New q-supercongruences derived from a quadratic transformation by Rahman

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Abstract. Employing the method of "creative microscoping" introduced by the first author and Wadim Zudilin, we deduce two q-supercongruences from a quadratic transformation by Rahman. These two results are generalizations of two q-congruences implicitly contained in a paper by Liu and Wang. By specializing $q \to 1$ in one of our q-supercongruences, we obtain the result: for primes $p \equiv 3 \pmod 8$,

$$\sum_{k=0}^{p-1} (6k+1) \frac{(\frac{1}{4})_k (\frac{3}{8})_k^2 (\frac{1}{2})_k^3}{(1)_k^4 (\frac{3}{4})_k^2} \equiv 0 \pmod{p^4},$$

where $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for $n \geqslant 1$. Meanwhile, by specializing $q \to -1$ in the same q-supercongruence, we get the result: for primes $p \equiv 3 \pmod 4$,

$$\sum_{k=0}^{p-1} (6k+1) \frac{(\frac{1}{2})_k^3 (\frac{1}{4})_k}{k!^4 4^k} \equiv 0 \pmod{p^4},$$

which was originally conjectured by Bing He and first confirmed by Chuanan Wei.

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1. Introduction

For any odd prime p and non-negative integer n, the *p-adic Gamma function* $\Gamma_p(n)$ may be defined as: $\Gamma_p(0) = 1$,

$$\Gamma_p(n) = (-1)^n \prod_{\substack{1<j<n \\ p\nmid j}} j, \quad n = 1, 2, \ldots.$$

This function can be uniquely extended to a continuous function: for any p-adic integer x,

$$\Gamma_p(x) = \lim_{x_n \to x} \Gamma_p(x_n),$$

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where x_n ranges over all positive integer sequences p-adically approximating x (see [9]). For some main properties of the p-adic Gamma function, the reader is referred to [8].

In 2017, motivated by the supercongruences in [7, 8], He [4] proved the following supercongruence:

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{(\frac{1}{2})_k^3(\frac{1}{4})_k}{k!^4 4^k}$$
$$\equiv \begin{cases} (-1)^{(p+3)/4}p\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})^2 \pmod{p^2}, & \text{if } p\equiv 1 \pmod 4, \\ 0 \pmod{p^2}, & \text{if } p\equiv 3 \pmod 4. \end{cases} \tag{1.1}$$

Here and in what follows, $(x)_n=x(x+1)\cdots(x+n-1)$ is the *Pochhammer symbol*. It should be remarked that the sign $(-1)^{(p+3)/4}$ was neglected by He. Shortly afterwards, Liu [5] further showed that (1.1) holds modulo p^3. It is easy to see that the left-hand side of (1.1) can be summed over k up to $p-1$, since $(1/2)_k/k!$ is congruent to 0 modulo p for k in the range $(p+1)/2\leqslant k\leqslant p-1$. He [4] further proposed the following conjecture: for any prime $p\equiv 3 \pmod 4$,

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac{1}{2})_k^3(\frac{1}{4})_k}{k!^4 4^k}\equiv 0 \pmod{p^4}. \tag{1.2}$$

In recent years, q-analogs of supercongruences have been investigated by quite a few authors. For example, employing the method of "creative microscoping" introduced by the first author and Zudilin [3], Liu and Wang [6] gave a q-analog of (1.1) modulo p^3: for all positive odd integers n, modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{N}[6k+1]\frac{(q;q^2)_k^3(q;q^4)_k}{(q^2;q^2)_k(q^4;q^4)_k^3}q^{k^2+k}\equiv \begin{cases} \dfrac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}[n]q^{(1-n)/4}, & \text{if } n\equiv 1 \pmod 4, \\ 0, & \text{if } n\equiv 3 \pmod 4, \end{cases} \tag{1.3}$$

where $N=n-1$ or $(n-1)/2$. Here and in what follows, the *q-shifted factorial* is defined as

$$(a;q)_n=\begin{cases} (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n=1,2,\dots, \\ 1, & \text{if } n=0. \end{cases}$$

