

Elementary proofs of some q -identities of Jackson and Andrews-Jain

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Abstract. We present elementary proofs of three q -identities of Jackson. They are Jackson's terminating q -analogue of Dixon's sum, Jackson's q -analogue of Clausen's formula, and a generalization of both of them. We also give an elementary proof of Jain's q -analogue of terminating Watson's summation formula, which is actually equivalent to Andrews's q -analogue of Watson's formula.

Keywords: Jackson, basic hypergeometric series, q -analogue of Dixon's sum, q -analogue of Clausen's formula

1 Introduction

We employ the notation and terminology in [13], and we always assume that $0 < |q| < 1$. The q -shifted factorial is defined by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n \in \mathbb{N}.$$

For brevity, we adopt the usual notation

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n.$$

The q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

Jackson [18] established the following three interesting theorems:

Theorem 1.1 *For $n \geq 0$, we have*

$$\sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix} (a, b; q)_k (a, b; q)_{2n-k} q^{k(2n-k+1)/2} = (a, b, q^{n+1}, abq^n; q)_n. \quad (1.1)$$

Theorem 1.2 For $n \geq 0$, we have

$$\sum_{k=0}^n \frac{(a, b; q)_k (a, b; q)_{n-k}}{(q, abq^{1/2}; q)_k (q, abq^{1/2}; q)_{n-k}} q^{k/2} = \frac{(a, b; q^{1/2})_n (ab; q)_n}{(ab, q^{1/2}; q^{1/2})_n (abq^{1/2}; q)_n}. \quad (1.2)$$

Theorem 1.3 For $n \geq 0$, we have

$$\begin{aligned} & \sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix} (a, b, c, d; q)_k (a, b, c, d; q)_{2n-k} q^{k(6n-3k+1)/2} \\ & = (-1)^n (a, b, c, d, q^{n+1}, abq^n, bcq^n, caq^n; q)_n d^n q^{n(3n-1)/2}, \end{aligned} \quad (1.3)$$

where $abcd = q^{1-3n}$.

Jackson obtained (1.1) and (1.3) by a rather curious transformation of his q -analogue of Dougall's theorem [16], and derived (1.2) as a special case of a q -theorem in his paper [17]. Note that Jackson had already stated (1.1) without proof in [15]. It was pointed out by Bailey [3] that (1.2) with n even is a special case of (1.3).

As usual, the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n z^n}{(q, b_1, b_2, \dots, b_r; q)_n}.$$

Andrews [2] obtained a q -analogue of a terminating version of Watson's summation formula for ${}_3F_2[1]$:

$${}_4\phi_3 \left[\begin{matrix} a, b, c^{1/2}, -c^{1/2} \\ (abq)^{1/2}, -(abq)^{1/2}, c \end{matrix}; q, q \right] = \frac{(aq, bq, q/c, abq/c; q^2)_{\infty}}{(q, abq, bq/c, aq/c; q^2)_{\infty}}, \quad (1.4)$$

where $b = q^{-n}$ and n is a nonnegative integer.

Jain [19] obtained a q -analogue of terminating Watson's summation formula due to Bailey [5] in the form

$$\sum_{k=0}^n \frac{(a, b; q)_k (q^{-2n}; q^2)_k q^k}{(q; q)_k (abq; q^2)_k (q^{-2n}; q)_k} = \frac{(aq, bq; q^2)_n}{(q, abq; q^2)_n}, \quad (1.5)$$

and pointed out that (1.5) for $n \rightarrow \infty$ gives the q -analogue of Gauss's second summation theorem due to Andrews [1]. It is not difficult to see that (1.4) and (1.5) are equivalent to each other.

The objective of this paper is to give elementary proofs of the above three theorems of Jackson and Jain's q -identity (1.5). We will show that (1.1)–(1.3) and (1.5) can be recovered from their special cases. In particular, (1.2) in the case n is even implies (1.3). The main idea is that to prove a terminating q -identity with parameters it is enough to prove it for infinitely many values of these parameters. In this way, we also obtain an interesting q -identity (4.5) which generalizes (1.2) in the case n is odd. Note that Chen

and Liu [7, 8] and Liu [21] developed a method of deriving hypergeometric identities by parameter augmentation, which is powerful to deal with nonterminating hypergeometric series.

The following lemma plays an important role throughout the paper.

