Contiguous relations and summation and transformation formulas for basic hypergeometric series

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Abstract. By using contiguous relations for basic hypergeometric series, we give simple proofs of Bailey's ${}_4\phi_3$ summation, Carlitz's ${}_5\phi_4$ summation, Sears' ${}_3\phi_2$ to ${}_5\phi_4$ transformation, Sears' ${}_4\phi_3$ transformations, Chen's bibasic summation, Gasper's split poised ${}_{10}\phi_9$ transformation, Chu's bibasic symmetric transformation. Along the same line, finite forms of Sylvester's identity, Jacobi's triple product identity, and Kang's identity are also obtained.

Keywords: q-shifted factorial; basic hypergeometric series; contiguous relations; Sylvester's identity

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1 Introduction

In the previous work [15–17], many terminating summation and transformation formulas for basic hypergeometric series are proved by using contiguous relations and mathematical induction. The present paper is a complement to [15–17]. The difference here is that we will usually apply mathematical induction twice rather than once. Recall that the *q-shifted factorial* is defined by

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \quad \text{and} \quad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} \quad \text{for} \quad n \in \mathbb{Z}.$$

We employ the abbreviated notation

$$(a_1, a_2, \dots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n, \quad \text{for} \quad n=\infty \quad \text{or} \quad n \in \mathbb{Z}.$$

The *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix} ; q,\ z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_k z^k}{(q, b_1, b_2, \dots, b_r; q)_k}.$$

An ${}_{r+1}\phi_r$ series is called *well-poised* if $a_1 q = a_2 b_1 = \cdots = a_{r+1} b_r$.

Let

$$F_k(a_1, a_2, \dots, a_{r+1}; q, z) := \frac{(a_1, \dots, a_{r+1}; q)_k}{(q, a_1 q/a_2, \dots, a_1 q/a_{r+1}; q)_k} z^k$$

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be the k-th term of a well-poised hypergeometric series. In [17], the following contiguous relations were established:

$$\begin{aligned} F_k(a_1, a_2, \ldots, a_r q, a_{r+1}; q, z) - F_k(a_1, a_2, \ldots, a_r, a_{r+1} q; q, z) \\ = \alpha F_{k-1}(a_1 q^2, a_2 q, \ldots, a_{r+1} q; q, z), \end{aligned} \tag{1.1}$$

$$\begin{aligned} F_k(a_1, a_2, \ldots, a_r, a_{r+1}; q, qz) - F_k(a_1, a_2, \ldots, a_r, a_{r+1} q; q, z) \\ = \beta F_{k-1}(a_1 q^2, a_2 q, \ldots, a_{r+1} q; q, z), \end{aligned} \tag{1.2}$$

where

$$\alpha = \frac{(a_r - a_{r+1})(1 - a_1/a_r a_{r+1})(1 - a_1)(1 - a_1 q)(1 - a_2) \cdots (1 - a_{r-1}) z}{(1 - a_1/a_r)(1 - a_1/a_{r+1})(1 - a_1 q/a_2) \cdots (1 - a_1 q/a_{r+1})},$$

$$\beta = -\frac{(1 - a_1)(1 - a_1 q)(1 - a_2) \cdots (1 - a_r) z}{(1 - a_1/a_{r+1})(1 - a_1 q/a_2) \cdots (1 - a_1 q/a_{r+1})},$$

and quite a few terminating summation and transformation formulas for basic hypergeometric series were proved by applying (1.1) or (1.2). Note that Krattenthaler [22] has indicated how to derive contiguous relations from special cases of known formulas (available at http://www.mat.univie.ac.at/˜kratt/papers.html).

In this paper we shall prove more such identities based on the contiguous relations (1.1), (1.2) or others. Suppose that

$$\sum_{k=0}^{n} F_{n,k}(a_1, \ldots, a_s) = S_n(a_1, \ldots, a_s), \tag{1.3}$$

where $F_{n,k}(a_1, \ldots, a_s) = 0$ if $k < 0$ or $k > n$. If one can show that the summand $F_{n,k}(a_1, \ldots, a_s)$ satisfies the following recurrence relation:

$$F_{n,k}(a_1 q, a_2 \ldots, a_s) - F_{n,k}(a_1, a_2, \ldots, a_s) = \gamma_n F_{n-v,k-1}(b_1, \ldots, b_s), \ v = 1 \text{ or } 2 \tag{1.4}$$

for some parameters $b_1, \ldots, b_s$, where γ_n is independent of k, then we can prove (1.3) by induction on n. Note that (1.5) is not a special case of Sister Celine's method [26, p. 58, (4.3.1)], as mentioned by [17].