For simplicity, we will also adopt the compact notation

$$(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$$

for $n=0,1,2,\dots,$ or $n=\infty$. The *q-integer* is defined by $[n]=(1-q^n)/(1-q)$, and $\Phi_n(q)$ stands for the n-th *cyclotomic polynomial*, i.e.,

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

with ζ being an n-th primitive root of unity. It is well known that $\Phi_n(q^2)=\Phi_n(q)\Phi_n(-q)$ for odd n. Moreover, for any two rational functions $A(q)$, $B(q)$, and a polynomial $P(q)$ with integer coefficients, the q-congruence $A(q)\equiv B(q) \pmod{P(q)}$ means that $P(q)$ divides the numerator of the reduced form of $A(q)-B(q)$ in the polynomial ring $\mathbb{Z}[q]$.

Liu and Wang [6] also provided a parametric generalization of (1.3) as follows: for positive odd n, modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{N}[6k+1]\frac{(q,e,q^3/e;q^4)_k(q;q^2)_k^3}{(q^2,e,q^3/e;q^2)_k(q^4;q^4)_k^3}q^{2k}$$
$$\equiv\begin{cases}\dfrac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}[n]q^{(1-n)/4}\displaystyle\sum_{k=0}^{(n-1)/4}\frac{(q;q^4)_k^3q^{4k}}{(q^4,eq^2,q^5/e;q^4)_k}, & \text{if } n\equiv 1 \pmod 4,\\ 0, & \text{if } n\equiv 3 \pmod 4,\end{cases}\tag{1.4}$$

where $N=(n-1)/2$ or $n-1$. It is clear that (1.3) follows from (1.4) by letting $e\to 0$ or ∞. In the same year, Wei [11] completely confirmed the supercongruence (1.2) by establishing the following generalization of the second case of (1.3): for any positive integer $n\equiv 3 \pmod 4$,

$$\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^2)_k^3(q;q^4)_k}{(q^2;q^2)_k(q^4;q^4)_k^3}q^{k^2+k}\equiv 0 \pmod{[n]\Phi_n(q)^3},\tag{1.5}$$

It is easy to see that (1.2) follows from (1.5) by taking $n=p$ and $q\to 1$. Wei [11] also presented a generalization of the first case of (1.3): for any positive integer $n\equiv 1 \pmod 4$,

$$\sum_{k=0}^{N}[6k+1]\frac{(q;q^2)_k^3(q;q^4)_k}{(q^2;q^2)_k(q^4;q^4)_k^3}q^{k^2+k}$$
$$\equiv\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}[n]q^{(1-n)/4}\left\{1-[n]^2\sum_{k=1}^{(n-1)/4}\frac{q^{4k}}{[4k]^2}\right\} \pmod{[n]\Phi_n(q)^3},\tag{1.6}$$

where $N=(n-1)/2$ or $n-1$.

Letting $q\mapsto q^2$ and $e=q^3$ in the second part of (1.4), we can obtain the following result: for $n\equiv 3 \pmod 4$,

$$\sum_{k=0}^{n-1}[6k+1]_{q^2}\frac{(q^2;q^8)_k(q^3;q^8)_k^2(q^2;q^4)_k^3}{(q^4;q^4)_k(q^3;q^4)_k^2(q^8;q^8)_k^3}q^{4k}$$
$$\equiv 0\begin{cases}\pmod{\Phi_n(q^2)^3}, & \text{if } n\equiv 3 \pmod 8,\\ \pmod{\Phi_n(q^2)\Phi_n(-q)^2}, & \text{if } n\equiv 7 \pmod 8.\end{cases}\tag{1.7}$$

This is because the denominator of the reduced form of $(q^3;q^8)_{n-1}/(q^3;q^4)_{n-1}$ contains the factor $\Phi_n(q)$ but is coprime with $\Phi_n(-q)$ when $n\equiv 7 \pmod 8$.

The first aim of this note is to give the following generalization of (1.7).

