frac{-1+p}{2} \\ & -\frac{1}{4}, & -\frac{1}{4}, & -\frac{1}{4}, & 1+\frac{p}{2}, & 1-\frac{p}{2} \end{matrix};-1\right] \\ &= \frac{\Gamma(1+\frac{p}{2})\Gamma(1-\frac{p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} {}_3F_2\left[\begin{matrix} -1, & \frac{-1-p}{2}, & \frac{-1+p}{2} \\ & -\frac{1}{4}, & -\frac{1}{4} \end{matrix};1\right]. \end{aligned} $$

It is easy to see that, for $k = 0, 1, \ldots, (p-1)/2$,

$$ \frac{\left(\frac{-1-p}{2}\right)_k \left(\frac{-1+p}{2}\right)_k}{\left(1+\frac{p}{2}\right)_k \left(1-\frac{p}{2}\right)_k} = \prod_{j=1}^{k} \frac{(2j-3)^2 - p^2}{4j^2 - p^2} \equiv \frac{\left(-\frac{1}{2}\right)_k^2}{k!^2} \pmod{p^2}. $$

Applying the property $\Gamma(x+1) = x\Gamma(x)$, we obtain

$$ \frac{\Gamma(1+\frac{p}{2})\Gamma(1-\frac{p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} = p(-1)^{(p-1)/2}, $$

and so

$$ \begin{aligned} -\sum_{k=0}^{(p+1)/2} (-1)^k (4k-1)^3 \frac{(-\frac{1}{2})_k^3}{k!^3} &\equiv p(-1)^{\frac{(p-1)}{2}} {}_3F_2\left[\begin{matrix} -1, & \frac{-1-p}{2}, & \frac{-1+p}{2} \\ & -\frac{1}{4}, & -\frac{1}{4} \end{matrix};1\right] \\ &= p(-1)^{\frac{(p-1)}{2}}(1 - 4(1-p^2)) \\ &\equiv -3p(-1)^{\frac{(p-1)}{2}} \pmod{p^2}, \end{aligned} $$

as desired.

4. Proof of Theorem 1.3

Let $H_n^{(2)} = \sum_{j=1}^{n} \frac{1}{j^2}$. The following lemma plays an important role in our proof.

Lemma 4.1. *For any integer $n \geqslant 2$ and odd positive integer m with $3 \leqslant m \leqslant 7$, we have*

$$ \sum_{k=0}^{n} (4k-1)^m \frac{\left(-\frac{1}{2}\right)_k^2 (-n)_k (n-1)_k}{(1)_k^2 \left(n+\frac{1}{2}\right)_k \left(\frac{3}{2}-n\right)_k} = f_m(n), \tag{4.1} $$

$$ \sum_{k=0}^{n} (4k-1)^m \frac{\left(-\frac{1}{2}\right)_k^2 (-n)_k (n-1)_k}{(1)_k^2 \left(n+\frac{1}{2}\right)_k \left(\frac{3}{2}-n\right)_k} \sum_{j=1}^{k} \left(\frac{1}{4j^2} - \frac{1}{(2j-3)^2}\right) = g_m(n), \tag{4.2} $$

where $f_m(n)$ and $g_m(n)$ are listed in the following table:

Table 1: Values of $f_m(n)$ and $g_m(n)$.

m	$f_m(n)$	$g_m(n)$
3	0	$\frac{(2n-1)^2\left(7n^2-7n+1\right)}{4n^2(n-1)^2}$
5	$-64n(n-1)(2n-1)$	$\frac{(2n-1)(256n^6-640n^5+320n^4+414n^3-493n^2+145n-1)}{4n^2(n-1)^2}+32n(n-1)(2n-1)H_n^{(2)}$
7	$-64n(n-1)(2n-1)(24n^2-24n+11)$	$\frac{(2n-1)(6144n^8-21504n^7+24320n^6-1920n^5-18496n^4+17582n^3-7557n^2+1433n-1)}{4n^2(n-1)^2}$ $+32n(n-1)(2n-1)\left(24n^2-24n+11\right)H_n^{(2)}$

Remark. **For fixed odd $m \geqslant 9$, there exist similar formulas for (4.1) and (4.2). But the formulas will become larger and larger and there are no general formulas for these m's.**

Proof. By using the Mathematica package `Sigma` due to Schneider [26], one can automatically discover and prove (4.1) and (4.2). For example, the steps to discover and prove (4.2) for $m=5$ are as follows.

