Some congruences related to a congruence of Van Hamme

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Some congruences related to a congruence of Van Hamme

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Abstract. We establish some supercongruences related to a supercongruence of Van Hamme, such as
\begin{equation}
\sum_{k=0}^{(p+1)/2} (-1)^k(4k+1)\left(\frac{\frac{3}{2}k}{k!^3}\right) \equiv p(-1)^{(p+1)/2} + p^3(2 - E_{p-3}) \pmod{p^4},
\end{equation}
\begin{equation}
\sum_{k=0}^{(p+1)/2} (4k+1)^5\left(\frac{\frac{4}{2}k}{k!^4}\right) \equiv 16p \pmod{p^4},
\end{equation}
where $p$ is an odd prime and $E_{p-3}$ is the $(p-3)$-th Euler number. Our proof uses some congruences of Z.-W. Sun, the Wilf–Zeilberger method, Whipple’s $\gamma F_6$ transformation, and the Mathematica package $\texttt{Sigma}$ developed by Schneider. We also put forward two related conjectures.

Keywords: supercongruence; Euler numbers; gamma function; Whipple’s $\gamma F_6$ transformation; WZ-pair.

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1. Introduction

In 1997, Van Hamme [32] conjectured that the Ramanujan-type formula for $1/\pi$:
\begin{equation}
\sum_{k=0}^{\infty} (-1)^k(4k+1)\left(\frac{\frac{3}{2}k}{k!^3}\right) = \frac{2}{\pi},
\end{equation}
due to Bauer [2], possesses a nice $p$-adic analogue:
\begin{equation}
\sum_{k=0}^{(p-1)/2} (-1)^k(4k+1)\left(\frac{\frac{3}{2}k}{k!^3}\right) \equiv p(-1)^{(p-1)/2} \pmod{p^3}. \tag{1.1}
\end{equation}
Here and throughout the paper, \( p \) is an odd prime and \((a)_k = a(a+1)\cdots(a+k-1)\) is the Pochhammer symbol. The supercongruence (1.1) was first proved by Mortenson [24] in 2008 and reproved by Zudilin [34] in 2009. Motivated by Zudilin’s work, in 2018, the first author [5] gave a \( q \)-analogue of (1.1) as follows:

\[
\frac{(p-1)/2}{\sum_{k=0}^{(p-1)/2} (-1)^k q^{k^2}[4k + 1] (q^2; q^2)_k^3} = \left[ p \right] q^{(p-1)^2/4} (-1)^{(p-1)/2} \pmod{[p]^3},
\]

where \((a; q)_n = (1 - a)(1 - aq)\cdots(1 - aq^{n-1})\) and \([n]_q = [n]_q = 1 + q + \cdots + q^{n-1}\). For more supercongruences and \( q \)-supercongruences, we refer the reader to [4,6–21,27,30,31].

On the other hand, there is a similar supercongruence as follows:

\[
\frac{(p+1)/2}{\sum_{k=0}^{(p+1)/2} (-1)^k (4k - 1)\frac{(-1/2)_k}{k!^3}} \equiv p(-1)^{(p+1)/2} \pmod{p^3},
\]

which is a special case of [9, Theorem 1.3] or [17, Theorem 4.9] (see also [16, Section 5]). Note that Sun [29] gave the following refinement of (1.1) modulo \( p^4 \):

\[
\frac{(p-1)/2}{\sum_{k=0}^{(p-1)/2} (-1)^k (4k + 1)\frac{(1/2)_k}{k!^3}} \equiv p(-1)^{(p-1)/2} + p^3 E_{p-3} \pmod{p^4},
\]

where \( E_n \) is the \( n \)-th Euler number which may be defined by

\[
\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \frac{2e^x}{e^{2x} + 1} \quad \text{for} \quad |x| < \frac{\pi}{2}.
\]

Inspired by Sun’s result (1.3), we shall prove the following refinement of (1.2) modulo \( p^4 \).