Of course, we need to check that $S_n(a_1, \ldots, a_s)$ satisfies the following recurrence relation

$$S_n(a_1 q, a_2, \ldots, a_s) - S_n(a_1, a_2, \ldots, a_s) = \gamma_n S_{n-v}(b_1, \ldots, b_s). \tag{1.5}$$

If $S_n(a_1, \ldots, a_s)$ appears as a *closed form* as in Bailey's ${}_4\phi_3$ summation formula (2.1), then the verification of (1.5) is quite easy. If $S_n(a_1, \ldots, a_s)$ is of the following form:

$$S_n(a_1, \ldots, a_s) = \sum_{k=0}^{n} G_{n,k}(a_1, \ldots, a_s),$$

where $G_{n,k}(a_1,\ldots,a_s)=0$ if $k<0$ or $k>n$, then we may try to apply q-Gosper's algorithm [21, p. 75] to find a sequence $H_{n,k}(a_1,\ldots,a_s)$ of closed forms such that

$$G_{n,k}(a_1q,a_2,\ldots,a_s)-G_{n,k}(a_1,a_2\ldots,a_s)-\gamma_n G_{n-v,k-1}(b_1,\ldots,b_s)=H_{n,k}-H_{n,k-1}, \tag{1.6}$$

where $H_{n,n}=H_{n,-1}=0$. If (1.6) exists, then by telescoping we get (1.5).

It is easy to see that almost all identities are trivial for $n=0$ or $n=1$. For the induction step, we need firstly to prove that (1.3) is true for some special a_1, and secondly give (1.4) (or, in addition, (1.6)). In this way, we shall give simple proofs of Bailey's ${}_4\phi_3$ summation, Carlitz's ${}_5\phi_4$ summation, Sears' ${}_3\phi_2$ to ${}_5\phi_4$ transformation, Sears' ${}_4\phi_3$ transformations, Chen's bibasic summation, Gasper's split poised ${}_{10}\phi_9$ transformation, Chu's bibasic symmetric transformation. Moreover, we will give a finite form of Sylvester's identity, as well as a finite form of Kang's identity [19].

Note that all terminating identities can be also proved automatically by using the q-Zeilberger algorithm (see, for example, [4, 21, 26]). Moreover, the so-called "Abel's method" (see, for example, [9, 10]) is also a nice method to deal with such identities. Gasper [12] has already developed elementary proofs for many summation formulas for basic hypergeometric series by considering contiguous relations. However, our method is a little different from Gasper's method, since the latter is mainly applied to prove nonterminating basic hypergeometric series, while our method is usually applied to prove terminating series.

2 Bailey's ${}_4\phi_3$ summation formula

Bailey [3] obtained the following ${}_4\phi_3$ summation formula, which was later generalized by Carlitz [5].

Theorem 2.1 (Bailey's ${}_4\phi_3$ summation formula). *For $n\geq 0$, there holds*

$${}_4\phi_3\left[\begin{matrix} q^{-n},\, b,\, c,\, -q^{1-n}/bc \\ q^{1-n}/b,\, q^{1-n}/c,\, -bc \end{matrix};q,\,q\right]=\begin{cases}\dfrac{(q,b^2,c^2;q^2)_m(bc;q)_{2m}}{(b,c;q)_{2m}(b^2c^2;q^2)_m}, & \text{if } n=2m,\\ 0, & \text{if } n=2m+1.\end{cases} \tag{2.1}$$

The following lemma is the $c=q$ case of Theorem 2.1.

Lemma 2.2. *For $n\geq 0$, there holds*

$$\sum_{k=0}^{n}(-1)^k\frac{(b;q)_k(b;q)_{n-k}}{(-b;q)_{n-k+1}(-bq;q)_k}=\begin{cases}\dfrac{(b;q)_{n+1}}{(-bq;q)_n(1-b^2q^n)}, & \text{if } n \text{ is even},\\ 0, & \text{if } n \text{ is odd}.\end{cases} \tag{2.2}$$

Proof. Since the k-th term and $(n-k)$-th term only differ by a factor $(-1)^n$, the left-hand side of (2.2) is equal to 0 if n is odd.

Now consider the case where n is even. Since

$$(1-b^2q^n) = (1-bq^{n-k}) - (1-bq^k) + (1+bq^{n-k})(1-bq^k),$$

we have

$$\begin{aligned}(1-b^2q^n)&\sum_{k=0}^{n}(-1)^k\frac{(b;q)_k(b;q)_{n-k}}{(-b;q)_{n-k+1}(-bq;q)_k}\\
&=\sum_{k=0}^{n}(-1)^k\frac{(b;q)_k(b;q)_{n+1-k}-(b;q)_{k+1}(b;q)_{n-k}}{(-b;q)_{n-k+1}(-bq;q)_k}+\sum_{k=0}^{n}(-1)^k\frac{(b;q)_{k+1}(b;q)_{n-k}}{(-b;q)_{n-k}(-bq;q)_k}. \quad (2.3)\end{aligned}$$

Noticing that in the first summation of (2.3) the k-th term and the $(n-k)$-th term only differ by a sign, we immediately get

$$\sum_{k=0}^{n}(-1)^k\frac{(b;q)_k(b;q)_{n+1-k}-(b;q)_{k+1}(b;q)_{n-k}}{(-b;q)_{n-k+1}(-bq;q)_k}=0. \quad (2.4)$$

For the same reason, we have

$$\sum_{k=0}^{n-1}(-1)^k\frac{(b;q)_{k+1}(b;q)_{n-k}}{(-b;q)_{n-k}(-b;q)_{k+1}}=0. \quad (2.5)$$

The proof then follows from combining (2.3)–(2.5). ∎

Proof of Theorem 2.1. It suffices to prove it for the cases $c=q^M$ $(M\geq 1)$. We proceed by induction on M. For $M=1$, Equation (2.1) reduces to (2.2).

