ON THE LEAST COMMON MULTIPLE OF $q$-BINOMIAL COEFFICIENTS

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Abstract

We first prove the following identity

$$\text{lcm}\left(\begin{bmatrix} n \\ 0 \end{bmatrix}_q, \begin{bmatrix} n \\ 1 \end{bmatrix}_q, \ldots, \begin{bmatrix} n \\ n \end{bmatrix}_q \right) = \frac{\text{lcm}(\begin{bmatrix} 1 \\ q \end{bmatrix}, \begin{bmatrix} 2 \\ q \end{bmatrix}, \ldots, \begin{bmatrix} n+1 \\ q \end{bmatrix})}{\begin{bmatrix} n+1 \\ q \end{bmatrix}},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the $q$-binomial coefficient and $[n]_q = \frac{1-q^n}{1-q}$. Then we show that this identity is indeed a $q$-analogue of that of Farhi [Amer. Math. Monthly 116 (2009), 836–839].

1. Introduction

An equivalent form of the prime number theorem states that $\log \text{lcm}(1,2,\ldots,n) \sim n$ as $n \to \infty$ (see, for example, [4]). Nair [7] gave a nice proof for the well-known estimate $\text{lcm}\{1,2,\ldots,n\} \geq 2^{n-1}$, while Hanson [3] already obtained $\text{lcm}\{1,2,\ldots,n\} \leq 3^n$. Recently, Farhi [1] established the following interesting result.

**Theorem 1 (Farhi)** For any positive integer $n$, there holds

$$\text{lcm}\left(\begin{bmatrix} n \\ 0 \end{bmatrix}, \begin{bmatrix} n \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} n \\ n \end{bmatrix} \right) = \frac{\text{lcm}(1,2,\ldots,n+1)}{n+1}. \quad (1)$$

As an application, Farhi shows that the inequality $\text{lcm}\{1,2,\ldots,n\} \geq 2^{n-1}$ follows immediately from (1).

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The purpose of this note is to give a $q$-analogue of (1) by using cyclotomic polynomials. Recall that a natural $q$-analogue of the nonnegative integer $n$ is given by $[n]_q = \frac{1-q^n}{1-q}$. The corresponding $q$-factorial is $[n]_q! = \prod_{k=1}^n [k]_q$ and the $q$-binomial coefficient $\binom{M}{N}_q$ is defined as

$$\binom{M}{N}_q = \begin{cases} \frac{[M]_q!}{[N]_q! [M-N]_q!}, & \text{if } 0 \leq N \leq M, \\ 0, & \text{otherwise.} \end{cases}$$

Let $lcm$ also denote the least common multiple of a sequence of polynomials in $\mathbb{Z}[q]$. Our main results can be stated as follows:

**Theorem 2** For any positive integer $n$, there holds

$$lcm \left( \binom{n}{0}_q, \binom{n}{1}_q, \ldots, \binom{n}{n}_q \right) = \frac{lcm([1]_q,[2]_q,\ldots,[n+1]_q)}{[n+1]_q}. \quad (2)$$

**Theorem 3** The identity (2) is a $q$-analogue of Farhi’s identity (1), i.e.,

$$\lim_{q \to 1} lcm \left( \binom{n}{0}_q, \binom{n}{1}_q, \ldots, \binom{n}{n}_q \right) = lcm \left( \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n} \right), \quad (3)$$

and

$$\lim_{q \to 1} \frac{lcm([1]_q,[2]_q,\ldots,[n+1]_q)}{[n+1]_q} = \frac{lcm(1,2,\ldots,n+1)}{n+1}. \quad (4)$$

Although it is clear that

$$\lim_{q \to 1} \binom{n}{k}_q = \binom{n}{k},$$

the identities (3) and (4) are not trivial. For example, we have

$$4 = \lim_{q \to 1} lcm \left( 1 + q, 1 + q^2 \right) \neq lcm \left( \lim_{q \to 1} (1 + q), \lim_{q \to 1} (1 + q^2) \right) = 2.$$

**2. Proof of Theorem 2**

Let $\Phi_n(x)$ be the $n$-th cyclotomic polynomial. The following easily proved result can be found in [5, (10)] and [2].
Lemma 4 The $q$-binomial coefficient $\left[\begin{array}{c} n \\ k \end{array}\right]_q$ can be factorized into

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = \prod_d \Phi_d(q),$$

where the product is over all positive integers $d \leq n$ such that $\lfloor k/d \rfloor + \lfloor (n-k)/d \rfloor < \lfloor n/d \rfloor$.

Lemma 5 Let $n$ and $d$ be two positive integers with $n \geq d$. Then there exists at least one positive integer $k$ such that

$$\lfloor k/d \rfloor + \lfloor (n-k)/d \rfloor < \lfloor n/d \rfloor \quad (5)$$

if and only if $d$ does not divide $n+1$.

Proof. Suppose that (5) holds for some positive integer $k$. Let

$$k \equiv a \pmod{d}, \quad (n-k) \equiv b \pmod{d}$$

for some $1 \leq a, b \leq d - 1$. Then $n \equiv a + b \pmod{d}$ and $d \leq a + b \leq 2d - 2$. Namely, $n + 1 \equiv a + b + 1 \not\equiv 0 \pmod{d}$. Conversely, suppose that $n + 1 \equiv c \pmod{d}$ for some $1 \leq c \leq d - 1$. Then $k = c$ satisfies (5). This completes the proof.

