On Jensen's and related combinatorial identities[1]

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Abstract. Motivated by the recent work of Chu [Electron. J. Combin. 17 (2010), #N24], we give simple proofs of Jensen's identity

$$\sum_{k=0}^{n}\binom{x+kz}{k}\binom{y-kz}{n-k}=\sum_{k=0}^{n}\binom{x+y-k}{n-k}z^k,$$

and Chu's and Mohanty-Handa's generalizations of Jensen's identity. We also give a quite simple proof of an equivalent form of Graham-Knuth-Patashnik's identity

$$\sum_{k\geq 0}\binom{m+r}{m-n-k}\binom{n+k}{n}x^{m-n-k}y^k=\sum_{k\geq 0}\binom{-r}{m-n-k}\binom{n+k}{n}(-x)^{m-n-k}(x+y)^k,$$

which was rediscovered, respectively, by Sun in 2003 and Munarini in 2005. Finally we give a multinomial coefficient generalization of this identity.

1 Introduction

Abel's identity (see, for example, [8, §3.1])

$$\sum_{k=0}^{n}\binom{n}{k}x(x+kz)^{k-1}(y-kz)^{n-k}=(x+y)^n$$

and Rothe's identity [23] (or Hagen-Rothe's identity, see, for example, [9, §5.4])

$$\sum_{k=0}^{n}\frac{x}{x-kz}\binom{x-kz}{k}\binom{y+kz}{n-k}=\binom{x+y}{n},$$

are famous in the literature and play an important role in enumerative combinatorics. Recently, Chu [6] gave elementary proofs of Abel's identity and Rothe's identity by using the binomial theorem and the Chu-Vandermonde convolution formula respectively.

Motivated by Chu's work, we shall study Jensen's identity [17], which is closely related to Rothe's identity, and can be stated as follows:

$$\sum_{k=0}^{n}\binom{x+kz}{k}\binom{y-kz}{n-k}=\sum_{k=0}^{n}\binom{x+y-k}{n-k}z^k. \tag{1}$$

[1]2010 Mathematics Subject Classification. Primary 05A10. Secondly 05A19.

Keywords and Phrases. Jensen's identity, Chu's identity, Mohanty-Handa's identity, Graham-Knuth-Patashnik's identity, multinomial coefficient.

Jensen's identity (1) has ever attracted much attention by different authors. Gould [11] obtained the following Abel-type analogue:

$$\sum_{k=0}^{n}\frac{(x+kz)^k}{k!}\frac{(y-kz)^{n-k}}{(n-k)!}=\sum_{k=0}^{n}\frac{(x+y)^k}{k!}z^{n-k}. \tag{2}$$

Carlitz [1] gave two interesting theorem related to (1) and (2) by mathematical induction. With the help of generating functions, Gould [12] derived the following variation of Jensen's identity (1):

$$\sum_{k=0}^{n}\binom{x+kz}{k}\binom{y-kz}{n-k}=\sum_{k=0}^{n}k\binom{x+y-k}{n-k}\frac{x+y-(n-k)z-k}{x+y-k}z^k.$$

E. G.-Rodeja F. [10] deduced Gould's identity (2) from (1) by establishing an identity which includes both. Cohen and Sun [7] also gave an expression which unifies (1) and (2). Chu [4] generalized Jensen's identity (1) to a multi-sum form:

$$\sum_{k_1+\cdots+k_s=n}\prod_{i=1}^{s}\binom{x_i+k_iz}{k_i}=\sum_{k=0}^{n}\binom{k+s-2}{k}\binom{x_1+\cdots+x_s+nz-k}{n-k}z^k. \tag{3}$$

Moreover, the identities (1) and (3) were respectively generalized by Mohanty and Handa [19] and Chu [5] to the case of multinomial coefficients (to be stated in Section 4).

The first purpose of this paper is to give simple proofs of Jensen's identity, Chu's identity (3), Mohanty-Handa's identity, and Chu's generalization of Mohanty-Handa's identity. We shall use the Chu-Vandermonde convolution formula

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k}=\binom{x+y}{n}$$

and the well-known identity

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^r=\begin{cases}0, & \text{if } 0\le r\le n-1,\\ n!, & \text{if } r=n.\end{cases} \tag{4}$$

Eq. (4) may be easily deduced from the Stirling numbers of the second kind [27, p. 34, (24a)]. The first case of (4) was already utilized by the author [13] to give a simple proof of Dixon's identity and by Chu [6] in his proofs of Abel's and Rothe's identities.