Assume that (2.1) holds for $c=q^M$. Let

$$F_{n,k}(b,c,q)=\frac{(q^{-n};q)_k(b;q)_k(c;q)_k(-q^{1-n}/bc;q)_kq^k}{(q;q)_k(q^{1-n}/b;q)_k(q^{1-n}/c;q)_k(-bc;q)_k}.$$

Applying the contiguous relation (1.1) with $a_r=b$ and $a_{r+1}=-q^{-n}/bc$, we have

$$F_{n,k}(bq,c,q)-F_{n,k}(b,c,q)=\alpha_nF_{n-2,k-1}(bq,cq,q), \quad (2.6)$$

where

$$\alpha_n=\frac{(b+q^{-n}/bc)(1-c^2)(1-q^{-n})(1-q^{1-n})q}{(1-q^{-n}/b)(1-q^{1-n}/b)(1-q^{1-n}/c)(1+bc)(1+bcq)}.$$

Let

$$S_n(b,c,q):=\sum_{k=0}^{n}F_{n,k}(b,c,q).$$

Then summing (2.6) over k from 0 to n gives

$$S_{n-2}(bq,cq,q)=\alpha_n^{-1}(S_n(bq,c,q)-S_n(b,c,q)),$$

from which one can readily check that (2.1) holds for $cq=q^{M+1}$. ∎

3 Carlitz's ${}_5\phi_4$ summation formula

By using the q-Pfaff-Saalschütz formula, Carlitz [5] obtained the following ${}_5\phi_4$ summation formula, which is a generalization of [20, Eq. (1)]. Simple proofs of Carlitz's formula have already been given by Guo [15] and Chu and Jia [10]. Here we give a simpler one.

Theorem 3.1 (Carlitz's ${}_5\phi_4$ summation formula)**.** *For $n \geq 0$, there holds*

$$
\begin{aligned}
&{}_5\phi_4\left[\begin{matrix} q^{-n},\, b,\, c,\, d,\, e \\ q^{1-n}/b,\, q^{1-n}/c,\, q^{1-n}/d,\, q^{1-n}/e \end{matrix}; q,\, q\right] \\
&\quad = q^{m(1+m-n)}(de)^{-m}\frac{(q^{-m};q)_{2m}(q^{1-n}/bc, q^{1-n}/bd, q^{1-n}/be;q)_m(q^{2m-n};q)_{n-2m}}{(q, q^{1-n}/b, q^{1-n}/c, q^{1-n}/d, q^{1-n}/e, q^{n-m}c;q)_m},
\end{aligned}
\tag{3.1}
$$

where $bcde = q^{1+m-2n}$ and $m = \lfloor n/2 \rfloor$.

Proof. It suffices to prove the cases $d = -q^{-M}$ $(M \geq \lfloor n/2 \rfloor)$. We proceed by induction on M. For $M = \lfloor n/2 \rfloor$, the identity (3.1) reduces to (2.1). Assume that (3.1) holds for $d = -q^{-M}$.

Let

$$F_{n,k}(b,c,d,q) = \frac{(q^{-n}, b, c, d, q^{1-\lfloor (3n+1)/2 \rfloor}/bcd;q)_k}{(q, q^{1-n}/b, q^{1-n}/c, q^{1-n}/d, q^{\lfloor (n+1)/2 \rfloor}bcd;q)_k}q^k.$$

Applying the contiguous relation (1.1), we have

$$F_{n,k}(b,c,dq,q) - F_{n,k}(b,c,d,q) = \alpha_n F_{n-2,k-1}(bq,cq,dq,q), \tag{3.2}$$

where

$$\alpha_n = -\frac{d(1-b)(1-c)(1-q^{\lfloor (3n+1)/2 \rfloor}bcd^2)(1-q^{\lfloor (n+1)/2 \rfloor}bc)(1-q^n)(1-q^{n-1})q^{\lfloor n/2 \rfloor - 1}}{(1-q^{\lfloor (n+1)/2 \rfloor}bcd)(1-q^{\lfloor (n+3)/2 \rfloor}bcd)(1-q^{n-1}b)(1-q^{n-1}c)(1-q^{n-1}d)(1-q^n d)}.$$

Let

$$S_n(b,c,d,q) := \sum_{k=0}^{n} F_{n,k}(b,c,d,q).$$

Then summing (3.2) over k from 0 to n gives

$$S_n(b,c,d,q) = S_n(b,c,dq,q) - \alpha_n S_{n-2}(bq,cq,dq,q),$$

from which one can verify that (3.1) holds for $d = -q^{-M-1}$. ∎

4 Sears' ${}_3\phi_2$ to ${}_5\phi_4$ transformation

The following transformation was first obtained by Sears [27, (4.1)]. See also Carlitz [5, (2.4)].