Proof of Theorem 2. By Lemma 4, we have

$$\text{lcm} \left( \left[\begin{array}{c} n \\ 0 \end{array}\right]_q, \left[\begin{array}{c} n \\ 1 \end{array}\right]_q, \ldots, \left[\begin{array}{c} n \\ n \end{array}\right]_q \right) = \prod_d \Phi_d(q), \quad (6)$$

where the product is over all positive integers $d \leq n$ such that for some $k$ ($1 \leq k \leq n$) there holds $\lfloor k/d \rfloor + \lfloor (n-k)/d \rfloor < \lfloor n/d \rfloor$. On the other hand, since

$$\left[\begin{array}{c} k \\ q \end{array}\right] = \frac{q^k - 1}{q - 1} = \prod_{d|k, d > 1} \Phi_d(q),$$

we have

$$\text{lcm}([1]_q, [2]_q, \ldots, [n+1]_q) = \prod_{d \leq n, d|(n+1)} \Phi_d(q). \quad (7)$$

By Lemma 5, one sees that the right-hand sides of (6) and (7) are equal. This proves the theorem.
3. Proof of Theorem 3

We need the following property.

**Lemma 6** For any positive integer \( n \), there holds

\[
\Phi_n(1) = \begin{cases} p, & \text{if } n = p^r \text{ is a prime power;} \\ 1, & \text{otherwise.} \end{cases}
\]

**Proof.** See for example [6, p. 160]. \( \Box \)

In view of (6), we have

\[
\lim_{q \to 1} \operatorname{lcm} \left( n^0_q, n^1_q, \ldots, n^n_q \right) = \prod_{d} \Phi_d(1),
\]

where the product is over all positive integers \( d \leq n \) such that for some \( k \) \((1 \leq k \leq n)\) there holds \( \lfloor k/d \rfloor + \lfloor (n - k)/d \rfloor < \lfloor n/d \rfloor \). By Lemma 6, the right-hand side of (8) can be written as

\[
\prod_{\text{primes } p \leq n} p^{\sum_{r=1}^{\infty} \max_{0 \leq k \leq n} \{ [n/p^r] - [k/p^r] - [(n-k)/p^r] \}}.
\]

We now claim that

\[
\sum_{r=1}^{\infty} \max_{0 \leq k \leq n} \{ [n/p^r] - [k/p^r] - [(n-k)/p^r] \} = \max_{0 \leq k \leq n} \sum_{r=1}^{\infty} \{ [n/p^r] - [k/p^r] - [(n-k)/p^r] \}.
\]

Let \( n = \sum_{i=0}^{M} a_ip^i \), where \( 0 \leq a_0, a_1, \ldots, a_M \leq p - 1 \) and \( a_M \neq 0 \). By Lemma 5, the left-hand side of (10) (denoted \( \text{LHS}(10) \)) is equal to the number of \( r \)'s such that \( p^r \leq n \) and \( p^r \nmid n + 1 \). It follows that

\[
\text{LHS}(10) = \begin{cases} 0, & \text{if } n = p^{M+1} - 1, \\ M - \min\{i: a_i \neq p - 1\}, & \text{otherwise.} \end{cases}
\]

It is clear that the right-hand side of (10) is less than or equal to \( \text{LHS}(10) \). If \( n = p^{M+1} - 1 \), then both sides of (10) are equal to 0. Assume that \( n \neq p^{M+1} - 1 \) and \( i_0 = \min\{i: a_i \neq p - 1\} \). Taking \( k = p^M - 1 \), we have

\[
[n/p^r] - [k/p^r] - [(n-k)/p^r] = \begin{cases} 0, & \text{if } r = 1, \ldots, i_0, \\ 1, & \text{if } r = i_0 + 1, \ldots, M, \end{cases}
\]
and so
\[ \sum_{r=1}^{\infty} \left\lfloor \frac{n}{p^r} \right\rfloor - \left\lfloor \frac{k}{p^r} \right\rfloor - \left\lfloor \frac{(n-k)}{p^r} \right\rfloor = M - i_0. \]
Thus (10) holds. Namely, the expression (9) is equal to
\[ \prod_{\text{primes } p \leq n} p^{\max_{0 \leq i \leq n} \sum_{r=1}^{\infty} (\left\lfloor \frac{n}{p^r} \right\rfloor - \left\lfloor \frac{k}{p^r} \right\rfloor - \left\lfloor \frac{(n-k)}{p^r} \right\rfloor)} = \text{lcm} \left( \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n} \right). \]
This proves (3). To prove (4), we apply (7) to get
\[ \lim_{q \to 1} \frac{\text{lcm}(1_q, 2_q, \ldots, n+1_q)}{n+1_q} = \prod_{d \leq n, d | (n+1)} \Phi_d(1), \]
which, by Lemma 6, is clearly equal to
\[ \frac{\text{lcm}(1, 2, \ldots, n+1)}{n+1}. \]

Finally, we mention that (10) has the following interesting conclusion.

**Corollary 7** Let \( p \) be a prime number and let \( k_1, k_2, \ldots, k_m \leq n, \) \( r_1 < r_2 < \cdots < r_m \) be positive integers such that
\[ \left\lfloor \frac{n}{p^{r_i}} \right\rfloor - \left\lfloor \frac{k_i}{p^{r_i}} \right\rfloor - \left\lfloor \frac{(n-k_i)}{p^{r_i}} \right\rfloor = 1 \quad \text{for } i = 1, 2, \ldots, m. \]
Then there exists a positive integer \( k \leq n \) such that
\[ \left\lfloor \frac{n}{p^{r_i}} \right\rfloor - \left\lfloor \frac{k}{p^{r_i}} \right\rfloor - \left\lfloor \frac{(n-k)}{p^{r_i}} \right\rfloor = 1 \quad \text{for } i = 1, 2, \ldots, m. \]

**References**