It is interesting that our proof of Chu's identity (3) will also leads to a very short proof of Graham-Knuth-Patashnik's identity, which was rediscovered several times in the past few years. The second purpose of this paper is to give a multinomial coefficient generalization of Graham-Knuth-Patashnik's identity.

2 Proof of Jensen's identity

By the Chu-Vandermonde convolution formula, we have

$$\sum_{k=0}^{n} \binom{x+kz}{k} \binom{y-kz}{n-k} = \sum_{k=0}^{n} \binom{x+kz}{k} \sum_{i=k}^{n} \binom{x+y+1}{n-i} \binom{-x-kz-1}{i-k}. \tag{5}$$

Interchanging the summation order in (5) and noticing that

$$\binom{x+kz}{k} \binom{-x-kz-1}{i-k} = (-1)^{i-k} \binom{i}{k} \binom{x+kz+i-k}{i},$$

we have

$$\begin{aligned} \sum_{k=0}^{n} \binom{x+kz}{k} \binom{y-kz}{n-k} &= \sum_{i=0}^{n} \binom{x+y+1}{n-i} \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \binom{x+kz+i-k}{i} \\ &= \sum_{i=0}^{n} \binom{x+y+1}{n-i} (z-1)^i, \end{aligned} \tag{6}$$

where the second equality holds because $\binom{x+kz+i-k}{i}$ is a polynomial in k of degree i with leading coefficient $(z-1)^i/i!$ and we can apply (4) to simplify. We now substitute $x \to -x-1$, $y \to -y+n-1$ and $z \to -z+1$ in (6) and observe that

$$\binom{-x}{k} = (-1)^k \binom{x+k-1}{k}. \tag{7}$$

Then we obtain

$$\sum_{k=0}^{n} \binom{x+kz}{k} \binom{y-kz}{n-k} = \sum_{i=0}^{n} \binom{x+y-i}{n-i} z^i,$$

as desired.

Combining (1) and (6), we get the following identity:

$$\sum_{k=0}^{n} \binom{x-k}{n-k} z^k = \sum_{k=0}^{n} \binom{x+1}{n-k} (z-1)^k,$$

which is equivalent to the following identity in Graham et al. [9, p. 218]:

$$\sum_{k \le m} \binom{m+r}{k} x^k y^{m-k} = \sum_{k \le m} \binom{-r}{k} (-x)^k (x+y)^{m-k}.$$

3 Proofs of Chu's and Graham-Knuth-Patashnik's identities

Comparing the coefficients of x^n in both sides of the equation

$$(1+x)^{a_1+\cdots+a_s} = (1+x)^{a_1}\cdots(1+x)^{a_s}$$

by the binomial theorem, we have

$$\binom{a_1+\cdots+a_s}{n} = \sum_{i_1+\cdots+i_s=n} \binom{a_1}{i_1}\cdots\binom{a_s}{i_s}. \tag{8}$$

Letting $a_i = -x_i - k_i z - 1$ $(1 \le i \le s-1)$ and $a_s = x_1+\cdots+x_s+nz+s-1$ in (8), we have

$$\begin{aligned}\binom{x_s+k_s z}{k_s} &= \binom{x_s+(n-k_1-\cdots-k_{s-1})z}{n-k_1-\cdots-k_{s-1}} \\ &= \sum_{j=k_1+\cdots+k_{s-1}}^{n} \sum_{j_1+\cdots+j_{s-1}=j} \binom{x_1+\cdots+x_s+nz+s-1}{n-j} \\ &\qquad \times \prod_{i=1}^{s-1}\binom{-x_i-k_i z-1}{j_i-k_i},\end{aligned}$$

where $k_1+\cdots+k_s=n$. It follows that

$$\begin{aligned}\sum_{k_1+\cdots+k_s=n}\prod_{i=1}^{s}\binom{x_i+k_i z}{k_i} &= \sum_{k_1+\cdots+k_{s-1}=0}^{n}\ \sum_{j=k_1+\cdots+k_{s-1}}^{n}\ \sum_{j_1+\cdots+j_{s-1}=j}\binom{x_1+\cdots+x_s+nz+s-1}{n-j} \\ &\qquad \times \prod_{i=1}^{s-1}\binom{x_i+k_i z}{k_i}\binom{-x_i-k_i z-1}{j_i-k_i}.\end{aligned} \tag{9}$$