**Theorem 1.1.** We have

\[
\frac{(p+1)/2}{\sum_{k=0}^{(p+1)/2} (-1)^k (4k - 1)\frac{(-1/2)_k}{k!^3}} \equiv p(-1)^{(p+1)/2} + p^3(2 - E_{p-3}) \pmod{p^4}.
\]

We shall also prove the following weaker supercongruence.

**Theorem 1.2.** We have

\[
\frac{(p+1)/2}{\sum_{k=0}^{(p+1)/2} (-1)^k (4k - 1)\frac{(-1/2)_k}{k!^3}} \equiv 3p(-1)^{(p-1)/2} \pmod{p^2}.
\]

To the best of our knowledge, there are no \( q \)-congruences involving the Euler numbers \( E_{p-3} \) in the literature. For this reason, we believe that it is
difficult to find a $q$-analogue of (1.4). Recently, the first author [6, Theorem 5.1] gave a $q$-analogue of (1.5) as follows:

$$\sum_{k=0}^{(p+1)/2} (-1)^k q^{2k^2} [4k - 1] q^2 [4k - 1]^2 (q^{-2}; q^4)_k^3 \equiv \frac{2 + q[p] q^{(p-1)^2/2} (-1)^{(p-1)/2}}{q^3} \mod [p]^3.$$

The first author and Schlosser [15] proved the following supercongruence

$$\sum_{k=0}^{(p+1)/2} \frac{(4k - 1)^3 (-\frac{1}{2})_k^4}{k!^4} \equiv -5p^4 \mod p^5.$$

In this paper, using the same method in [20], we shall prove the following related result.

**Theorem 1.3.** We have

1. $$\sum_{k=0}^{(p+1)/2} \frac{(4k - 1)^3 (-\frac{1}{2})_k^4}{k!^4} \equiv 0 \mod p^4,$$
2. $$\sum_{k=0}^{(p+1)/2} \frac{(4k - 1)^5 (-\frac{1}{2})_k^4}{k!^4} \equiv 16p \mod p^4,$$
3. $$\sum_{k=0}^{(p+1)/2} \frac{(4k - 1)^7 (-\frac{1}{2})_k^4}{k!^4} \equiv 80p \mod p^4.$$

We can also give similar supercongruences for $$\sum_{k=0}^{(p+1)/2} (4k - 1)^m (-\frac{1}{2})_k^4 / k!^4$$ for some other odd integers $m \geq 9$. But the proofs will become more complicated and we omit these results here.

## 2. Proof of Theorem 1.1

We first give the following result due to Sun [28].

**Lemma 2.1.** We have

$$\sum_{k=1}^{(p-1)/2} \frac{4^k}{(2k - 1)(2k)} \equiv E_{p-1} - 1 + (-1)^{(p-1)/2} \mod p.$$ \hspace{1cm} (2.1)

We also need the following congruence, which was given in the proof of [29, Theorem 1.1] implicitly.

**Lemma 2.2.** For $1 \leq k \leq (p-1)/2$, we have

$$(-1)^{(p+1)/2+k} \frac{2(1/2)^{2(p+1)/2} (1/2)^{(p-1)/2} + k}{(1/2)^{(p+1)/2} (1/2)^{(p-1)/2} - k (1/2)^{2k}} \equiv \frac{p^3 4^k}{2k(2k - 1)(2k)} \mod p^4.$$ \hspace{1cm} (2.2)
Proof of Theorem 1.1. For all non-negative integers \( n \) and \( k \), define the functions

\[
F(n, k) = (-1)^{n+k} \frac{(4n-1)(-\frac{1}{2})^n(-\frac{1}{2})^{n+k}}{(1)^2_n(1)_{n-k}(-\frac{1}{2})^2_k},
\]

\[
G(n, k) = (-1)^{n+k} \frac{2(-\frac{1}{2})^n(-\frac{1}{2})^{n+k-1}}{(1)^2_{n-1}(1)_{n-k}(-\frac{1}{2})^2_k},
\]

where we assume that \( \frac{1}{(1)_m} = 0 \) for \( m = -1, -2, \ldots \). The functions \( F(n, k) \) and \( G(n, k) \) form a Wilf–Zeilberger pair (WZ-pair). Namely, they satisfy the following relation

\[
F(n, k - 1) - F(n, k) = G(n + 1, k) - G(n, k).
\]  \hfill (2.3)