Theorem 4.1 (Sears' ${}_3\phi_2$ to ${}_5\phi_4$ transformation)**.** *For $a = q^{-n}$, $n = 0, 1, 2, \ldots$, there holds*

$$
{}_3\phi_2\left[\begin{matrix} a,\, b,\, c \\ aq/b,\, aq/c \end{matrix}\,;q,\ \frac{aqx}{bc}\right] = \frac{(ax;q)_\infty}{(x;q)_\infty}\,{}_5\phi_4\left[\begin{matrix} a^{\frac{1}{2}},\, -a^{\frac{1}{2}},\, (aq)^{\frac{1}{2}},\, -(aq)^{\frac{1}{2}},\, aq/bc \\ aq/b,\, aq/c,\, ax,\, q/x, \end{matrix}\,;q,\ q\right]. \tag{4.1}
$$

Proof. Let

$$
F_{n,k}(x,b,c,q) = \frac{(q^{-n}, b, c; q)_k}{(q^{1-n}/b, q^{1-n}/c, q; q)_k}\left(\frac{q^{1-n}x}{bc}\right)^k.
$$

Applying the contiguous relation (1.2) with $a_{r+1} = c$ and $z = xq^{-n}/bc$, we have

$$
F_{n,k}(x,b,c,q) - F_{n,k}(x,b,cq,q) = \beta_n F_{n-2,k-1}(xq^{-1}, bq, cq, q),
$$

where

$$
\beta_n = -\frac{(1-q^{-n})(1-q^{1-n})(1-b)xq^{-n}/bc}{(1-q^{-n}/c)(1-q^{1-n}/b)(1-q^{1-n}/c)}.
$$

Let

$$
G_{n,k}(x,b,c,q) = \frac{(q^{-n}x;q)_n(q^{-n};q)_{2k}(q^{1-n}/bc;q)_k q^k}{(q, q^{1-n}/b, q^{1-n}/c, q^{-n}x, q/x; q)_k}.
$$

It is easy to verify that

$$
G_{n,k}(x,b,c,q) - G_{n,k}(x,b,cq,q) - \beta_n G_{n-2,k-1}(xq^{-1}, bq, cq, q) = 0.
$$

On the other hand, for $c = q^{1-n}/b$, the identity (4.1) reduces to the terminating q-binomial theorem (see, for example, [1, (3.3.6)]).

5 Sears' ${}_4\phi_3$ transformations

Sears [28] derived the following two transformations of ${}_4\phi_3$ series. The first one is a q-analogue of Whipple's formula [29]. It is also one of the fundamental formulas in the theory of the basic hypergeometric series. There are several different proofs in the literature. See Andrews and Bowman [2], Ismail [18], Liu [24] and Fang [11]. Here we shall give another simple proof of Sears' ${}_4\phi_3$ transformations.

Theorem 5.1 (Sears' ${}_4\phi_3$ transformations)**.** *If $n \geq 0$, there holds*

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n},\, a,\, b,\, c \\ d,\, e,\, f \end{matrix}\,;q,\ q\right] \\
&= \frac{(e/a, f/a; q)_n}{(e, f; q)_n} a^n\,{}_4\phi_3\left[\begin{matrix} q^{-n},\, a,\, d/b,\, d/c \\ d,\, aq^{1-n}/e,\, aq^{1-n}/f, \end{matrix}\,;q,\ q\right] &(5.1)\\
&= \frac{(a, ef/ab, ef/ac; q)_n}{(e, f, ef/abc; q)_n}\,{}_4\phi_3\left[\begin{matrix} q^{-n},\, e/a,\, f/a,\, ef/abc \\ ef/ab,\, ef/ac,\, q^{1-n}/a, \end{matrix}\,;q,\ q\right], &(5.2)
\end{aligned}
$$

where $def = abcq^{1-n}$.

Proof. Let

$$F_{n,k}(a,b,d,e,f,q)=\frac{(q^{-n},a,b,def/abq^{1-n};q)_k}{(q,d,e,f;q)_k}q^k$$

be the k-th term in the left-hand side of (5.1). One may check that

$$F_{n,k}(aq,b,d,e,f,q)-F_{n,k}(a,b,d,e,f,q)=\alpha_n F_{n-1,k-1}(aq,bq,dq,eq,fq,q), \tag{5.3}$$

where

$$\alpha_n=\frac{(1-q^{-n})(1-b)(a-defq^{n-2}/ab)q}{(1-d)(1-e)(1-f)}.$$

Let

$$G_{n,k}(a,b,d,e,f,q)=\frac{(e/a,f/a;q)_n a^n}{(e,f;q)_n}\frac{(q^{-n},a,d/b,abq^{1-n}/ef;q)_k}{(q,d,aq^{1-n}/e,aq^{1-n}/f;q)_k}q^k.$$

It is easy to verify that

$$\begin{aligned}&G_{n,k}(aq,b,d,e,f,q)-G_{n,k}(a,b,d,e,f,q)-\alpha_n G_{n-1,k-1}(aq,bq,dq,eq,fq,q)\\&\quad=H_{n,k}-H_{n,k-1},\end{aligned} \tag{5.4}$$

where

$$H_{n,k}=\frac{a^n(a^2q^{k-n+2}/ef-1)(e/a,f/a;q)_n(q^{-n};q)_{k+1}(aq,d/b,abq^{2-n}/ef;q)_k}{(e,f;q)_n(q,d;q)_k(aq^{1-n}/e,aq^{1-n}/f;q)_{k+1}}.$$

It follows from (5.3) and (5.4) that (5.1) is true for all $a=q^{-M}$ $(M\geq 0)$.