Lemma 1.4 *Let $f(a, b) = b(1 - ax_1)(1 - ax_2)(1 - ax_3)(1 - ab^2x_1x_2x_3)$. Then the divided difference*

$$\frac{f(a, b) - f(b, a)}{a - b}$$

can be factorized into

$$- (1 - abx_1x_2)(1 - abx_1x_3)(1 - abx_2x_3).$$

2 Jackson's terminating q -analogue of Dixon's sum

When considered as a basic hypergeometric series, Equation (1.1) reads

$${}_3\phi_2 \left[\begin{matrix} q^{-2n}, & a, & b \\ q^{1-2n}/a, & q^{1-2n}/b \end{matrix} ; q, q^{2-n}/ab \right] = \frac{(a, b; q)_n (q, ab; q)_{2n}}{(q, ab; q)_n (a, b; q)_{2n}},$$

which is now called Jackson's terminating q -analogue of Dixon's sum. See Bailey [3], Carlitz [6], and Gasper and Rahman [13, p. 50].

In this section we give an elementary proof of (1.1). The following lemma is its special case $b = q$.

Lemma 2.1 *For $n \geq 0$, we have*

$$\sum_{k=0}^{2n} (-1)^k (a; q)_k (a; q)_{2n-k} q^{k(2n-k+1)/2} = (a, aq^{n+1}; q)_n.$$

Proof. Since

$$(1 - a^2q^{2n}) = (1 - aq^{2n-k}) + aq^{2n-k}(1 - aq^k),$$

we have

$$\begin{aligned} & (1 - a^2q^{2n}) \sum_{k=0}^{2n} (-1)^k (a; q)_k (a; q)_{2n-k} q^{k(2n-k+1)/2} \\ &= \sum_{k=0}^{2n} (-1)^k (a; q)_k (a; q)_{2n+1-k} q^{k(2n-k+1)/2} \\ & \quad + aq^n \sum_{k=0}^{2n} (-1)^k (a; q)_{k+1} (a; q)_{2n-k} q^{(k+1)(2n-k)/2}. \end{aligned} \tag{2.1}$$

Noticing that $(a; q)_k(a; q)_{2n+1-k}q^{k(2n-k+1)/2}$ is symmetric in k and $2n+1-k$, we have

$$\sum_{k=1}^{2n} (-1)^k (a; q)_k (a; q)_{2n+1-k} q^{k(2n-k+1)/2} = 0,$$

or equivalently,

$$\sum_{k=0}^{2n-1} (-1)^k (a; q)_{k+1} (a; q)_{2n-k} q^{(k+1)(2n-k)/2} = 0.$$

Therefore, the right-hand side of (2.1) is equal to

$$(a; q)_{2n+1} + aq^n (a; q)_{2n+1} = (1 + aq^n)(a; q)_{2n+1}.$$

Namely,

$$\sum_{k=0}^{2n} (-1)^k (a; q)_k (a; q)_{2n-k} q^{k(2n-k+1)/2} = (a; q)_{2n+1} / (1 - aq^n),$$

as desired. ■

Proof of Theorem 1.1. It is enough to prove the cases $b = q^m$, $m \geq 1$. We proceed by induction on m . For $m = 1$, Equation (1.1) holds by Lemma 2.1.

Suppose (1.1) holds for $b (= q^m)$. Substituting a with aq in (1.1) and multiplying by $(1 - a)^2$ on its both sides, we obtain

$$\begin{aligned} & \sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix} (a; q)_{k+1} (a; q)_{2n-k+1} (b; q)_k (b; q)_{2n-k} q^{k(2n-k+1)/2} \\ &= (1 - a)(a; q)_{n+1}(b, q^{n+1}, abq^{n+1}; q)_n. \end{aligned} \quad (2.2)$$

Since

$$(1 - bq^k)(1 - bq^{2n-k}) = (1 - b/a)(1 - abq^{2n}) + (b/a)(1 - aq^k)(1 - aq^{2n-k}),$$

we get

$$\begin{aligned} (a, bq; q)_k (a, bq; q)_{2n-k} &= \frac{(1 - b/a)(1 - abq^{2n})}{(1 - b)^2} (a, b; q)_k (a, b; q)_{2n-k} \\ &\quad + \frac{b/a}{(1 - b)^2} (a; q)_{k+1} (a; q)_{2n-k+1} (b; q)_k (b; q)_{2n-k}. \end{aligned}$$