Theorem 1.1. *Let $n \equiv 3 \pmod 4$ be a positive integer. Then*

$$\sum_{k=0}^{n-1}[6k+1]_{q^2}\frac{(q^2;q^8)_k(q^3;q^8)_k^2(q^2;q^4)_k^3}{(q^4;q^4)_k(q^3;q^4)_k^2(q^8;q^8)_k^3}q^{4k} \equiv 0\begin{cases}\pmod{\Phi_n(q^2)^4}, & \text{if } n\equiv 3 \pmod 8,\\ \pmod{\Phi_n(q^2)^2\Phi_n(-q)^2}, & \text{if } n\equiv 7 \pmod 8.\end{cases} \tag{1.8}$$

In particular, letting $n = p$ be a prime and taking the limits as $q \to 1$ in (1.8), we obtain the following supercongruence.

Corollary 1.2. *Let $p \equiv 3 \pmod 4$ be a prime. Then*

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac14)_k(\frac38)_k^2(\frac12)_k^3}{(1)_k^4(\frac34)_k^2} \equiv 0\begin{cases}\pmod{p^4}, & \text{if } p\equiv 3 \pmod 8,\\ \pmod{p^2}, & \text{if } p\equiv 7 \pmod 8.\end{cases} \tag{1.9}$$

On the other hand, letting $n = p$ be a prime and taking $q \to -1$ in (1.8), we recover He's conjecture (1.2). Namely, the q-supercongruence (1.8) can also be considered as a new q-analogue of (1.2).

Letting $q \mapsto q^2$ and $e = q^3$ in the first part of (1.4), we may get the following conclusion: for $n \equiv 1 \pmod 4$,

$$\sum_{k=0}^{n-1}[6k+1]_{q^2}\frac{(q^2;q^8)_k(q^3;q^8)_k^2(q^2;q^4)_k^3}{(q^4;q^4)_k(q^3;q^4)_k^2(q^8;q^8)_k^3}q^{4k} \equiv 0 \pmod{\Phi_n(q^2)}. \tag{1.10}$$

The second aim of this note is to give the following generalization of (1.10).

Theorem 1.3. *Let $n > 1$ an integer with $n \equiv 1 \pmod 4$. Then, modulo $\Phi_n(q^2)^3$,*

$$\sum_{k=0}^{n-1}[6k+1]_{q^2}\frac{(q^2;q^8)_k(q^3;q^8)_k^2(q^2;q^4)_k^3}{(q^4;q^4)_k(q^3;q^4)_k^2(q^8;q^8)_k^3}q^{4k} \equiv \frac{(q^{10},q^4,q^5,q^5;q^8)_{(n-1)/4}}{(q^2,q^8,q^7,q^7;q^8)_{(n-1)/4}}\left\{1+[2n]^2q^{(1-n)/2}\sum_{k=1}^{(n-1)/4}\left(\frac{q^{8k-3}}{[8k-3]^2}-\frac{q^{8k-6}}{[8k-6]^2}\right)\right\}. \tag{1.11}$$

For n prime and $q \to 1$ in (1.11), we obtain the following supercongruence.

Corollary 1.4. *Let $p \equiv 1 \pmod 4$ be a prime. Then*

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac14)_k(\frac38)_k^2(\frac12)_k^3}{(1)_k^4(\frac34)_k^2} \equiv p\frac{(\frac12)_{(p-1)/4}(\frac58)_{(p-1)/4}^2}{(1)_{(p-1)/4}(\frac78)_{(p-1)/4}^2}\left\{1+4p^2\sum_{k=1}^{(p-1)/4}\left(\frac{1}{(8k-3)^2}-\frac{1}{(8k-6)^2}\right)\right\}.$$

Similarly, for n prime and $q \to -1$ in (1.11), we get the following supercongruence, which also follows from (1.3) and coincides with Liu's generalization of (1.1) modulo p^3 by properties of the p-adic Gamma function.