To prove (5.2), let

$$P_{n,k}(a,b,d,e,f,q)=\frac{(a,ef/ab,bq^{1-n}/d;q)_n}{(e,f,q^{1-n}/d;q)_n}\frac{(q^{-n},e/a,f/a,q^{1-n}/d;q)_k}{(q,ef/ab,bq^{1-n}/d,q^{1-n}/a;q)_k}$$

Then we have

$$\begin{aligned}&P_{n,k}(aq,b,d,e,f,q)-P_{n,k}(a,b,d,e,f,q)-\alpha_n P_{n-1,k-1}(aq,bq,dq,eq,fq,q)\\&\quad=Q_{n,k}-Q_{n,k-1},\end{aligned} \tag{5.5}$$

where

$$Q_{n,k}=\frac{q^n(efq^{k-1}-a^2b)(aq,ef/ab;q)_{n-1}(bq^{1-n}/d;q)_n(q^{-n};q)_{k+1}(e/a,f/a,q^{1-n}/d;q)_k}{ab(e,f,q^{1-n}/d;q)_n(q,ef/ab,bq^{1-n}/d,q^{1-n}/a;q)_k}.$$

It follows from (5.3) and (5.5) that (5.2) is true for all $a=q^{-M}$ $(M\geq 0)$. ∎

6 Chen's bibasic summation formula

Chu [9] established the following bibasic summation formula:

$$
\begin{aligned}
&\sum_{k=-m}^{n} \frac{(1-\alpha a_k b_k)(b_k - a_k/\alpha d)}{(1-\alpha a_0 b_0)(b_0 - a_0/\alpha d)} \frac{\prod_{j=0}^{k-1}(1-a_j)(1-a_j/d)(1-cb_j)(1-\alpha^2 db_j/c)}{\prod_{j=1}^{k}(1-\alpha b_j)(1-\alpha a_j/c)(1-\alpha db_j)(1-ca_j/\alpha d)} \\
&= \frac{(1-a_0)(1-a_0/d)(1-cb_0)(1-\alpha^2 db_0/c)}{\alpha(1-\alpha a_0 b_0)(1-c/\alpha)(b_0 - a_0/\alpha d)(1-\alpha d/c)} \\
&\quad \cdot \left(\prod_{j=1}^{n} \frac{(1-a_j)(1-a_j/d)(1-cb_j)(1-\alpha^2 db_j/c)}{(1-\alpha b_j)(1-\alpha a_j/c)(1-\alpha db_j)(1-ca_j/\alpha d)} \right. \\
&\quad \left. - \prod_{j=-m}^{0} \frac{(1-\alpha b_j)(1-\alpha a_j/c)(1-\alpha db_j)(1-ca_j/\alpha d)}{(1-a_j)(1-a_j/d)(1-cb_j)(1-\alpha^2 db_j/c)} \right),
\end{aligned} \tag{6.1}
$$

which is a generalization of Gasper and Rahman's bibasic summation formula [14, (3.6.13)].

Setting $c = \alpha/B_n$ and $b_k = B_k/\alpha$ for $k = 0, 1, \ldots$, Chen [6] noticed that (6.1) can be simplified as

$$
F(n) = \sum_{k=0}^{n} \frac{(1-a_k B_k)(dB_k - a_k)}{(1-a_0 B_n)(dB_n - a_0)} \frac{\sum_{j=0}^{k-1}(B_n - B_j)(1-dB_n B_j)}{\prod_{j=1}^{k}(1-B_n a_j)(dB_n - a_j)} G(k), \tag{6.2}
$$

where

$$
F(n) = \frac{(1-B_0)(1-dB_0)}{(1-B_n)(1-dB_n)}, \qquad G(n) = \frac{\prod_{j=0}^{n-1}(1-a_j)(d-a_j)}{\prod_{j=1}^{n}(1-B_j)(1-dB_j)}.
$$

Then applying Krattenthaler's matrix inverse [23] to (6.2), Chen [6] obtained

$$
G(n) = \sum_{k=0}^{n} \frac{\prod_{j=0}^{n-1}(1-B_k a_j)(dB_k - a_j)}{\prod_{j=0,\, j\neq k}(1-dB_k B_j)(B_k - B_j)} F(k). \tag{6.3}
$$

In particular, when $a_k = ap^k$ and $B_k = bq^k$, the identity (6.3) reduces to the following bibasic summation formula.