Therefore, by the inductive hypothesis and (2.2), we have

$$\begin{aligned}
& \sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix} (a, bq; q)_k (a, bq; q)_{2n-k} q^{k(2n-k+1)/2} \\
&= \frac{(1-b/a)(1-abq^{2n})}{(1-b)^2} (a, b, q^{n+1}, abq^n; q)_n \\
&\quad + \frac{b/a}{(1-b)^2} (1-a)(a; q)_{n+1} (b, q^{n+1}, abq^{n+1}; q)_n \\
&= (a, bq, q^{n+1}, abq^{n+1}; q)_n.
\end{aligned}$$

Namely, Equation (1.1) holds for $bq (= q^{m+1})$. This completes the proof. \blacksquare

It is easy to see that

$$\begin{aligned}
& \sum_{k=0}^{2n+1} (-1)^k (a; q)_k (a; q)_{2n+1-k} q^{k(2n-k+3)/2} \\
&= \sum_{k=0}^{2n+1} (-1)^k (a; q)_k (a; q)_{2n+2-k} q^{k(2n-k+3)/2} \\
&\quad + aq^{2n+1} \sum_{k=0}^{2n+1} (-1)^k (a; q)_k (a; q)_{2n+1-k} q^{k(2n-k+1)/2}. \tag{2.3}
\end{aligned}$$

By Lemma 2.1, we have

$$\sum_{k=0}^{2n+2} (-1)^k (a; q)_k (a; q)_{2n+2-k} q^{k(2n-k+3)/2} = (a; q)_{n+1} (aq^{n+2}; q)_{n+1}. \tag{2.4}$$

By symmetry, it is obvious that

$$\sum_{k=0}^{2n+1} (-1)^k (a; q)_k (a; q)_{2n+1-k} q^{k(2n-k+1)/2} = 0. \tag{2.5}$$

Combining (2.3)–(2.5) yields

$$\sum_{k=0}^{2n+1} (-1)^k (a; q)_k (a; q)_{2n+1-k} q^{k(2n-k+3)/2} = (1-q^{n+1})(a; q)_{n+1} (aq^{n+2}; q)_n.$$

Then, by the same argument as in the proof Theorem 1.1, we obtain

Theorem 2.2 *For $n \geq 0$, we have*

$$\begin{aligned}
& \sum_{k=0}^{2n+1} (-1)^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix} (a, b; q)_k (a, b; q)_{2n+1-k} q^{k(2n-k+3)/2} \\
&= (a, b, q^{n+1}; q)_{n+1} (abq^{n+1}; q)_n. \tag{2.6}
\end{aligned}$$

Setting $a = b = q^{-2n-1}$ in (2.6), we get

$$\sum_{k=0}^{2n+1} (-1)^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix}^3 q^{(k-n)(3k-3n-1)/2} = (-1)^n \frac{(q; q)_{3n+1}}{(q, q, q; q)_n},$$

an identity due to Bailey [4].

3 Jackson's q -analogue of Clausen's formula

It is clear that (1.2) is equivalent to the following product formula [17, 18]:

$$\begin{aligned} & {}_2\phi_1 \left[\begin{matrix} a, b \\ abq^{1/2} \end{matrix}; q, z \right] {}_2\phi_1 \left[\begin{matrix} a, b \\ abq^{1/2} \end{matrix}; q, zq^{1/2} \right] \\ &= {}_4\phi_3 \left[\begin{matrix} a, b, a^{1/2}b^{1/2}, -a^{1/2}b^{1/2} \\ ab, a^{1/2}b^{1/2}q^{1/4}, -a^{1/2}b^{1/2}q^{1/4} \end{matrix}; q^{1/2}, z \right], \quad |z| < 1. \end{aligned} \quad (3.1)$$

which is a q -analogue of Clausen's formula [9]:

$$\left\{ {}_2F_1 \left[\begin{matrix} a, b \\ a+b+1/2 \end{matrix}; z \right] \right\}^2 = {}_3F_2 \left[\begin{matrix} 2a, 2b, a+b \\ 2a+2b, a+b+1/2 \end{matrix}; z \right], \quad |z| < 1.$$

See for example [11, 12].

Several proofs of (3.1) have been given by Singh [23], Nassrallah [22], and Jain and Srivastava [20]. In this section we add a new one.