This WZ-pair is similar to one WZ-pair in [34] and can be found in the spirit of [3, 25]. Summing (2.3) over \( n \) from 0 to \( (p + 1)/2 \), we obtain

\[
\sum_{n=0}^{(p+1)/2} F(n, k - 1) - \sum_{n=0}^{(p+1)/2} F(n, k) = G \left( \frac{p+3}{2}, k \right) - G(0, k) = G \left( \frac{p+3}{2}, k \right).
\]  \hfill (2.4)

Summing (2.4) further over \( k = 1, 2, \ldots, (p+1)/2 \), we get

\[
\sum_{n=0}^{(p+1)/2} F(n, 0) = F \left( \frac{p+1}{2}, \frac{p+1}{2} \right) + \sum_{k=1}^{(p+1)/2} G \left( \frac{p+3}{2}, k \right),
\]  \hfill (2.5)

where we have used \( F(n, k) = 0 \) for \( n < k \).

It is easy to see that

\[
F \left( \frac{p+1}{2}, \frac{p+1}{2} \right) = \frac{(2p+1)(-\frac{1}{2})^{p+1}}{(1)^2_{(p+1)/2}} = \frac{-2p(2p+1)}{4^p(p+1)^2} \binom{2p}{p} \left( \frac{p-1}{(p-1)/2} \right)
\]

\[
\equiv (-1)^{(p+1)/2} 2^{p+1} \equiv (-1)^{(p-1)/2} (p^3 - p) \pmod{p^4},
\]  \hfill (2.6)

where we have used Wolstenholme’s congruence [33]:

\[
\binom{2p}{p} \equiv 2 \pmod{p^3} \quad \text{for } p > 3,
\]

and Morley’s congruence [23]:

\[
\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \pmod{p^3} \quad \text{for } p > 3.
\]
Moreover, we have
\[
\sum_{k=1}^{(p+1)/2} G \left( \frac{p+3}{2}, k \right) = G \left( \frac{p+3}{2}, 1 \right) + \sum_{k=1}^{(p-1)/2} G \left( \frac{p+3}{2}, k+1 \right)
\]
\[
= (-1)^{(p-1)/2} \frac{p^3}{(p+1)^32^{3(p-1)}} \left( \frac{p-1}{(p-1)/2} \right)^3
\]
\[
+ \sum_{k=1}^{(p-1)/2} (-1)^{(p+5)/2+k} 2(-\frac{1}{2})^{(p+3)/2}(\frac{1}{2})^{(p+3)/2+k}
\]
\[
\times \left( \frac{1}{(p+1)/2} \right)^2 \left( \frac{1}{(p+1)/2} \right)^2 \left( \frac{1}{(p+1)/2} \right)^2 \left( \frac{1}{(p+1)/2} \right)^2 \left( p+1 \right)^2 \mod p^4.
\]
By (2.2), modulo \(p^4\) we may write the right-hand side of the above congruence as
\[
p^3 - \frac{p^3}{(p+1)^2} \sum_{k=1}^{(p-1)/2} \frac{4k}{(2k-1)(2k)}.
\]
Therefore, by (2.1), we obtain
\[
\sum_{k=1}^{(p+1)/2} G \left( \frac{p+3}{2}, k \right) \equiv p^3 - p^3(F_{p-3} - 1 + (-1)^{(p-1)/2}) \mod p^4. \quad (2.7)
\]
Substituting (2.6) and (2.7) into (2.5), we arrive at (1.4). \(\square\)

3. Proof of Theorem 1.2

Recall that the \textit{gamma function} \(\Gamma(z)\), for any complex number \(z\) with the real part positive, may be defined by
\[
\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx,
\]
and can be uniquely analytically extended to a meromorphic function defined for all complex numbers \(z\), except for non-positive integers. It is worthwhile to mention that the gamma function has the property \(\Gamma(z+1) = z\Gamma(z)\).