Interchanging the summation order in (9) and observing that

$$\binom{x_i+k_i z}{k_i}\binom{-x_i-k_i z-1}{j_i-k_i} = (-1)^{j_i-k_i}\binom{j_i}{k_i}\binom{x_i+k_i z+j_i-k_i}{j_i}$$

and $\binom{x_i+k_i z+j_i-k_i}{j_i}$ is a polynomial in k_i of degree j_i with leading coefficient $(z-1)^{j_i}/j_i!$, by (4) we get

$$\begin{aligned}\sum_{k_1+\cdots+k_s=n}\prod_{i=1}^{s}\binom{x_i+k_i z}{k_i} &= \sum_{j=0}^{n}\binom{x_1+\cdots+x_s+nz+s-1}{n-j}\sum_{j_1+\cdots+j_{s-1}=j}(z-1)^j \\ &= \sum_{j=0}^{n}\binom{j+s-2}{j}\binom{x_1+\cdots+x_s+nz+s-1}{n-j}(z-1)^j.\end{aligned} \tag{10}$$

Substituting $x_i \to -x_i - 1$ $(i = 1, \ldots, s)$ and $z \to -z + 1$ in (10) and using (7), we immediately get Chu's identity (3).

Comparing (3) with (10) and replacing s by $s + 2$, we obtain

$$\sum_{k=0}^{n} \binom{k+s}{k} \binom{x-k}{n-k} z^k = \sum_{k=0}^{n} \binom{k+s}{k} \binom{x+s+1}{n-k} (z-1)^k. \tag{11}$$

It is easy to see that the identity (11) is equivalent to each of the following known identities:

- Graham-Knuth-Patashnik's identity [9, p. 218]

$$\sum_{k \geq 0} \binom{m+r}{m-n-k} \binom{n+k}{n} x^{m-n-k} y^k = \sum_{k \geq 0} \binom{-r}{m-n-k} \binom{n+k}{n} (-x)^{m-n-k} (x+y)^k. \tag{12}$$

- Sun's identity [29]

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \binom{n+k}{a} (1+x)^{n+k-a} = \sum_{k=0}^{n} \binom{n}{k} \binom{m+k}{a} x^{m+k-a}. \tag{13}$$

- Munarini's identity [20]

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{\beta - \alpha + n}{n-k} \binom{\beta + k}{k} (1+x)^k = \sum_{k=0}^{n} \binom{\alpha}{n-k} \binom{\beta+k}{k} x^k. \tag{14}$$

For example, substituting $n \to m - n$, $s \to n$, $x \to -n - r - 1$ and $z \to -y/x$ in (11), we are led to (12). Replacing k by $m - k$ and $n - k$ respectively in both sides of (13), we get

$$\sum_{k=0}^{m+n-a} (-1)^{m-k} \binom{m}{k} \binom{m+n-k}{a} (1+x)^{m+n-k-a} = \sum_{k=0}^{m+n-a} \binom{n}{k} \binom{m+n-k}{a} x^{m+n-k-a},$$

which is equivalent to (11) by changing k to $m + n - a - k$.

Moreover, the following special case

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} (1+x)^k = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} x^k \tag{15}$$

was reproved by Simons [26], Hirschhorn [15], Chapman [2], Prodinger [21], Wang and Sun [30].

4 Mohanty-Handa's identity and Chu's generalization

Let m be a fixed positive integer. For $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{N}^m$ and $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{C}^m$, set $|\mathbf{a}| = a_1+\cdots+a_m$, $\mathbf{a}! = a_1! \cdots a_m!$, $\mathbf{a}+\mathbf{b} = (a_1+b_1, \ldots, a_m+b_m)$, $\mathbf{a}\cdot\mathbf{b} = a_1b_1+\cdots+a_mb_m$, and $\mathbf{b}^{\mathbf{a}} = b_1^{a_1} \cdots b_m^{a_m}$. For any variable x and $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}^m$, the *multinomial coefficient* $\binom{x}{\mathbf{n}}$ is defined by

$$\binom{x}{\mathbf{n}} = \begin{cases} x(x-1)\cdots(x-|\mathbf{n}|+1)/\mathbf{n}!, & \text{if } \mathbf{n} \in \mathbb{N}^m, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, we let $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, \ldots, 1)$.