Lemma 3.1 *For $n \geq 0$, we have*

$$\sum_{k=0}^n \frac{(q^{1/2}; q)_k (q^{1/2}; q)_{n-k}}{(1-aq^k)(q; q)_k (q; q)_{n-k}} q^{k/2} = \frac{(aq^{1/2}; q)_n}{(a; q)_{n+1}}. \quad (3.2)$$

Proof. Clearly, Equation (3.2) is equivalent to

$$\sum_{k=0}^n (a; q)_k (aq^{k+1}; q)_{n-k} \frac{(q^{1/2}; q)_k (q^{1/2}; q)_{n-k}}{(q; q)_k (q; q)_{n-k}} q^{k/2} = (aq^{1/2}; q)_n. \quad (3.3)$$

When $a = q^{-m}$ ($0 \leq m \leq n$), the left-hand side of (3.3) has only one nonzero term

$$(q^{-m}; q)_m (q; q)_{n-m} \frac{(q^{1/2}; q)_m (q^{1/2}; q)_{n-m}}{(q; q)_m (q; q)_{n-m}} q^{m/2} = (q^{-m+1/2}; q)_n.$$

Noticing that both sides of (3.3) are polynomials in a , we complete the proof. ■

Remark. Equation (3.3) is in fact the Lagrange interpolation formula for $(aq^{1/2}; q)_n$ at the values q^{-i} ($0 \leq i \leq n$) of a . See Fu and Lascoux [10] and Zeng [24].

Proof of Theorem 1.2. It suffices to show that (1.2) holds for all $b = q^{m+1/2}$, $m \geq 1$. We proceed by induction on m . For $m = 0$, $b = q^{1/2}$, Equation (1.2) reduces to

$$\sum_{k=0}^n \frac{(q^{1/2}; q)_k (q^{1/2}; q)_{n-k}}{(1 - aq^k)(1 - aq^{n-k})(q; q)_k (q; q)_{n-k}} q^{k/2} = \frac{(aq^{1/2}; q)_n}{(1 - aq^{n/2})(a; q)_{n+1}}. \quad (3.4)$$

Changing k to $n - k$ in (3.2), we obtain

$$\sum_{k=0}^n \frac{(q^{1/2}; q)_k (q^{1/2}; q)_{n-k}}{(1 - aq^{n-k})(q; q)_k (q; q)_{n-k}} q^{(n-k)/2} = \frac{(aq^{1/2}; q)_n}{(a; q)_{n+1}}. \quad (3.5)$$

Then (3.2) + (3.5) yields

$$\sum_{k=0}^n \frac{(q^{k/2} + q^{(n-k)/2})(1 - aq^{n/2})(q^{1/2}; q)_k (q^{1/2}; q)_{n-k}}{(1 - aq^k)(1 - aq^{n-k})(q; q)_k (q; q)_{n-k}} = 2 \frac{(aq^{1/2}; q)_n}{(a; q)_{n+1}}.$$

Since the above summation is symmetric in k and $n - k$, we get (3.4).

Now suppose (1.2) holds for $b = q^{m+1/2}$. Substituting a with aq in (1.2), we obtain

$$\sum_{k=0}^n \frac{(aq, b; q)_k (aq, b; q)_{n-k}}{(q, abq^{3/2}; q)_k (q, abq^{3/2}; q)_{n-k}} q^{k/2} = \frac{(aq, b; q^{1/2})_n (abq; q)_n}{(abq, q^{1/2}; q^{1/2})_n (abq^{3/2}; q)_n}. \quad (3.6)$$

Puttting $\{x_1, x_2, x_3\} = \{q^k, q^{n-k}, q^{1/2}\}$ in Lemma 1.4, we have

$$\begin{aligned} & \frac{b(1 - aq^{1/2})(1 - ab^2q^{n+1/2})(1 - a)^2}{(1 - abq^n)(b - a)} \sum_{k=0}^n \frac{(aq, b; q)_k (aq, b; q)_{n-k}}{(q, abq^{3/2}; q)_k (q, abq^{3/2}; q)_{n-k}} q^{k/2} \\ & + \frac{a(1 - bq^{1/2})(1 - a^2bq^{n+1/2})(1 - b)^2}{(1 - abq^n)(a - b)} \sum_{k=0}^n \frac{(a, bq; q)_k (a, bq; q)_{n-k}}{(q, abq^{3/2}; q)_k (q, abq^{3/2}; q)_{n-k}} q^{k/2} \\ & = (1 - abq^{1/2})^2 \sum_{k=0}^n \frac{(a, b; q)_k (a, b; q)_{n-k}}{(q, abq^{1/2}; q)_k (q, abq^{1/2}; q)_{n-k}} q^{k/2}, \end{aligned} \quad (3.7)$$