Define the following sum:

In[1]:=mySum=$\sum_{k=0}^{n}(4k-1)^5\frac{\left(-\frac{1}{2}\right)_k^2(-n)_k(n-1)_k}{(1)_k^2\left(n+\frac{1}{2}\right)_k\left(\frac{3}{2}-n\right)_k}\sum_{j=1}^{k}\left(\frac{1}{4j^2}-\frac{1}{(2j-3)^2}\right)$

Compute the recurrence for this sum:

In[2]:=rec=GenerateRecurrence[mySum,n][[1]]

Out[2]=$(1+n)(1+2n)\mathrm{SUM}[n]-(-1+n)(-1+2n)\mathrm{SUM}[1+n]$

$$== \frac{(-1+2n)(1+2n)\left(78-183n^2+109n^4\right)}{2(-1+n)^2n(1+n)^2}+\frac{n(-1+2n)^3(1+2n)^2(3+2n)(3+4n)^4\left(-\frac{1}{2}\right)_n^2(-1+n)_n(-n)_n}{2(1+n)^3(1+4n)(1)_n^2\left(\frac{3}{2}-n\right)_n\left(\frac{1}{2}+n\right)_n}$$
$$-\frac{n(-1+2n)^3(1+2n)^2(3+2n)(3+4n)^4\left(-\frac{1}{2}\right)_n^2(-1+n)_n(-n)_n}{2(1+n)^3(1+4n)(1)_n^2\left(\frac{1}{2}(3-2n)\right)_n\left(\frac{1}{2}(1+2n)\right)_n}$$
$$+\left(-\frac{n(-1+2n)^3(1+2n)(3+4n)^4\left(-\frac{1}{2}\right)_n^2(-1+n)_n(-n)_n}{2(1+n)(1+4n)(1)_n^2\left(\frac{3}{2}-n\right)_n\left(\frac{1}{2}+n\right)_n}+\frac{n(-1+2n)^3(1+2n)(3+4n)^4\left(-\frac{1}{2}\right)_n^2(-1+n)_n(-n)_n}{2(1+n)(1+4n)(1)_n^2\left(\frac{1}{2}(3-2n)\right)_n\left(\frac{1}{2}(1+2n)\right)_n}\right)\left(\sum_{\iota_1=1}^{n}\frac{1}{\iota_1^2}\right)$$
$$+\left(\frac{2n(-1+2n)^3(1+2n)(3+4n)^4\left(-\frac{1}{2}\right)_n^2(-1+n)_n(-n)_n}{(1+n)(1+4n)(1)_n^2\left(\frac{3}{2}-n\right)_n\left(\frac{1}{2}+n\right)_n}-\frac{2n(-1+2n)^3(1+2n)(3+4n)^4\left(-\frac{1}{2}\right)_n^2(-1+n)_n(-n)_n}{(1+n)(1+4n)(1)_n^2\left(\frac{1}{2}(3-2n)\right)_n\left(\frac{1}{2}(1+2n)\right)_n}\right)\left(\sum_{\iota_1=1}^{n}\frac{1}{(-1+2\iota_1)^2}\right)$$

Now we solve this recurrence:

In[3]:=recSol=SolveRecurrence[rec,SUM[n]]

Out[3]=$\{\{0,(-1+n)n(-1+2n)\},$

$$\left\{1,\frac{(-1+2n)\left(-1+145n-493n^2+670n^3-448n^4+128n^5\right)}{4(-1+n)^2n^2}+32(-1+n)n(-1+2n)\left(\sum_{\iota_1=1}^{n}\frac{1}{\iota_1^2}\right)\right\}\}$$

Finally, we combine the solutions to represent mySum:

In[4]:=FindLinearCombination[recSol,mySum,1]

Out[4]= $\frac{(-1+2n)(-1+145n-493n^2+414n^3+320n^4-640n^5+256n^6)}{4(-1+n)^2n^2} + 32(-1+n)n(-1+2n)\left(\sum_{\iota_1=1}^{n} \frac{1}{\iota_1^2}\right)$

Thus, we discover and prove (4.2) for $m = 5$. □

Proof of Theorem 1.3. Observe that

$$\frac{(2j-3)^2 - x^2}{(2j)^2 - x^2} = \left(\frac{2j-3}{2j}\right)^2 - \frac{3(4j-3)}{(2j)^4}x^2 + O(x^4),$$

and

$$\prod_{j=1}^{k}(a_j + b_j x^2) = \left(\prod_{j=1}^{k} a_j\right)\left(1 + x^2 \sum_{j=1}^{k} \frac{b_j}{a_j}\right) + O(x^4).$$