Corollary 1.5. *Let $p \equiv 1 \pmod 4$ be a prime. Then*

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac{1}{2})_k^3(\frac{1}{4})_k}{k!^4 4^k} \equiv p\frac{(\frac{1}{2})_{(p-1)/4}}{(1)_{(p-1)/4}} \pmod{p^3}.$$

Since $(q^{10};q^8)_{(n-1)/4} \equiv (q^5;q^8)_{(n-1)/4} \equiv 0 \pmod{\Phi_n(q)}$ and $(q^2,q^8,q^7;q^8)_{(n-1)/4}$ is coprime with $\Phi_n(q)$ for $n \equiv 5 \pmod 8$, from Theorem 1.3 we immediately deduce the following q-supercongruence.

Corollary 1.6. *Let $n \equiv 5 \pmod 8$ be a positive integer. Then*

$$\sum_{k=0}^{n-1}[6k+1]_{q^2}\frac{(q^2;q^8)_k(q^3;q^8)_k^2(q^2;q^4)_k^3}{(q^4;q^4)_k(q^3;q^4)_k^2(q^8;q^8)_k^3}q^{4k} \equiv 0 \pmod{\Phi_n(q)^3}. \tag{1.12}$$

The rest of the note is organized as follows. We shall prove Theorems 1.1 and 1.3 in Sections 2 and 3, respectively, by making use of the "creative microscoping" method introduced in [3], a quadratic transformation formula of Rahman, and the Chinese remainder theorem for coprime polynomials. Finally, in Section 4, we give some concluding remarks on our main results.

2. Proof of Theorem 1.1

Recall that the *basic hypergeometric series* ${}_{r+1}\phi_r$ involving $r+1$ upper parameters $a_1,\dots,a_{r+1}$, r lower parameters $b_1,\dots,b_r$, base q, and argument z is given by

$${}_{r+1}\phi_r\left[\begin{matrix}a_1,a_2,\dots,a_{r+1}\\ b_1,b_2,\dots,b_r\end{matrix};q,z\right]=\sum_{k=0}^{\infty}\frac{(a_1,a_2,\dots,a_{r+1};q)_k z^k}{(q,b_1,\dots,b_r;q)_k}.$$

Then a quadratic transformation formula of Rahman [1, (3.8.13)] can be stated as follows:

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(1-aq^{3k})(a,d,aq/d;q^2)_k(b,c,aq/bc;q)_k}{(1-a)(q,aq/d,d;q)_k(aq^2/b,aq^2/c,bcq;q^2)_k}q^k\\ &\quad=\frac{(aq^2,bq,cq,aq^2/bc;q^2)_\infty}{(q,aq^2/b,aq^2/c,bcq;q^2)_\infty}\,{}_3\phi_2\left[\begin{matrix}b,\ \ c,\ \ aq/bc\\ dq,\ \ aq^2/d\end{matrix};q^2,q^2\right].\end{aligned}\tag{2.1}$$

We first give the following q-identity.

Lemma 2.1. *Let $n \equiv 3 \pmod 4$ be a positive integer. Let e be an indeterminate. Then*

$$\sum_{k=0}^{n-1}\frac{1-q^{12k+2-6n}}{1-q^{2-6n}}\frac{(q^{2-6n},eq^3,q^{3-6n}/e;q^8)_k(q^2/a,q^2/b,abq^{2-6n};q^4)_k}{(q^4,q^{3-6n}/e,eq^3;q^4)_k(aq^{8-6n},bq^{8-6n},q^8/ab;q^8)_k}q^{4k}=0. \tag{2.2}$$

Proof. Setting $q \mapsto q^4$, $a = q^{2-6n}$, $b = q^2/a$, $c = q^2/b$ and $d = eq^3$ in (2.1), then for $n \equiv 3 \pmod 4$, the left-hand side of (2.1) is equal to

$$\frac{(q^{10-6n}, q^6/a, q^6/b, abq^{6-6n}; q^8)_\infty}{(q^4, aq^{8-2n}, bq^{8-2n}, q^8/ab; q^4)_\infty} {}_3\phi_2\left[\begin{matrix} q^2/a, & q^2/b, & abq^{2-6n} \\ & eq^7, & q^{7-6n}/e \end{matrix}; q^8, q^8\right].$$

It is easy to check that $(q^{10-6n}; q^8)_\infty = 0$ for $n \equiv 3 \pmod 4$, and so the left-hand of the (2.2) is equal to 0. This completes the proof. □

We now give a parametric version of Theorem 1.1.