while putting $\{x_1, x_2, x_3\} = \{1, q^{n/2}, q^{(n+1)/2}\}$ in Lemma 1.4, we have

$$\begin{aligned} & \frac{b(1 - aq^{1/2})(1 - ab^2q^{n+1/2})(1 - a)^2}{(1 - abq^n)(b - a)} \frac{(aq, b; q^{1/2})_n (abq; q)_n}{(abq, q^{1/2}; q^{1/2})_n (abq^{3/2}; q)_n} \\ & + \frac{a(1 - bq^{1/2})(1 - a^2bq^{n+1/2})(1 - b)^2}{(1 - abq^n)(a - b)} \frac{(a, bq; q^{1/2})_n (abq; q)_n}{(abq, q^{1/2}; q^{1/2})_n (abq^{3/2}; q)_n} \\ & = (1 - abq^{1/2})^2 \frac{(a, b; q^{1/2})_n (ab; q)_n}{(ab, q^{1/2}; q^{1/2})_n (abq^{1/2}; q)_n}. \end{aligned} \quad (3.8)$$

Comparing (3.7) and (3.8) and using (3.6) and (1.2), we obtain

$$\sum_{k=0}^n \frac{(a, bq; q)_k (a, bq; q)_{n-k}}{(q, abq^{3/2}; q)_k (q, abq^{3/2}; q)_{n-k}} q^{k/2} = \frac{(a, bq; q^{1/2})_n (abq; q)_n}{(abq, q^{1/2}; q^{1/2})_n (abq^{3/2}; q)_n},$$

i.e., Equation (1.2) holds for bq . This completes the proof. \blacksquare

For the case n is even in (1.2), we have

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix} \frac{(a, b; q)_k (a, b; q)_{2n-k}}{(abq^{1/2}; q)_k (abq^{1/2}; q)_{2n-k}} q^{k/2} = \frac{(a, b, q^{n+1}, abq^n, aq^{1/2}, bq^{1/2}; q)_n}{(abq^{1/2}, q^{1/2}; q)_n (abq^{1/2}; q)_{2n}} \quad (3.9)$$

(see Jackson [18, Equation (3)]), while for the odd case, we have

$$\begin{aligned} & \sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix} \frac{(a, b; q)_k (a, b; q)_{2n+1-k}}{(abq^{1/2}; q)_k (abq^{1/2}; q)_{2n+1-k}} q^{k/2} \\ &= \frac{(a, b, q^{n+1}; q)_{n+1} (abq^{n+1}, aq^{1/2}, bq^{1/2}; q)_n}{(abq^{1/2}; q)_n (q^{1/2}; q)_{n+1} (abq^{1/2}; q)_{2n+1}}. \end{aligned} \quad (3.10)$$

4 Jackson's Theorem 1.3

We have seen that (1.1) and (1.2) can be recovered from their special cases $b = q$ and $b = q^{1/2}$, respectively. We will show that (1.3) can also be derived from the case $b = q^{1/2-n}$, which reduces to Jackson's q -analogue of Clausen's formula.

Substituting for $d = q^{1-3n}/abc$, Equation (1.3) may be written as

$$\begin{aligned} & \sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix} \frac{(a, b, c; q)_k (a, b, c; q)_{2n-k}}{(abcq^n; q)_k (abcq^n; q)_{2n-k}} q^{k(2n-k+1)/2} \\ &= \frac{(a, b, c, abq^n, bcq^n, acq^n, q^{n+1}; q)_n}{(abcq^n; q)_n (abcq^n; q)_{2n}}. \end{aligned} \quad (4.1)$$

It is clear that (1.1) can be obtained from (4.1) by taking $c = 0$.

Proof of Theorem 1.3. It suffices to show that (4.1) holds for all $b = q^{1/2+m}$, $m \geq -n$. It is easy to see that the case $b = q^{1/2-n}$ of (4.1) reduces to (3.9) on a and c , as was pointed out by Bailey [3].