Note that the Chu-Vandermonde convolution formula has the following trivial generalization

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{x}{\mathbf{k}} \binom{y}{\mathbf{n}-\mathbf{k}} = \binom{x+y}{\mathbf{n}}, \tag{16}$$

as mentioned by Zeng [32], while (4) can be easily generalized as

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} (-1)^{|\mathbf{n}|-|\mathbf{k}|} \binom{\mathbf{n}}{\mathbf{k}} \mathbf{k}^{\mathbf{r}} = \begin{cases} 0, & \text{if } r_i < n_i \text{ for some } 1 \le i \le m. \\ \mathbf{n}!, & \text{if } \mathbf{r} = \mathbf{n}, \end{cases} \tag{17}$$

where

$$\binom{\mathbf{n}}{\mathbf{k}} := \prod_{i=1}^{m} \binom{n_i}{k_i}.$$

In 1969, Mohanty and Handa [19] established the following multinomial coefficient generalization of Jensen's identity

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{x+\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}} \binom{y-\mathbf{k}\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{x+y-|\mathbf{k}|}{\mathbf{n}-\mathbf{k}} \binom{|\mathbf{k}|}{\mathbf{k}} \mathbf{z}^{\mathbf{k}}. \tag{18}$$

Here and in what follows, $\mathbf{k} = (k_1, \ldots, k_m)$. Twenty years later, Mohanty-Handa's identity was generalized by Chu [5] as follows:

$$\sum_{\mathbf{k}_1+\cdots+\mathbf{k}_s=\mathbf{n}} \prod_{i=1}^{s} \binom{x_i+\mathbf{k}_i\cdot\mathbf{z}}{\mathbf{k}_i} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{|\mathbf{k}|+s-2}{\mathbf{k}} \binom{x_1+\cdots+x_s+\mathbf{n}\cdot\mathbf{z}-|\mathbf{k}|}{\mathbf{n}-\mathbf{k}} \mathbf{z}^{\mathbf{k}}, \tag{19}$$

which is also a generalization of (3). Here $\mathbf{k}_i = (k_{i1}, \ldots, k_{im})$, $i = 1, \ldots, m$.

Remark. Note that the corresponding multinomial coefficient generalization of Rothe's identity was already obtained by Raney [22] (for a special case) and Mohanty [18]. The reader is referred to Strehl [28] for a historical note on Raney-Mohanty's identity.

We will give an elementary proof of Chu's identity (19) similar to that of (3).

Lemma 4.1 *For* $\mathbf{n} \in \mathbb{N}^m$ *and* $s \geq 1$*, there holds*

$$\sum_{\mathbf{k}_1+\cdots+\mathbf{k}_s=\mathbf{n}} \prod_{i=1}^{s} \binom{|\mathbf{k}_i|}{\mathbf{k}_i} = \binom{|\mathbf{n}|+s-1}{\mathbf{n}}. \tag{20}$$

Proof. For any nonnegative integers $a_1, \ldots, a_s$ such that $a_1 + \cdots + a_s = |\mathbf{n}|$, by the Chu-Vandermonde convolution formula (16), the following identity holds

$$\sum_{\mathbf{k}_1+\cdots+\mathbf{k}_s=\mathbf{n}} \prod_{i=1}^{s} \binom{a_i}{\mathbf{k}_i} = \binom{|\mathbf{n}|}{\mathbf{n}}. \tag{21}$$

Moreover, for $\mathbf{k}_1 + \cdots + \mathbf{k}_s = \mathbf{n}$, we have

$$\prod_{i=1}^{s} \binom{a_i}{\mathbf{k}_i} \neq 0 \quad \text{if and only if} \quad |\mathbf{k}_i| = a_i \ (i = 1, \ldots, s).$$

Thus, the identity (21) may be rewritten as

$$\sum_{\substack{\mathbf{k}_1+\cdots+\mathbf{k}_s=\mathbf{n} \\ |\mathbf{k}_1|=a_1,\ldots,|\mathbf{k}_s|=a_s}} \prod_{i=1}^{s} \binom{a_i}{\mathbf{k}_i} = \binom{|\mathbf{n}|}{\mathbf{n}}.$$