Theorem 2.2. *Let $n \equiv 3 \pmod 4$ be a positive integer. Then, modulo $\Phi_n(q)(a-q^{2n})(b-q^{2n})(1-abq^{2n})$,*

$$\sum_{k=0}^{n-1} [6k+1]_{q^2} \frac{(q^2, eq^3, q^3/e; q^8)_k (q^2/a, q^2/b, abq^2; q^4)_k}{(q^4, q^3/e, eq^3; q^4)_k (aq^8, bq^8, q^8/ab; q^8)_k} q^{4k} \equiv 0. \tag{2.3}$$

Proof. For $a = q^{2n}$ or $a = q^{-2n}/b$, the left-hand of the (2.3) can be written as

$$\sum_{k=0}^{n-1} [6k+1]_{q^2} \frac{(q^2, eq^3, q^3/e; q^8)_k (q^{2-2n}, q^2/b, bq^{2+2n}; q^4)_k}{(q^4, q^3/e, eq^3; q^4)_k (q^{8+2n}, bq^8, q^{8-2n}/b; q^8)_k} q^{4k} \tag{2.4}$$

Letting $q \mapsto q^4$, $a = q^2$, $b = q^{2-2n}$, $c = q^2/b$ and $d = eq^3$ in (2.1), we see that (2.4) is equal to

$$\frac{(q^{10}, q^{6-2n}, q^6/b, bq^{6+2n}; q^8)_\infty}{(q^4, q^{8+2n}, bq^8, q^{8-2n}/b; q^8)_\infty} {}_3\phi_2\left[\begin{matrix} q^{2-2n}, & q^2/b, & bq^{2+2n} \\ & eq^7, & q^7/e \end{matrix}; q^8, q^8\right] = 0, \tag{2.5}$$

where we have used the fact that $(q^{6-2n}, q^8)_\infty = 0$ for $n \equiv 3 \pmod 4$. This establishes the truth of (2.3) modulo $a - q^{2n}$ and $1 - abq^{2n}$. Noticing that the left-hand side of (2.3) is symmetric in a and b, we see that the q-congruence (2.3) also holds modulo $(b - q^{2n})$. The proof then follows from the fact that $a - q^{2n}$, $b - q^{2n}$, and $1 - abq^{2n}$ are pairwise coprime polynomials in q. □

Proof of Theorem 1.1. Since $1 - q^{2n}$ contains the factor $\Phi_n(q^2)$, which is coprime with $(q^8; q^8)_{n-1}$, taking $a = b = 1$ in (2.3), we obtain the following q-supercongruence: for $n \equiv 3 \pmod 4$,

$$\sum_{k=0}^{n-1} [6k+1]_{q^2} \frac{(q^2, eq^3, q^3/e; q^8)_k (q^2; q^4)_k^3}{(q^4, q^3/e, eq^3; q^4)_k (q^8; q^8)_k^3} q^{4k} \equiv 0 \pmod{\Phi_n(q^2)^4}. \tag{2.6}$$

If $n \equiv 3 \pmod 8$, then the denominator of the reduced form of $(q^3; q^8)_k/(q^3; q^4)_k$ is coprime with $\Phi_n(q^2)$ for $0 \leqslant k \leqslant n-1$. If $n \equiv 7 \pmod 8$, then the denominator of the reduced form of $(q^3; q^8)_k/(q^3; q^4)_k$ is coprime with $\Phi_n(-q)$ for $0 \leqslant k \leqslant n-1$, but contains the factor $\Phi_n(q)$ (not $\Phi_n(q)^2$ of course) for $k = (n+1)/4$. Hence, letting $e = 1$ in (2.6), we get the desired q-supercongruence (1.8). □

3. Proof of Theorem 1.3

Like before, we first establish a parametric generalization of Theorem 1.3.