Suppose (4.1) holds for $b = q^{1/2+m}$. Substituting a with aq in (4.1), we obtain

$$\begin{aligned} & \sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix} \frac{(aq, b, c; q)_k (aq, b, c; q)_{2n-k}}{(abcq^{n+1}; q)_k (abcq^{n+1}; q)_{2n-k}} q^{k(2n-k+1)/2} \\ &= \frac{(aq, b, c, abq^{n+1}, bcq^n, acq^{n+1}, q^{n+1}; q)_n}{(abcq^{n+1}; q)_n (abcq^{n+1}; q)_{2n}}. \end{aligned} \quad (4.2)$$

Puttting $\{x_1, x_2, x_3\} = \{q^k, q^{2n-k}, cq^n\}$ in Lemma 1.4, we have

$$\begin{aligned}
& \frac{b(1-acq^n)(1-ab^2cq^{3n})(1-a)^2}{(b-a)(1-abq^{2n})} \\
& \times \sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix} \frac{(aq, b, c; q)_k (aq, b, c; q)_{2n-k}}{(abcq^{n+1}; q)_k (abcq^{n+1}; q)_{2n-k}} q^{k(2n-k+1)/2} \\
& + \frac{a(1-bcq^n)(1-a^2bcq^{3n})(1-b)^2}{(a-b)(1-abq^{2n})} \\
& \times \sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix} \frac{(a, bq, c; q)_k (a, bq, c; q)_{2n-k}}{(abcq^{n+1}; q)_k (abcq^{n+1}; q)_{2n-k}} q^{k(2n-k+1)/2} \\
& = (1-abcq^n)^2 \sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix} \frac{(a, b, c; q)_k (a, b, c; q)_{2n-k}}{(abcq^n; q)_k (abcq^n; q)_{2n-k}} q^{k(2n-k+1)/2}. \tag{4.3}
\end{aligned}$$

while putting $\{x_1, x_2, x_3\} = \{1, q^n, cq^{2n}\}$ in Lemma 1.4, we have

$$\begin{aligned}
& \frac{b(1-acq^n)(1-ab^2cq^{3n})(1-a)^2}{(b-a)(1-abq^{2n})} \frac{(aq, b, c, abq^{n+1}, bcq^n, acq^{n+1}, q^{n+1}; q)_n}{(abcq^{n+1}; q)_n (abcq^{n+1}; q)_{2n}} \\
& + \frac{a(1-bcq^n)(1-a^2bcq^{3n})(1-b)^2}{(a-b)(1-abq^{2n})} \frac{(a, bq, c, abq^{n+1}, bcq^{n+1}, acq^n, q^{n+1}; q)_n}{(abcq^{n+1}; q)_n (abcq^{n+1}; q)_{2n}} \\
& = (1-abcq^n)^2 \frac{(a, b, c, abq^n, bcq^n, acq^n, q^{n+1}; q)_n}{(abcq^n; q)_n (abcq^n; q)_{2n}}. \tag{4.4}
\end{aligned}$$

It follows from (4.1)–(4.4) that

$$\begin{aligned}
& \sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix} \frac{(a, bq, c; q)_k (a, bq, c; q)_{2n-k}}{(abcq^{n+1}; q)_k (abcq^{n+1}; q)_{2n-k}} q^{k(2n-k+1)/2} \\
& = \frac{(a, bq, c, abq^{n+1}, bcq^{n+1}, acq^n, q^{n+1}; q)_n}{(abcq^{n+1}; q)_n (abcq^{n+1}; q)_{2n}}.
\end{aligned}$$

Namely, Equation (4.1) holds for $bq = q^{3/2+m}$. This completes the proof. ■

It is interesting that there holds a q -identity similar to (4.1) as follows:

Theorem 4.1 *For $n \geq 0$, we have*

$$\begin{aligned}
& \sum_{k=0}^{2n+1} (-1)^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix} \frac{(a, b, c; q)_k (a, b, c; q)_{2n+1-k}}{(abcq^{n+1}; q)_k (abcq^{n+1}; q)_{2n+1-k}} q^{k(2n-k+3)/2} \\
& = \frac{(a, b, c, q^{n+1}; q)_{n+1} (abq^{n+1}, bcq^{n+1}, acq^{n+1}; q)_n}{(abcq^{n+1}; q)_n (abcq^{n+1}; q)_{2n+1}}. \tag{4.5}
\end{aligned}$$

When $c = 0$, Equation (4.5) reduces to (2.6), and when $c = q^{-1/2-n}$, Equation (4.5) reduces to (3.10). The proof of Theorem 4.1 is exactly the same as that of Theorem 1.3. Namely, we need to use Lemma 1.4 twice, taking $\{x_1, x_2, x_3\} = \{q^k, q^{2n+1-k}, cq^{n+1}\}$ and $\{x_1, x_2, x_3\} = \{1, q^n, cq^{2n+1}\}$, respectively.