It follows that

$$\begin{aligned}
\sum_{\mathbf{k}_1+\cdots+\mathbf{k}_s=\mathbf{n}} \prod_{i=1}^{s} \binom{|\mathbf{k}_i|}{\mathbf{k}_i} &= \sum_{a_1+\cdots+a_s=|\mathbf{n}|} \sum_{\substack{\mathbf{k}_1+\cdots+\mathbf{k}_s=\mathbf{n} \\ |\mathbf{k}_1|=a_1,\ldots,|\mathbf{k}_s|=a_s}} \prod_{i=1}^{s} \binom{a_i}{\mathbf{k}_i} \\
&= \sum_{a_1+\cdots+a_s=|\mathbf{n}|} \binom{|\mathbf{n}|}{\mathbf{n}} \\
&= \binom{|\mathbf{n}|+s-1}{|\mathbf{n}|} \binom{|\mathbf{n}|}{\mathbf{n}},
\end{aligned}$$

as desired. ∎

By repeatedly using the convolution formula (16), we may rewrite the left-hand side of (19) as

$$\begin{aligned}
&\sum_{\mathbf{k}_1+\cdots+\mathbf{k}_{s-1}=\mathbf{0}}^{\mathbf{n}} \sum_{\mathbf{j}=\mathbf{k}_1+\cdots+\mathbf{k}_{s-1}}^{\mathbf{n}} \sum_{\mathbf{j}_1+\cdots+\mathbf{j}_{s-1}=\mathbf{j}} \binom{x_1+\cdots+x_s+\mathbf{n}\cdot\mathbf{z}+m-1}{\mathbf{n}-\mathbf{j}} \\
&\times \prod_{i=1}^{s-1} \binom{x_i+\mathbf{k}_i\cdot\mathbf{z}}{\mathbf{k}_i} \binom{-x_i-\mathbf{k}_i\cdot\mathbf{z}-1}{\mathbf{j}_i-\mathbf{k}_i}.
\end{aligned} \tag{22}$$

Interchanging the summation order in (22), observing that

$$\binom{x_i+\mathbf{k}_i\cdot\mathbf{z}}{\mathbf{k}_i} \binom{-x_i-\mathbf{k}_i\cdot\mathbf{z}-1}{\mathbf{j}_i-\mathbf{k}_i} = (-1)^{|\mathbf{j}_i|-|\mathbf{k}_i|} \binom{\mathbf{j}_i}{\mathbf{k}_i} \binom{x_i+\mathbf{k}_i\cdot\mathbf{z}+|\mathbf{j}_i|-|\mathbf{k}_i|}{\mathbf{j}_i}$$

and

$$\binom{x_i + \mathbf{k}_i \cdot \mathbf{z} + |\mathbf{j}_i| - |\mathbf{k}_i|}{\mathbf{j}_i}$$

is a polynomial in $k_{i\,1}, \dots, k_{i\,m}$ with the coefficient of $\mathbf{k}_i^{\mathbf{j}_i}$ being $\binom{|\mathbf{j}_i|}{\mathbf{j}_i}(\mathbf{z}-\mathbf{1})^{\mathbf{j}_i}/\mathbf{j}_i!$, applying (17), we get

$$\begin{aligned}
&\sum_{\mathbf{k}_1+\cdots+\mathbf{k}_s=\mathbf{n}} \prod_{i=1}^{s} \binom{x_i + \mathbf{k}_i \cdot \mathbf{z}}{\mathbf{k}_i} \\
&= \sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{n}} \binom{x_1 + \cdots + x_s + \mathbf{n} \cdot \mathbf{z} + s - 1}{\mathbf{n} - \mathbf{j}} (\mathbf{z}-\mathbf{1})^{\mathbf{j}} \sum_{\mathbf{j}_1+\cdots+\mathbf{j}_{s-1}=\mathbf{j}} \prod_{i=1}^{m} \binom{|\mathbf{j}_i|}{\mathbf{j}_i} \\
&= \sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{n}} \binom{|\mathbf{j}| + s - 2}{\mathbf{j}} \binom{x_1 + \cdots + x_s + \mathbf{n} \cdot \mathbf{z} + s - 1}{\mathbf{n} - \mathbf{j}} (\mathbf{z}-\mathbf{1})^{\mathbf{j}}, \qquad (23)
\end{aligned}$$

where the second equality follows from (20). Substituting $x_i \to -x_i - 1$ $(i = 1, \dots, s)$ and $\mathbf{z} \to -\mathbf{z} + \mathbf{1}$ in (23) and observing that $\binom{-x}{\mathbf{k}} = (-1)^{|\mathbf{k}|}\binom{x+|\mathbf{k}|-1}{\mathbf{k}}$, we immediately get (19).