Lemma 3.1. *Let $n > 1$ be an integer with $n \equiv 1 \pmod 4$. Let a and b be indeterminates. Then, modulo $(1-abq^{2n})(a-q^{2n})(b-q^{2n})$,*

$$\sum_{k=0}^{n-1}[6k+1]_{q^2}\frac{(q^2;q^8)_k(q^3;q^8)_k^2(q^2/a,q^2/b,abq^2;q^4)_k}{(q^4;q^4)_k(q^3;q^4)_k^2(aq^8,bq^8,q^8/ab;q^8)_k}q^{4k}$$
$$\equiv \frac{(q^{10},q^4,bq^5,q^{5-2n}/b;q^8)_{(n-1)/4}}{(bq^2,bq^8,q^7,q^{3-2n};q^8)_{(n-1)/4}}q^{(1-n)/2}b^{(n-1)/4}\frac{(b-q^{2n})(ab^2-a^2b-1+abq^{2n})}{(a-b)(1-ab^2)}$$
$$+\frac{(q^{10},q^4,aq^5,q^{5-2n}/a;q^8)_{(n-1)/4}}{(aq^2,aq^8,q^7,q^{3-2n};q^8)_{(n-1)/4}}q^{(1-n)/2}a^{(n-1)/4}\frac{(a-q^{2n})(1-abq^{2n})}{(a-b)(1-ab^2)} \tag{3.1}$$

Proof. For $a=q^{2n}$ or $a=q^{-2n}/b$, the left-hand side of (3.1) can be written as

$$\sum_{k=0}^{n-1}[6k+1]_{q^2}\frac{(q^2;q^8)_k(q^3;q^8)_k^2(q^{2-2n},q^2/b,bq^{2+2n};q^4)_k}{(q^4;q^4)_k(q^3;q^4)_k^2(q^{8+2n},bq^8,q^{8-2n}/b;q^8)_k}q^{4k} \tag{3.2}$$

In view of (2.1), setting $q\mapsto q^4$, $a=q^2$, $b=q^{2-2n}$, $c=q^2/b$ and $d=q^3$, we see that (3.2) is equal to

$$\frac{(q^{10},q^{6-2n},q^6/b,bq^{6+2n};q^8)_\infty}{(q^4,q^{8+2n},bq^8,q^{8-2n}/b;q^8)_\infty}\,{}_3\phi_2\left[\begin{matrix}q^{2-2n}, & q^2/b, & bq^{2+2n}\\ & q^7, & q^7\end{matrix};q^8,q^8\right]$$
$$=\frac{(q^{10},q^{6-2n},bq^5,q^{5-2n}/b;q^8)_{(n-1)/4}}{(q^{8-2n}/b,bq^8,q^7,q^{3-2n};q^8)_{(n-1)/4}}$$
$$=\frac{(q^{10},q^4,bq^5,q^{5-2n}/b;q^8)_{(n-1)/4}}{(bq^2,bq^8,q^7,q^{3-2n};q^8)_{(n-1)/4}}q^{(1-n)/2}b^{(n-1)/4},$$

where we have utilized the q-Pfaff–Saalschütz summation (see [1, II.12])

$${}_3\phi_2\left[\begin{matrix}a, & b, & q^{-n}\\ & c, & abc^{-1}q^{1-n}\end{matrix};q,q\right]=\frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n}$$

with the parameter replacements $q\mapsto q^8$, $a=q^2/b$, $b=bq^{2+2n}$, $c=q^7$, and $n=(n-1)/4$.