We need the following hypergeometric identity, which is a specialization of Whipple’s \( \text{\textit{7F}}_6 \) transformation (see [1, p. 28]):
\[
\begin{align*}
\text{\textit{6F}}_5 & \left[ a, \quad 1 + \frac{1}{2}a, \quad b, \quad c, \quad d, \quad e; -1 \right] \\
& = \frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)} \text{\textit{3F}}_2 \left[ 1 + a - b - c, \quad d, \quad e; 1 \right]
\end{align*}
\]  
\quad (3.1)
where
\[ \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k \cdots (a_r)_k}{k!(b_1)_k \cdots (b_r)_k} z^k. \]

Motivated by McCarthy and Osburn [22] and Mortenson [24], we take the following choice of variables in (3.1). Letting \( a = -\frac{1}{2}, b = c = \frac{3}{4}, d = -\frac{1-p}{2}, \) and \( e = -\frac{1+p}{2}, \) we conclude immediately that
\[ \sum_{k=0}^{\infty} \frac{(-1)^k (4k+1)^3}{k!} \equiv 0 \mod p^2. \]

Applying the property \( \Gamma(x + 1) = x \Gamma(x), \) we obtain
\[ \frac{\Gamma(1 + \frac{3}{2}) \Gamma(1 - \frac{5}{2})}{\Gamma(1 + \frac{1}{2}) \Gamma(1 - \frac{3}{2})} = p(-1)^{(p-1)/2}, \]
and so
\[ - \sum_{k=0}^{(p+1)/2} (-1)^k (4k+1)^3 \equiv p(-1)^{(p-1)/2} \left( -1, -\frac{1-p}{2}, -\frac{1+p}{2}; -1 \right) \]
\[ = p(-1)^{(p-1)/2} (1 - 4(1 - p^2)) \equiv -3p(-1)^{(p-1)/2} \mod p^2, \]
as desired.

### 4. Proof of Theorem 1.3

Let \( H_n^{(2)} = \sum_{j=1}^{n} \frac{1}{j^2}. \) The following lemma plays an important role in our proof.

**Lemma 4.1.** For any integer \( n \geq 2 \) and odd positive integer \( m \) with \( 3 \leq m \leq 7, \) we have
\[ \sum_{k=0}^{n} (4k+1)^m \frac{(-1)^2_k}{(1)_k(n+\frac{1}{2})_k(\frac{3}{2} - n)_k} = f_m(n), \]
and
\[ \sum_{k=0}^{n} (4k+1)^m \frac{(-1)^2_k}{(1)_k(n+\frac{1}{2})_k(\frac{3}{2} - n)_k} \sum_{j=1}^{k} \frac{1}{4j^2} = g_m(n), \]
where \( f_m(n) \) and \( g_m(n) \) are listed in the following table:

URL: http://mc.manuscriptcentral.com/gitr
Table 1: Values of $f_m(n)$ and $g_m(n)$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$f_m(n)$</th>
<th>$g_m(n)$</th>
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<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>$(2n-1)^2\left(7n^2-7n+1\right)$</td>
</tr>
<tr>
<td>5</td>
<td>$-64(n-1)(2n-1)$</td>
<td>$(2n-1)(256n^6-640n^3+320n^4+414n^3-493n^2+145n-1) + 32(n-1)(2n-1)H_n^{(2)}$</td>
</tr>
<tr>
<td>7</td>
<td>$-64(n-1)(2n-1)(24n^2-24n+1)$</td>
<td>$(2n-1)(6144n^8-21504n^7+24320n^6-1920n^5-18496n^4+17582n^3-7557n^2+1433n-1) + 32(n-1)(2n-1)(24n^2-24n+11)H_n^{(2)}$</td>
</tr>
</tbody>
</table>

Remark. For fixed odd $m \geq 9$, there exist similar formulas for (4.1) and (4.2). But the formulas will become larger and larger and there are no general formulas for these $m$‘s.