Theorem 6.1 (Chen's bibasic summation formula)**.** *If $n \geq 0$, there holds*

$$
\sum_{k=0}^{n} \frac{(1-b)(1-db)(abq^k, a/dbq^k; p)_n (-1)^k q^{\binom{k+1}{2}}}{(1-bq^k)(1-dbq^k)(q, db^2q^k; q)_k (q, db^2 q^{2k+1}; q)_{n-k}} = \frac{(a, a/d; p)_n}{(bq, dbq; q)_n}. \tag{6.4}
$$

Proof. Let

$$
F_{n,k}(a,b,d,q) = \frac{(1-b)(1-db)(abq^k, a/dbq^k; p)_n (-1)^k q^{\binom{k+1}{2}}}{(1-bq^k)(1-dbq^k)(q, db^2q^k; q)_k (q, db^2 q^{2k+1}; q)_{n-k}}.
$$

Similarly to the contiguous relation (1.1), we have

$$F_{n,k}(a,b,d,q)-\alpha_n F_{n-1,k}(a,b,d,q)=\beta_n F_{n-1,k-1}(a,bq,d,q), \tag{6.5}$$

where

$$\begin{aligned}\alpha_n &= \frac{(1-abp^{n-1})(1-ap^{n-1}/bd)}{(1-q^n)(1-db^2q^n)},\\ \beta_n &= \frac{(abq^np^{n-1}-1)(bdq^n-ap^{n-1})(1-bd)(1-b)}{db(1-bdq)(1-bq)(1-db^2q^n)(1-q^n)}.\end{aligned}$$

The proof then follows from summing (6.5) over k from 0 to n and proceeding by induction on n. ∎

Note that, the $p=q$ case of (6.4) may be rewritten as

$${}_8\phi_7\left[\begin{matrix} db^2,\ q\sqrt{db^2},\ -q\sqrt{db^2},\ b,\ db,\ dbq/a,\ abq^n,\ q^{-n} \\ \sqrt{db^2},\ -\sqrt{db^2},\ dbq,\ bq,\ ab,\ dbq^{1-n}/a,\ db^2q^{1+n}\end{matrix};q,\ q\right]=\frac{(q,a,a/d,db^2q;q)_n}{(ab,bq,dbq,a/db;q)_n},$$

which is a special case of Jackson's ${}_8\phi_7$ summation formula [20] (see [14, (II.22)]).

7 Gasper's split poised ${}_{10}\phi_9$ transformation

In this section, we show that Gasper's split poised ${}_{10}\phi_9$ transformation formula (see [12]) can also be proved by the same method.

Theorem 7.1 (Gasper's split poised ${}_{10}\phi_9$ transformation)**.** *For $n\ge 0$, there holds*

$$\begin{aligned}&{}_{10}\phi_9\left[\begin{matrix} a,\ qa^{\frac12},\ -qa^{\frac12},\ b,\ c,\ a/bc,\ C/Aq^n,\ 1/BCq^n,\ B/Aq^n,\ q^{-n} \\ a^{\frac12},\ -a^{\frac12},\ aq/b,\ aq/c,\ bcq,\ 1/Cq^n,\ BC/Aq^n,\ 1/Bq^n,\ 1/Aq^n\end{matrix};q,\ q\right]\\ &=\frac{(aq,bq,cq,aq/bc,Aq/B,Aq/C,BCq;q)_n}{(Aq,Bq,Cq,Aq/BC,aq/b,aq/c,bcq;q)_n}\\ &\quad\times{}_{10}\phi_9\left[\begin{matrix} A,\ qA^{\frac12},\ -qA^{\frac12},\ B,\ C,\ A/BC,\ c/aq^n,\ 1/bcq^n,\ b/aq^n,\ q^{-n} \\ A^{\frac12},\ -A^{\frac12},\ Aq/B,\ Aq/C,\ BCq,\ 1/cq^n,\ bc/aq^n,\ 1/bq^n,\ 1/aq^n\end{matrix};q,\ q\right],\end{aligned} \tag{7.1}$$

where $\lambda=a^2q/bcd$.

Proof. Let

$$F_{n,k}(a,b,c,A,B,C,q)=\frac{(a,qa^{\frac12},-qa^{\frac12},b,c,a/bc,C/Aq^n,1/BCq^n,B/Aq^n,q^{-n};q)_k}{(a^{\frac12},-a^{\frac12},aq/b,aq/c,bcq,1/Cq^n,BC/Aq^n,1/Bq^n,1/Aq^n;q)_k}q^k.$$

Similarly to the contiguous relation (1.1) with $a_r=c$ and $a_{r+1}=a/bcq$, we have

$$F_{n,k}(a,b,cq,A,B,C,q)-F_{n,k}(a,b,c,A,B,C,q)=\alpha_n F_{n-1,k-1}(aq^2,bq,cq,A,B,C,q), \tag{7.2}$$

where

$$\alpha_n = \frac{(c-a/bcq)(1-bq)(1-aq)(1-aq^2)(1-b)}{(1-a/c)(1-bcq)(1-aq/b)(1-aq/c)(1-bcq^2)} \times \frac{(1-C/Aq^n)(1-1/BCq^n)(1-B/Aq^n)(1-q^{-n})q}{(1-1/Cq^n)(1-BC/Aq^n)(1-1/Bq^n)(1-1/Aq^n)}.$$