Since the polynomials $1-abq^{2n}$ and $a-q^{2n}$ are comprime with each other, we immediately get the following q-congruence: modulo $(1-abq^{2n})(a-q^{2n})$,

$$\sum_{k=0}^{n-1}[6k+1]_{q^2}\frac{(q^2;q^8)_k(q^3;q^8)_k^2(q^2/a,q^2/b,abq^2;q^4)_k}{(q^4;q^4)_k(q^3;q^4)_k^2(aq^8,bq^8,q^8/ab;q^8)_k}q^{4k}$$
$$\equiv\frac{(q^{10},q^4,bq^5,q^{5-2n}/b;q^8)_{(n-1)/4}}{(bq^2,bq^8,q^7,q^{3-2n};q^8)_{(n-1)/4}}q^{(1-n)/2}b^{(n-1)/4}.$$

Interchanging the indeterminates a and b in the above q-congruence, we obtain the following result: modulo $b-q^{2n}$,

$$\sum_{k=0}^{n-1}[6k+1]_{q^2}\frac{(q^2;q^8)_k(q^3;q^8)_k^2(q^2/a,q^2/b,abq^2;q^4)_k}{(q^4;q^4)_k(q^3;q^4)_k^2(aq^8,bq^8,q^8/ab;q^8)_k}q^{4k}$$
$$\equiv\frac{(q^{10},q^4,aq^5,q^{5-2n}/a;q^8)_{(n-1)/4}}{(aq^2,aq^8,q^7,q^{3-2n};q^8)_{(n-1)/4}}q^{(1-n)/2}a^{(n-1)/4}.$$

Moreover, we can easily check that

$$\frac{(b-q^{2n})(ab^2-a^2b-1+abq^{2n})}{(a-b)(1-ab^2)}\equiv 1 \pmod{(1-abq^{2n})(a-q^{2n})}, \tag{3.3}$$
$$\frac{(a-q^{2n})(1-abq^{2n})}{(a-b)(1-ab^2)}\equiv 1 \pmod{b-q^{2n}}. \tag{3.4}$$

Since $(a-q^{2n})(1-abq^{2n})$ and $b-q^{2n}$ are coprime polynomials in q, making use of the Chinese remainder theorem for coprime polynomials along with (3.3) and (3.4), we get the desired q-congruence. □

Proof of Theorem 1.3. It is easy to see that $1-q^{2n}$ contains the factor $\Phi_n(q^2)$. Putting $b=1$ in (3.1), and applying the following identity

$$(1-x)(1+a^2-a-ax)=(1-a)^2+(1-ax)(a-x) \tag{3.5}$$

we arrive at the following q-congruence: modulo $\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})$,

$$\sum_{k=0}^{n-1}[6k+1]_{q^2}\frac{(q^2;q^8)_k(q^3;q^8)_k^2(q^2/a,q^2/b,abq^2;q^4)_k}{(q^4;q^4)_k(q^3;q^4)_k^2(aq^8,bq^8,q^8/ab;q^8)_k}q^{4k}$$
$$\equiv\frac{(q^{10},q^4,q^5,q^{5-2n};q^8)_{(n-1)/4}}{(q^2,q^8,q^7,q^{3-2n};q^8)_{(n-1)/4}}q^{(1-n)/2}+q^{(1-n)/2}\frac{(1-aq^{2n})(a-q^{2n})}{(1-a)^2}$$
$$\times\left\{\frac{(q^{10},q^4,q^5,q^{5-2n};q^8)_{(n-1)/4}}{(q^2,q^8,q^7,q^{3-2n};q^8)_{(n-1)/4}}-\frac{(q^{10},q^4,aq^5,q^{5-2n}/a;q^8)_{(n-1)/4}}{(aq^2,aq^8,q^7,q^{3-2n};q^8)_{(n-1)/4}}a^{(n-1)/4}\right\} \tag{3.6}$$

It is clear that

$$\frac{(q^{10},q^4,q^5,q^{5-2n};q^8)_{(n-1)/4}}{(q^2,q^8,q^7,q^{3-2n};q^8)_{(n-1)/4}}-\frac{(q^{10},q^4,aq^5,q^{5-2n}/a;q^8)_{(n-1)/4}}{(aq^2,aq^8,q^7,q^{3-2n};q^8)_{(n-1)/4}}a^{(n-1)/4}$$
$$\equiv\frac{(q^{10},q^4,q^5,q^5;q^8)_{(n-1)/4}}{(q^2,q^2,q^7,q^7;q^8)_{(n-1)/4}}-\frac{(q^{10},q^4,aq^5,q^5/a;q^8)_{(n-1)/4}}{(aq^2,q^2/a,q^7,q^7;q^8)_{(n-1)/4}} \pmod{\Phi_n(q^2)}. \tag{3.7}$$