Proof. By using the Mathematica package Sigma due to Schneider [26], one can automatically discover and prove (4.1) and (4.2). For example, the steps to discover and prove (4.2) for $m = 5$ are as follows.

Define the following sum:

$$\text{ln[1]:=mySum}=\sum_{n=0}^{\infty}(4k-1)^n\left(\frac{1}{2}\right)^n\frac{(-1)^{n-k}k!}{(k-n)!}\frac{1}{(2^n-3)^n}$$

Compute the recurrence for this sum:

$$\text{ln[2]:=rec=GenerateRecurrence[mySum,n]}$$

$$\text{Out[2]}=(1+n)(1+2n)\text{SUM}[n] - (-1+n)(1-2n)\text{SUM}[1+n]$$

$$=\left(\frac{(-1+2n)(1+2n)(78-183n^2+109n^4)}{2(-1+n)2n(1+n)^2}\right) + \frac{n(-1+2n)^3(1+2n)^2(3+2n)(3+4n)^4(-\frac{1}{2})^n(-1+n)_{n}(-n)_{n}}{2(1+n)^3(1+4n)(1)(\frac{3}{2}-n)_{n}(\frac{3}{2}-n)_{n}}$$

$$- \frac{n(-1+2n)^3(1+4n)(1)(\frac{3}{2}-n)_{n}(\frac{3}{2}-n)_{n}}{2(1+n)(1+4n)(1)(\frac{3}{2}-n)_{n}(\frac{3}{2}-n)_{n}}$$

$$+ \left(\frac{n(-1+2n)^3(1+4n)(1)(\frac{3}{2}-n)_{n}(\frac{3}{2}-n)_{n}}{2(1+n)(1+4n)(1)(\frac{3}{2}-n)_{n}(\frac{3}{2}-n)_{n}}\right) - \frac{n(-1+2n)^3(1+2n)(3+4n)^4(-\frac{1}{2})^n(-1+n)_{n}(-n)_{n}}{2(1+n)(1+4n)(1)(\frac{3}{2}-n)_{n}(\frac{3}{2}-n)_{n}}$$

Now we solve this recurrence:

$$\text{ln[3]:=recSol= SolveRecurrence[rec,SUM[n]]}$$

$$\text{Out[3]}=\left\{0,\left(-1+n\right)\left(n-1+2n\right),\right\}$$

$$\left\{\frac{1}{4(-1+n)^2n^2}\left(-1+435n^2+670n^3+448n^4+128n^5\right) + 32(-1+n)(n-1+2n)\left(\frac{n}{\sum_{k=1}^{n}\frac{1}{k^{n-1}}}\right)\right\}$$

Finally, we combine the solutions to represent mySum:

$$\text{ln[4]:=FindLinearCombination[recSol,mySum,1]}$$

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The proof then follows from (4.7), (4.8) and Lemma 4.1.

Thus, we discover and prove (4.2) for $m = 5$. \hfill \Box

Proof of Theorem 1.3. Observe that

$$
\frac{(2j - 3)^2 - x^2}{(2j)^2 - x^2} = \left(\frac{2j - 3}{2j}\right)^2 - \frac{3(4j - 3)}{(2j)^4}x^2 + O(x^4),
$$

and

$$
\prod_{j=1}^{k}(a_j + b_jx^2) = \left(\prod_{j=1}^{k}a_j\right)\left(1 + x^2\sum_{j=1}^{k}b_j/a_j\right) + O(x^4).
$$

Hence, for $0 \leq k \leq (p + 1)/2$, we have

$$
\frac{(-1/2)^k(-1/2)^k}{(1 + \frac{p}{2})k(1 - \frac{p}{2})k} = \prod_{j=1}^{k}\frac{(2j - 3)^2 - p^2}{(2j)^2 - p^2}
$$

$$
\equiv \frac{(-1/2)^k}{k!^2}\left(1 + p^2\sum_{j=1}^{k}\left(\frac{1}{(2j)^2} - \frac{1}{(2j - 3)^2}\right)\right) \pmod{p^4}.
$$