It should be mentioned that Theorem 4.1 can also be deduced from Jackson's q -analogue of Dougall's theorem [16] by utilizing a curious transformation, just as the way that Theorem 1.3 was obtained by Jackson [18].

5 Jain's q -analogue of terminating Watson's formula

The case $b = q^2$ of (1.5) reduces to

$$\sum_{k=0}^n \frac{(q^{-2n}; q^2)_k (a; q)_k (1 - q^{k+1}) q^k}{(q^{-2n}; q)_k (aq; q^2)_{k+1}} = \frac{1 - q^{2n+1}}{1 - aq^{2n+1}}. \quad (5.1)$$

It is easy to check that, for $k = 0, 1, \dots, n$,

$$\begin{aligned} & \frac{(q^{-2n}; q^2)_k (a; q)_k (1 - q^{k+1}) q^k}{(q^{-2n}; q)_k (aq; q^2)_{k+1}} \\ &= \frac{q^{2n+1}}{1 - aq^{2n+1}} \left[\frac{(q^{-2n}; q^2)_{k+1} (a; q)_{k+1}}{(q^{-2n}; q)_k (aq; q^2)_{k+1}} - \frac{(q^{-2n}; q^2)_k (a; q)_k}{(q^{-2n}; q)_{k-1} (aq; q^2)_k} \right]. \end{aligned}$$

Hence, summing over k from 0 to n gives (5.1).

Proof of (1.5). It suffices to show that (1.5) holds for all $b = q^{2m}$ ($m \geq 1$). The case $b = q^2$ reduces to (5.1) and has been proved.

Suppose (1.5) holds for $b = q^{2m}$. Substituting a with aq^2 in (1.5), we obtain

$$\sum_{k=0}^n \frac{(aq^2, b; q)_k (q^{-2n}; q^2)_k q^k}{(q; q)_k (abq^3; q^2)_k (q^{-2n}; q)_k} = \frac{(aq^3, bq; q^2)_n}{(q, abq^3; q^2)_n}, \quad (5.2)$$

Letting $\{x_1, x_2, x_3\} = \{0, q^k, q^{k+1}\}$ in Lemma 1.4, we have

$$\begin{aligned} & \frac{b(1-a)(1-aq)}{b-a} \sum_{k=0}^n \frac{(aq^2, b; q)_k (q^{-2n}; q^2)_k q^k}{(q; q)_k (abq^3; q^2)_k (q^{-2n}; q)_k} \\ & - \frac{a(1-b)(1-bq)}{b-a} \sum_{k=0}^n \frac{(a, bq^2; q)_k (q^{-2n}; q^2)_k q^k}{(q; q)_k (abq^3; q^2)_k (q^{-2n}; q)_k} \\ &= (1-abq) \sum_{k=0}^n \frac{(a, b; q)_k (q^{-2n}; q^2)_k q^k}{(q; q)_k (abq; q^2)_k (q^{-2n}; q)_k}. \end{aligned} \quad (5.3)$$

On the other hand, letting $\{x_1, x_2, x_3\} = \{0, 1, q^{2n+1}\}$ in Lemma 1.4, we have

$$\begin{aligned} & \frac{b(1-a)(1-aq)}{b-a} \frac{(aq^3, bq; q^2)_n}{(abq^3; q^2)_n} - \frac{a(1-b)(1-bq)}{b-a} \frac{(aq, bq^3; q^2)_n}{(abq^3; q^2)_n} \\ &= (1-abq) \frac{(aq, bq; q^2)_n}{(abq; q^2)_n}. \end{aligned} \tag{5.4}$$

From (5.2)–(5.4) and the inductive hypothesis (1.5) it follows that

$$\sum_{k=0}^n \frac{(a, bq^2; q)_k (q^{-2n}; q^2)_k q^k}{(q; q)_k (abq^3; q^2)_k (q^{-2n}; q)_k} = \frac{(a, bq^3; q^2)_n}{(abq^3; q^2)_n}.$$

Namely, Equation (1.5) holds for $bq^2 = q^{2m+2}$. This completes the proof. \blacksquare

The same method can also be used to prove a symmetric q -Pfaff-Saalschütz identity. See Guo and Zeng [14].

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