A FAMILY OF q-SUPERCONGRUENCES FROM WATSON'S ${}_8\phi_7$ TRANSFORMATION

VICTOR J. W. GUO

ABSTRACT. We prove a family of q-supercongruences modulo the fourth power of a cyclotomic polynomial. Our results may be regarded as a uniform generalization of a q-supercongruence by Wei, two q-supercongruences by Liu and Wang, and two previous q-supercongruences by the author himself. Our proof makes use of the method of "creative microscoping" introduced by the author and Zudilin, Watson's ${}_8\phi_7$ transformation formula, and the Chinese remainder theorem for coprime polynomials.

1. INTRODUCTION

In his second letter written to Hardy on February 27th, 1913, Ramanujan mentioned the following infinite series:

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5}=\frac{2}{\Gamma(\frac{3}{4})^4}, \tag{1.1}$$

where $\Gamma(x)$ is the *Gamma function* and $(x)_k=x(x+1)\cdots(x+k-1)$ denotes the *Pochhammer symbol.* Van Hamme [23, eq. (A.2)] conjectured that (1.1) possesses a p-adic analogue as follows:

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^5}{k!^5}\equiv\begin{cases}-\dfrac{p}{\Gamma_p(\frac{3}{4})^4} \pmod{p^3}, & \text{if } p\equiv 1 \pmod 4,\\ 0 \pmod{p^3}, & \text{if } p\equiv 3 \pmod 4,\end{cases} \tag{1.2}$$

where p is an odd prime and $\Gamma_p(x)$ stands for the *p-adic Gamma function.* The congruence (1.2) was first confirmed by McCarthy and Osburn [19]. Swisher [22] proved that (1.2) is also true modulo p^5 for $p>5$ and $p\equiv 1 \pmod 4$, and Liu [15] and Wei [26] extended the second part of (1.2) to the moduli p^4 and p^5 cases for $p>3$, respectively.

Van Hamme [23, (C.2)] himself proved that, for odd primes p,

$$\sum_{k=0}^{(p-1)/2}(4k+1)\frac{(\frac{1}{2})_k^4}{k!^4}\equiv p \pmod{p^3}, \tag{1.3}$$

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and Long [18] showed that (1.3) is also true modulo p^4 for any prime $p>3$. Besides, applying the fact that the Calabi–Yau threefold given by

$$x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}+w+\frac{1}{w}=0$$

over $\mathbb{Q}$ is modular (see [1]), Kilbourn [14] confirmed Van Hamme's (M.2) supercongruence: for any odd prime p,

$$\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{2})_k^4}{k!^4}\equiv a_p \pmod{p^3}, \tag{1.4}$$

where

$$\sum_{n=1}^{\infty}a_nq^n=q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4.$$

By making use of Whipple's ${}_7F_6$ transformation, Long [18] established the supercongruence: for primes $p>3$,

$$\sum_{k=0}^{(p-1)/2}(4k+1)\frac{(\frac{1}{2})_k^6}{k!^6}\equiv p\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{2})_k^4}{k!^4} \pmod{p^4}, \tag{1.5}$$

which in light of (1.4) may be rewritten as: for primes $p>3$,

$$\sum_{k=0}^{(p-1)/2}(4k+1)\frac{(\frac{1}{2})_k^6}{k!^6}\equiv pa_p \pmod{p^4}.$$

Motivated by Long's work, Barman and Saikia [2, Theorem 1.2] proved the following generalization of (1.3): for any prime $p\geqslant 5$ with $p\equiv 1 \pmod{d}$,

$$\sum_{k=0}^{p-1}(2dk+1)\frac{(\frac{1}{d})_k^4}{k!^4}\equiv (-1)^{1+(p-1)/d}p\Gamma_p(\tfrac{1}{d})^2\Gamma_p(\tfrac{d-2}{d}) \pmod{p^4}. \tag{1.6}$$

The author and Wang [11, Theorem 1.2] presented a q-analogue of (1.3): for positive odd integers n,

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}\equiv q^{(1-n)/2}[n]+\frac{(n^2-1)(1-q)^2}{24}q^{(1-n)/2}[n]^3 \pmod{[n]\Phi_n(q)^3}. \tag{1.7}$$

Here and in what follows, for indeterminates a and q, $[n]=[n]_q=(1-q^n)/(1-q)$ denotes the *q-integer*; $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ stands for the *q-shifted factorial*, and for brevity, we adopt the notation $(a_1,\ldots,a_m;q)_n=(a_1;q)_n\cdots(a_m;q)_n$. Moreover, $\Phi_n(q)$ represents the nth *cyclotomic polynomial* in q, which is irreducible over the ring of integers and can be factorized over the field of complex numbers as

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}} (q-\zeta^k),$$

where ζ is an nth primitive root of unity. From (1.7) the author and Wang [11] deduced that, for primes $p>3$ and positive r,

$$\sum_{k=0}^{(p^r-1)/2} (4k+1)\frac{(\frac{1}{2})_k^4}{k!^4} \equiv p^r \pmod{p^{r+3}},$$

thus confirming a previous result observed by Long [18].