Letting $n = (p + 1)/2$ in (4.1) and using (4.3), we get

$$
\sum_{k=0}^{(p+1)/2}(4k - 1)^m\frac{(-1/2)^k}{k!^4}\left(1 + p^2\sum_{j=1}^{k}\left(\frac{1}{(2j)^2} - \frac{1}{(2j - 3)^2}\right)\right)
$$

$$
\equiv f_m\left(\frac{p + 1}{2}\right) \pmod{p^4}. \hfill (4.4)
$$

Furthermore, it follows from (4.3) that

$$
\frac{(-1/2)^k(-1/2)^k}{(1 + \frac{p}{2})k(1 - \frac{p}{2})k} \equiv \frac{(-1/2)^k}{k!^2} \pmod{p^2}. \hfill (4.5)
$$

Letting $n = (p + 1)/2$ in (4.2) and noticing (4.5), we obtain

$$
\sum_{k=0}^{(p+1)/2}(4k - 1)^m\frac{(-1/2)^k}{k!^4}\sum_{j=1}^{k}\left(\frac{1}{(2j)^2} - \frac{1}{(2j - 3)^2}\right) \equiv g_m\left(\frac{p + 1}{2}\right) \pmod{p^2}. \hfill (4.6)
$$

Finally, combining (4.4) and (4.6) we are led to

$$
\sum_{k=0}^{(p+1)/2}(4k - 1)^m\frac{(-1/2)^k}{k!^4} \equiv f_m\left(\frac{p + 1}{2}\right) - p^2g_m\left(\frac{p + 1}{2}\right) \pmod{p^4}. \hfill (4.7)
$$

Note that

$$
H_{(p+1)/2}^{(2)} = \left(\frac{2}{p + 1}\right)^2 + H_{(p-1)/2}^{(2)} \equiv 4 \pmod{p}. \hfill (4.8)
$$

The proof then follows from (4.7), (4.8) and Lemma 4.1. \hfill \Box
5. Two open problems

We end the paper with the following two conjectures, which are generalizations of Theorems 1.2 and 1.3. Note that there are similar unsolved conjectures in [4].

**Conjecture 5.1.** For any odd positive integer $m$, there exists an integer $c_m$ such that, for any odd prime $p$ and positive integer $r$, there hold

\[
\sum_{k=0}^{\frac{p^r+1}{2}} (-1)^k (4k-1)^m \frac{(-\frac{1}{2})^k}{k!^3} \equiv c_m p^r (-1)^{\frac{(p-1)r}{2}} \pmod{p^r+2},
\]

\[
\sum_{k=0}^{p^r-1} (-1)^k (4k-1)^m \frac{(-\frac{1}{2})^k}{k!^3} \equiv c_m p^r (-1)^{\frac{(p-1)r}{2}} \pmod{p^r+2}.
\]

In particular, we have $c_1 = -1, c_3 = 3, c_5 = 23, c_7 = -5, c_9 = 1647, and c_{11} = -96973.$

**Conjecture 5.2.** For any odd positive integer $m$, there exists an integer $d_m$ such that, for any odd prime $p$ and positive integer $r$, there hold

\[
\sum_{k=0}^{\frac{p^r+1}{2}} (4k-1)^m \frac{(-\frac{1}{2})^k}{k!^4} \equiv d_m p^r \pmod{p^r+3},
\]

\[
\sum_{k=0}^{p^r-1} (4k-1)^m \frac{(-\frac{1}{2})^k}{k!^4} \equiv d_m p^r \pmod{p^r+3}.
\]

In particular, we have $d_1 = d_3 = 0, d_5 = 16, d_7 = 80, d_9 = 192, d_{11} = 640, d_{13} = -3472,$ and $d_{15} = 138480.$

Note that, Conjecture 5.1 is true for $m = 1$ by [9, Theorem 1.3], and Conjecture 5.2 is also true for $m = 1$ by [15, Theorem 1.1].

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**References**


[23] F. Morley, Note on the congruence $2^{4n} \equiv (-1)^n(2n)!/(n!)^2$, where $2n + 1$ is a prime, Ann. of Math. 9 (1894/95), 168–170.