Hence, for $0 \le k \le (p+1)/2$, we have

$$\begin{aligned}\frac{(\frac{-1-p}{2})_k(\frac{-1+p}{2})_k}{(1+\frac{p}{2})_k(1-\frac{p}{2})_k} &= \prod_{j=1}^{k} \frac{(2j-3)^2 - p^2}{(2j)^2 - p^2} \\ &\equiv \frac{(-\frac{1}{2})_k^2}{k!^2}\left(1 + p^2 \sum_{j=1}^{k}\left(\frac{1}{(2j)^2} - \frac{1}{(2j-3)^2}\right)\right) \pmod{p^4}. \qquad (4.3)\end{aligned}$$

Letting $n = (p+1)/2$ in (4.1) and using (4.3), we get

$$\begin{aligned}&\sum_{k=0}^{(p+1)/2} (4k-1)^m \frac{(-\frac{1}{2})_k^4}{k!^4}\left(1 + p^2 \sum_{j=1}^{k}\left(\frac{1}{(2j)^2} - \frac{1}{(2j-3)^2}\right)\right) \\ &\equiv f_m\left(\frac{p+1}{2}\right) \pmod{p^4}. \qquad (4.4)\end{aligned}$$

Furthermore, it follows from (4.3) that

$$\frac{(\frac{-1-p}{2})_k(\frac{-1+p}{2})_k}{(1+\frac{p}{2})_k(1-\frac{p}{2})_k} \equiv \frac{(-\frac{1}{2})_k^2}{k!^2} \pmod{p^2}. \qquad (4.5)$$

Letting $n = (p+1)/2$ in (4.2) and noticing (4.5), we obtain

$$\sum_{k=0}^{(p+1)/2} (4k-1)^m \frac{(-\frac{1}{2})_k^4}{k!^4} \sum_{j=1}^{k}\left(\frac{1}{(2j)^2} - \frac{1}{(2j-3)^2}\right) \equiv g_m\left(\frac{p+1}{2}\right) \pmod{p^2}. \qquad (4.6)$$

Finally, combining (4.4) and (4.6) we are led to

$$\sum_{k=0}^{(p+1)/2} (4k-1)^m \frac{(-\frac{1}{2})_k^4}{k!^4} \equiv f_m\left(\frac{p+1}{2}\right) - p^2 g_m\left(\frac{p+1}{2}\right) \pmod{p^4}. \qquad (4.7)$$

Note that

$$H_{(p+1)/2}^{(2)} = \left(\frac{2}{p+1}\right)^2 + H_{(p-1)/2}^{(2)} \equiv 4 \pmod{p}. \qquad (4.8)$$

The proof then follows from (4.7), (4.8) and Lemma 4.1. □

5. Two open problems

We end the paper with the following two conjectures, which are generalizations of Theorems 1.2 and 1.3. Note that there are similar unsolved conjectures in [4].

Conjecture 5.1. *For any odd positive integer m, there exists an integer c_m such that, for any odd prime p and positive integer r, there hold*

$$\sum_{k=0}^{\frac{p^r+1}{2}}(-1)^k(4k-1)^m\frac{(-\frac{1}{2})_k^3}{k!^3}\equiv c_m p^r(-1)^{\frac{(p-1)r}{2}} \pmod{p^{r+2}},$$

$$\sum_{k=0}^{p^r-1}(-1)^k(4k-1)^m\frac{(-\frac{1}{2})_k^3}{k!^3}\equiv c_m p^r(-1)^{\frac{(p-1)r}{2}} \pmod{p^{r+2}}.$$

In particular, we have $c_1=-1$, $c_3=3$, $c_5=23$, $c_7=-5$, $c_9=1647$, and $c_{11}=-96973$.

Conjecture 5.2. *For any odd positive integer m, there exists an integer d_m such that, for any odd prime p and positive integer r, there hold*

$$\sum_{k=0}^{\frac{p^r+1}{2}}(4k-1)^m\frac{(-\frac{1}{2})_k^4}{k!^4}\equiv d_m p^r \pmod{p^{r+3}},$$

$$\sum_{k=0}^{p^r-1}(4k-1)^m\frac{(-\frac{1}{2})_k^4}{k!^4}\equiv d_m p^r \pmod{p^{r+3}}.$$

In particular, we have $d_1=d_3=0$, $d_5=16$, $d_7=80$, $d_9=192$, $d_{11}=640$, $d_{13}=-3472$, and $d_{15}=138480$.

Note that, Conjecture 5.1 is true for $m=1$ by [9, Theorem 1.3], and Conjecture 5.2 is also true for $m=1$ by [15, Theorem 1.1].

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