Moreover, the author and Zudilin [13] devised a new method which they named "creative microscoping" to deal with many q-supercongruences conveniently. Employing this method along with the Chinese remainder theorem for coprime polynomials, the author [5] provided a new proof of (1.7). Meanwhile, the author and Schlosser [7] gave the following q-supercongruence: for odd integers $n>1$,

$$\sum_{k=0}^{(n+1)/2} [4k-1]\frac{(q^{-1};q^2)_k^4}{(q^2;q^2)_k^4} q^{4k} \equiv -(1+3q+q^2)[n]^4 \pmod{[n]^4\Phi_n(q)}. \tag{1.8}$$

Before long, they [8] gave a generalization of the modulus $[n]\Phi_n(q)^3$ case of (1.8) as follows: Let d,n,r be integers such that $d\geqslant 2$, $r\leqslant d-2$, $n\geqslant d-r$, $\gcd(d,r)=1$, and $n\equiv -r \pmod{d}$. Then

$$\sum_{k=0}^{M}[2dk+r]\frac{(q^r;q^d)_k^4}{(q^d;q^d)_k^4}q^{(d-2r)k}$$
$$\equiv \begin{cases} 0 \pmod{[n]\Phi_n(q)^3} & \text{if } d=2,\\ q^{r(n+r-dn)/d}\dfrac{(q^{2r};q^d)_{(dn-n-r)/d}}{(q^d;q^d)_{(dn-n-r)/d}}[dn-n] \pmod{[n]\Phi_n(q)^3} & \text{if } d\geqslant 3, \end{cases} \tag{1.9}$$

where $M=(dn-n-r)/d$ or $n-1$.

On the other hand, the author and Schlosser [9, Theorem 2.1] gave a partial q-analogue of (1.5) as follows: for any positive odd integer n,

$$\sum_{k=0}^{(n-1)/2} [4k+1]\frac{(q;q^2)_k^6}{(q^2;q^2)_k^6} q^k \equiv [n]q^{(1-n)/2}\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}q^{2k} \pmod{[n]\Phi_n(q)^2}. \tag{1.10}$$

Shortly afterwards, Wei [25] proved the following complete q-analogue of (1.5): modulo $[n]\Phi_n(q)^3$,

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{(q;q^2)_k^6}{(q^2;q^2)_k^6}q^k \equiv [n]q^{(1-n)/2}\left\{1+[n]^2\frac{(n^2-1)(1-q)^2}{24}\right\}\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}q^{2k}, \tag{1.11}$$

by establishing the following more general form: for positive odd integers n, modulo $[n]\Phi_n(q)^3$,

$$\sum_{k=0}^{M}[4k+1]\frac{(q;q^2)_k^4(xq,yq;q^2)_k}{(q^2;q^2)_k^4(q^2/x,q^2/y;q^2)_k}\left(\frac{q}{xy}\right)^k$$
$$\equiv [n]q^{(1-n)/2}\left\{1+[n]^2\frac{(n^2-1)(1-q)^2}{24}\right\}\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^3(q/xy;q^2)_kq^{2k}}{(q^2,q^2,q^2/x,q^2/y;q^2)_k}, \tag{1.12}$$

where $M=(n-1)/2$ or $n-1$. Moreover, Liu and Wang [17, Theorem 3] obtained the following similar result: for any positive integer $n\equiv 1 \pmod 4$, modulo $[n]\Phi_n(q)^3$,

$$\sum_{k=0}^{M}[8k+1]\frac{(q;q^4)_k^4(xq,yq;q^4)_k}{(q^4;q^4)_k^4(q^4/x,q^4/y;q^4)_k}\left(\frac{q^5}{xy}\right)^k$$
$$\equiv [n]q^{(1-n)/2}\left\{1+[n]^2\frac{(n^2-1)(1-q)^2}{24}+[n]^2\sum_{k=1}^{(n-1)/4}\frac{q^{4k-2}}{[4k-2]^2}\right\}$$
$$\times\sum_{k=0}^{(n-1)/4}\frac{(q;q^4)_k^3(q/xy;q^4)_kq^{4k}}{(q^4,q^2,q^4/x,q^4/y;q^4)_k}, \tag{1.13}$$

where $M=(n-1)/4$ or $n-1$. For more recent work on q-supercongruences, we refer the reader to [10,12,21,26–28].

The purpose of this paper is to give the following uniform generalization of (1.9)–(1.13).