By the L'Hôpital rule, there holds

$$\lim_{a\to 1}\frac{(1-aq^{2n})(a-q^{2n})}{(1-a)^2}\left\{\frac{(q^{10},q^4,q^5,q^5;q^8)_{(n-1)/4}}{(q^2,q^2,q^7,q^7;q^8)_{(n-1)/4}}-\frac{(q^{10},q^4,aq^5,q^5/a;q^8)_{(n-1)/4}}{(aq^2,q^2/a,q^7,q^7;q^8)_{(n-1)/4}}\right\}$$
$$=\frac{(q^{10},q^4,q^5,q^5;q^8)_{(n-1)/4}}{(q^2,q^2,q^7,q^7;q^8)_{(n-1)/4}}[2n]^2\sum_{k=1}^{(n-1)/4}\left(\frac{q^{8k-3}}{[8k-3]^2}-\frac{q^{8k-6}}{[8k-6]^2}\right).$$

Hence, substituting (3.7) into (3.6) , taking the limits as $a \to 1$, and noticing that

$$\frac{(q^{10}, q^4, q^5, q^{5-2n}; q^8)_{(n-1)/4}}{(q^2, q^8, q^7, q^{3-2n}; q^8)_{(n-1)/4}} = \frac{(q^{10}, q^4, q^5, q^5; q^8)_{(n-1)/4}}{(q^2, q^8, q^7, q^7; q^8)_{(n-1)/4}} q^{(n-1)/2},$$

and

$$\frac{(q^{10}, q^4, q^5, q^{5-2n}; q^8)_{(n-1)/4}}{(q^2, q^2, q^7, q^{3-2n}; q^8)_{(n-1)/4}} \equiv \frac{(q^{10}, q^4, q^5, q^5; q^8)_{(n-1)/4}}{(q^2, q^8, q^7, q^7; q^8)_{(n-1)/4}} q^{(n-1)/2} \pmod{\Phi_n(q^2)},$$

we complete the proof of the theorem. □

4. Concluding remarks

It should be mentioned that the q-supercongruences (1.8) and (1.11) do not hold modulo $[n]$ in general. However, the $q \mapsto q^2$ and $e = q^3$ case of (1.4) implies that both (1.8) and (1.11) hold modulo $[n]_{-q}$. Thus, (1.8) holds modulo $[n]_{-q}\Phi_n(-q)^3$ and the $n = p^r$ and $q \to -1$ case indicates that, for any prime $p \equiv 3 \pmod 4$ and positive odd integer r,

$$\sum_{k=0}^{p^r-1} (6k+1) \frac{(\frac{1}{2})_k^3 (\frac{1}{4})_k}{k!^4 4^k} \equiv 0 \pmod{p^{r+3}},$$

which is a generalization of (1.2), and was first obtained by Wei [11] as a limiting case from (1.5).

We end this paper with the following two conjectures.

Conjecture 4.1. *Let $p \equiv 3 \pmod 8$ be a prime and $r > 1$ an odd integer. Then*

$$\sum_{k=0}^{p^r-1} (6k+1) \frac{(\frac{1}{4})_k (\frac{3}{8})_k^2 (\frac{1}{2})_k^3}{(1)_k^4 (\frac{3}{4})_k^2} \equiv 0 \pmod{p^{r+2}}.$$

Conjecture 4.2. *Let $p \equiv 5 \pmod 8$ be a prime and r a positive integer. Then*

$$\sum_{k=0}^{p^r-1} (6k+1) \frac{(\frac{1}{4})_k (\frac{3}{8})_k^2 (\frac{1}{2})_k^3}{(1)_k^4 (\frac{3}{4})_k^2} \equiv 0 \pmod{p^{r+2}}.$$

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