Let

$$G_{n,k}(a,b,c,A,B,C,q) = \frac{(aq,bq,cq,aq/bc,Aq/B,Aq/C,BCq;q)_n}{(Aq,Bq,Cq,Aq/BC,aq/b,aq/c,bcq;q)_n} \times F_{n,k}(A,B,C,a,b,c,q).$$

Then we may verify that

$$G_{n,k}(a,b,cq,A,B,C,q) - G_{n,k}(a,b,c,A,B,C,q) - \alpha_n G_{n-1,k-1}(aq^2,bq,cq,A,B,C,q) = H_{n,k} - H_{n,k-1}, \tag{7.3}$$

where

$$H_{n,k} = \frac{a(1-b)(1-q^{n-k})(1-aq^{n-k+1})(1-bc^2q/a)}{b(a-c)(1-aq^{n-k}/bc)(1-bcq^{n-k+1})(1-cq)} G_{n,k}(a,b,c,A,B,C,q).$$

Summing (7.2) and (7.3) over k from 0 to n respectively, one sees that both sides of (7.1) satisfy the same recurrence relation

$$S_n(a,b,cq,A,B,C,q) - S_n(a,b,c,A,B,C,q) = \alpha_n S_{n-1}(aq^2,bq,cq,A,B,C,q). \tag{7.4}$$

On the other hand, for the case $c=1$, Eq. (7.1) reduces to

$$1 = \frac{(q,Aq/B,Aq/C,BCq;q)_n}{(Aq,Bq,Cq,Aq/BC;q)_n} \sum_{k=0}^{n} \frac{(1-Aq^{2k})(A,B,C,A/BC;q)_k}{(1-A)(q,Aq/B,Aq/C,BCq;q)_k} q^k,$$

which is almost trivial since

$$\frac{(1-Aq^{2k})(A,B,C,A/BC;q)_k}{(1-A)(q,Aq/B,Aq/C,BCq;q)_k} q^k = \frac{(Aq,Bq,Cq,Aq/BC;q)_k}{(q,Aq/B,Aq/C,BCq;q)_k} - \frac{(Aq,Bq,Cq,Aq/BC;q)_{k-1}}{(q,Aq/B,Aq/C,BCq;q)_{k-1}}.$$

Thus, by (7.4), Eq. (7.1) holds for all $c = q^{-M}$ $(M \geq 0)$. This completes the proof. ∎

Chu [9, Corollary 24] has generalized Theorem 7.1 to a bibasic symmetric transformation as follows.

Theorem 7.2 (Chu's bibasic symmetric transformation). *For $n \geq 0$, there holds*

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{(1-ap^{2k})(a,b,c,a/bc;p)_k(C/Aq^n,1/BCq^n,B/Aq^n,q^{-n};q)_k}{(1-a)(p,ap/b,ap/c,bcp;p)_k(1/Cq^n,BC/Aq^n,1/Bq^n,1/Aq^n;q)_k} p^k \\
&\quad = \frac{(ap,bp,cp,ap/bc;p)_n(q,Aq/B,Aq/C,BCq;q)_n}{(p,ap/b,ap/c,bcp;p)_n(Aq,Bq,Cq,Aq/BC;q)_n} \\
&\quad\quad \times \sum_{k=0}^{n} \frac{(1-Aq^{2k})(A,B,C,A/BC;q)_k(c/ap^n,1/bcp^n,b/ap^n,p^{-n};p)_k}{(1-A)(q,Aq/B,Aq/C,BCq;q)_k(1/cp^n,bc/ap^n,1/bp^n,1/ap^n;p)_k} q^k. \qquad (7.5)
\end{aligned}$$

We point out that (7.5) can be proved in the same way as (7.1). Let $F_{n,k}(a,b,c,A,B,C;p,q)$ and $G_{n,k}(a,b,c,A,B,C;p,q)$ be the k-th summands in the left-hand side and right-hand side of (7.5) respectively. Then there exist relations for $F_{n,k}$ and $G_{n,k}$ exactly similar to (7.2) and (7.3). For instance, we have

$$F_{n,k}(a,b,cp,A,B,C;p,q) - F_{n,k}(a,b,c,A,B,C;p,q) = \alpha_n F_{n-1,k-1}(ap^2,bp,cp,A,B,C;p,q),$$

where

$$\begin{aligned}
\alpha_n = &\frac{(c-a/bcp)(1-bp)(1-ap)(1-ap^2)(1-b)}{(1-a/c)(1-bcp)(1-ap/b)(1-ap/c)(1-bcp^2)} \\
&\times \frac{(1-C/Aq^n)(1-1/BCq^n)(1-B/Aq^n)(1-q^{-n})p}{(1-1/Cq^n)(1-BC/Aq^n)(1-1/Bq^n)(1-1/Aq^n)}.
\end{aligned}$$