Comparing (19) with (23) and replacing s by $s+2$, we obtain the following result.

Theorem 4.2 *For $\mathbf{n} \in \mathbb{N}^m$ and $\mathbf{z} \in \mathbb{C}^m$, there holds*

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{|\mathbf{k}| + s}{\mathbf{k}} \binom{x - |\mathbf{k}|}{\mathbf{n} - \mathbf{k}} \mathbf{z}^{\mathbf{k}} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{|\mathbf{k}| + s}{\mathbf{k}} \binom{x + s + 1}{\mathbf{n} - \mathbf{k}} (\mathbf{z}-\mathbf{1})^{\mathbf{k}}. \qquad (24)$$

It is easy to see that (24) is a multinomial coefficient generalization of (11). Substituting $s \to \beta$, $x \to \alpha - \beta - 1$ and $\mathbf{z} \to \mathbf{1} + \mathbf{x}$ in (24), we get

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} (-1)^{|\mathbf{n}|-|\mathbf{k}|} \binom{\beta - \alpha + |\mathbf{n}|}{\mathbf{n} - \mathbf{k}} \binom{\beta + |\mathbf{k}|}{\mathbf{k}} (\mathbf{1} + \mathbf{x})^{\mathbf{k}} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{\alpha}{\mathbf{n} - \mathbf{k}} \binom{\beta + |\mathbf{k}|}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}, \qquad (25)$$

which is a generalization of Munarini's identity (14). If $\alpha = \beta = |\mathbf{n}|$, then (25) reduces to

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} (-1)^{|\mathbf{n}|-|\mathbf{k}|} \binom{|\mathbf{n}|}{\mathbf{n} - \mathbf{k}} \binom{|\mathbf{n}| + |\mathbf{k}|}{\mathbf{k}} (\mathbf{1} + \mathbf{x})^{\mathbf{k}} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{|\mathbf{n}|}{\mathbf{n} - \mathbf{k}} \binom{|\mathbf{n}| + |\mathbf{k}|}{\mathbf{k}} \mathbf{x}^{\mathbf{k}},$$

which is a generalization of Simons' identity (15). Note that Shattuck [25] and Chen and Pang [3] have given different combinatorial proofs of (14). It is natural to ask the following problem.

Problem 4.3 Is there a combinatorial interpretation of (25)?

In fact, such a proof was recently found by Yang [31].

5 Concluding remarks

We know that binomial coefficient identities usually have nice q-analogues. However, there are only curious (not natural) q-analogues of Abel's and Rothe's identities (see [24] and references therein) up to now. There seems to have no q-analogues of Jensen's identity in the literature.

It is interesting that Hou and Zeng [16] gave a q-analogue of Sun's identity (13):

$$\sum_{k=0}^{m}(-1)^{m-k}\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}n+k\\a\end{bmatrix}(-xq^a;q)_{n+k-a}q^{\binom{k+1}{2}-mk+\binom{a}{2}} = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m+k\\a\end{bmatrix}x^{m+k-a}q^{mn+\binom{k}{2}}, \tag{26}$$

where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ and

$$\begin{bmatrix}\alpha\\k\end{bmatrix} = \begin{cases}\dfrac{(q^{\alpha-k+1};q)_k}{(q;q)_k}, & \text{if } k \geq 0,\\ 0, & \text{if } k < 0.\end{cases}$$

Clearly, (26) may be written as a q-analogue of Munarini's identity (14):

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}\beta-\alpha+n\\n-k\end{bmatrix}\begin{bmatrix}\beta+k\\k\end{bmatrix}q^{\binom{n-k}{2}-\binom{n}{2}}(-x;q)_k\\ &\quad= \sum_{k=0}^{n}\begin{bmatrix}\alpha\\n-k\end{bmatrix}\begin{bmatrix}\beta+k\\k\end{bmatrix}q^{\binom{n-k+1}{2}+(\beta-\alpha)(n-k)}x^k,\end{aligned} \tag{27}$$

as mentioned by Guo and Zeng [14]. We end this paper with the following problem.

Problem 5.1 Is there a q-analogue of (25)? Or equivalently, is there a multi-sum generalization of (27)?

Acknowledgments. This work was partially supported by the Fundamental Research Funds for the Central Universities, Shanghai Rising-Star Program (#09QA1401700), Shanghai Leading Academic Discipline Project (#B407), and the National Science Foundation of China (#10801054).

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