8 A finite form of Sylvester's identity and Jacobi's triple product identity

From now on, we assume that $|q| < 1$. In the same vein of (1.4), we will prove a finite form of Sylvester's identity and Jacobi's triple product identity. Recall that the *q-binomial coefficients* are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \quad n,k \in \mathbb{Z}.$$

Theorem 8.1 (A finite form of Sylvester's and Jacobi's identities). *For $n \geq 0$, there holds*

$$\sum_{k=0}^{n} (-1)^k x^k q^{k(3k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(1-xq^{2k+1})}{(xq^{k+1};q)_{n+1}} = 1. \qquad (8.1)$$

Proof. Let

$$F_{n,k}(x,q) = (-1)^k x^k q^{k(3k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(1-xq^{2k+1})}{(xq^{k+1};q)_{n+1}}.$$

Noticing the trivial relation

$$\begin{bmatrix} n \\ k \end{bmatrix}(1 - xq^{n+1}) = \begin{bmatrix} n-1 \\ k \end{bmatrix}(1 - xq^{n+k+1}) + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}(1 - xq^{k+1})q^{n-k}, \tag{8.2}$$

we have

$$F_{n,k}(x, q) = \frac{1}{1 - xq^{n+1}} F_{n-1,k}(x, q) - \frac{xq^{n+1}}{1 - xq^{n+1}} F_{n-1,k-1}(xq^2, q). \qquad \blacksquare$$

Letting $n \to \infty$ in (8.1), we immediately obtain Sylvester's identity [1, (9.2.3)]:

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^k q^{k(3k+1)/2}(1 - xq^{2k+1})}{(q;q)_k (xq^{k+1};q)_\infty} = 1. \tag{8.3}$$

On the other hand, if we make the substitutions $n \to m+n$, $x \to -xq^{-2m}$ and $k \to m+k$ in (8.1), and apply the following relation

$$\frac{(-xq^{-2m})^{m+k} q^{(3(m+k)^2+m+k)/2}}{(xq^{k-m+1};q)_{m+n+1}} = \frac{x^{2k} q^{2k^2+k}}{(1/x;q)_{m-k}(xq;q)_{n+k+1}}, \tag{8.4}$$

we obtain

$$\sum_{k=-m}^{n} \begin{bmatrix} m+n \\ m+k \end{bmatrix} \frac{(1 - xq^{2k+1}) x^{2k} q^{2k^2+k}}{(1/x;q)_{m-k}(xq;q)_{n+k+1}} = 1. \tag{8.5}$$

Letting $m, n \to \infty$ in (8.5), we immediately get

$$\sum_{k=-\infty}^{\infty} (1 - xq^{2k+1}) x^{2k} q^{2k^2+k} = (1/x, xq, q; q)_\infty, \tag{8.6}$$

which is Jacobi's triple product identity (see [1, p. 21]).

Note that finite forms of the quintuple product identity (see [14, p. 147])

$$\sum_{k=-\infty}^{\infty} (z^2 q^{2k+1} - 1) z^{3k+1} q^{k(3k+1)/2} = (q, z, q/z; q)_\infty (qz^2, q/z^2; q^2)_\infty,$$

which is related to Jacobi's triple product identity, were given by [7, 17, 25].

9 A finite form of Kang's identity

In this section, we give a finite form of Kang's [19] generalization of Sylvester's identity. Similarly to the previous section, we only give the corresponding contiguous relation for the kth terms in our proof.

Theorem 9.1. *For $n \geq 0$, there holds*

$$\sum_{k=0}^{n}(-1)^k x^k q^{k(3k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(cq^{-k}, dq^{-k}; q)_k (1 - xq^{2k+1})}{(cx, dx; q)_{k+1}(xq^{k+1}; q)_{n+1}} = \frac{(cdx; q)_n}{(cx, dx; q)_{n+1}}. \tag{9.1}$$

Proof. Let

$$F_{n,k}(c, d, x, q) = (-1)^k x^k q^{k(3k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(cq^{-k}, dq^{-k}; q)_k (1 - xq^{2k+1})}{(cx, dx; q)_{k+1}(xq^{k+1}; q)_{n+1}}.$$

Then

$$\begin{aligned} F_{n,k}(c, d, x, q) = & \frac{1}{1 - xq^{n+1}} F_{n-1,k}(c, d, x, q) \\ & - \frac{(1 - cq^{-1})(1 - dq^{-1})xq^{n+1}}{(1 - cx)(1 - dx)(1 - xq^{n+1})} F_{n-1,k-1}(cq^{-1}, dq^{-1}, xq^2, q). \end{aligned}$$

∎

Letting $n \to \infty$ in (9.1), we obtain the following identity due to Kang [19, (6.9)]:

$$\sum_{k=0}^{\infty}(-1)^k x^k q^{k(3k+1)/2} \frac{(aq^{-k}, bq^{-k}, xq; q)_k (1 - xq^{2k+1})}{(q; q)_k (ax, bx; q)_{k+1}} = \frac{(abx, xq; q)_\infty}{(ax, bx; q)_\infty}$$

for $|ax| < 1$ and $|bx| < 1$.

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