Theorem 1.1. *Let $d\geqslant 2$ be an integer and r a nonzero integer. Let $n>1$ be an integer with $\gcd(d,n)=1$. Let λ be the unique integer satisfying $\lambda n\equiv r \pmod d$ and $0\leqslant(\lambda n-r)\leqslant dn-d$, and let x and y be indeterminates. Then*

$$\sum_{k=0}^{M}[2dk+r]\frac{(q^r;q^d)_k^4(xq^r,yq^r;q^d)_k}{(q^d;q^d)_k^4(q^d/x,q^d/y;q^d)_k}(xy)^{-k}q^{(2d-3r)k}$$
$$\equiv [\lambda n]q^{r(r-\lambda n)/d}\frac{(q^{2r};q^d)_{(\lambda n-r)/d}}{(q^d;q^d)_{(\lambda n-r)/d}}\left(1-[\lambda n]^2\sum_{j=1}^{(\lambda n-r)/d}\frac{q^{dj}}{[dj]^2}\right)$$

$$\times \sum_{k=0}^{(\lambda n-r)/d} \frac{(q^r;q^d)_k^3 (q^{d-r}/xy;q^d)_k}{(q^d,q^d/x,q^d/y,q^{2r};q^d)_k} q^{dk} \pmod{[n]\Phi_n(q)^3}, \tag{1.14}$$

where $M=(\lambda n-r)/d$ *or* $n-1$.

For d,n,r satisfying the condition in (1.9), we have $\lambda=d-1$. It is easy to see that

$$\frac{(q^{2r};q^d)_{(dn-n-r)/d}}{(q^d;q^d)_{(dn-n-r)/d}} \equiv 0 \pmod{\Phi_n(q)},$$

and so (1.9) follows from (1.14) by letting $xy=q^{d-r}$. For $(d,r)=(2,1)$ and odd n, we have $\lambda=1$, using the same technique given by Shi and Pan [20], we have

$$\sum_{j=1}^{(n-1)/2} \frac{q^{2j}}{[2j]^2} \equiv \frac{1}{2}\sum_{j=1}^{n-1} \frac{q^j}{[j]^2} \equiv \frac{(1-n^2)(1-q)^2}{24} \pmod{\Phi_n(q)}, \tag{1.15}$$

and thus (1.12) follows immediately form (1.14). For $(d,r)=(2,1)$ and $n\equiv 1 \pmod 4$, we have $\lambda=1$, and using (1.15) again, we see that (1.14) reduces to (1.13). It should be mentioned that the $r=\pm 1$ and $n\equiv r \pmod d$ case of (1.14) has already been given by Wang and Xu [24, Theorem 2].

The rest of the paper is arranged as follows. In the next section, we shall give more concrete q-supercongruences from Theorem 1.1. In section 3, we give some lemmas by using the method of creative microscoping and Watson's ${}_8\phi_7$ transformation. Finally, the proof of Theorem 1.1 will be given in Section 4.

2. More consequences from Theorem 1.1

Let $d=2$, $r=1+2n$, $x=y=1$ in Theorem 1.1. In this case $\lambda=3$, using (1.15) we obtain the following conclusion.

Corollary 2.1. *Let* $n\geqslant 3$ *be an odd integer. Then*

$$\begin{aligned}
&\sum_{k=0}^{M} [4k+2n+1] \frac{(q^{1+2n};q^2)_k^6}{(q^2;q^2)_k^6} q^{(1-6n)k} \\
&\quad\equiv [3n] q^{(2n+1)(1-n)/2} \frac{(q^{2+4n};q^2)_{(n-1)/2}}{(q^2;q^2)_{(n-1)/2}} \left\{ 1+[3n]^2 \frac{(n^2-1)(1-q)^2}{24} \right\} \\
&\qquad\times \sum_{k=0}^{(n-1)/2} \frac{(q^{1+2n};q^2)_k^3 (q^{1-2n};q^2)_k}{(q^2;q^2)_k^3 (q^{2+4n};q^2)_k} q^{2k} \pmod{[n]\Phi_n(q)^3},
\end{aligned}$$

where $M=(n-1)/2$ *or* $n-1$.

Let $d=2$, $r=1-2n$, $x=y=1$ in Theorem 1.1. This time $\lambda=-1$, using (1.15) once more, we get the following q-supercongruence.

Corollary 2.2. *Let $n \geqslant 3$ be an odd integer. Then*

$$\begin{aligned}
&\sum_{k=0}^{M}[4k-2n+1]\frac{(q^{1-2n};q^2)_k^6}{(q^2;q^2)_k^6}q^{(6n+1)k} \\
&\quad\equiv [-n]q^{(2n-1)(n-1)/2}\frac{(q^{2-4n};q^2)_{(n-1)/2}}{(q^2;q^2)_{(n-1)/2}}\left\{1+[-n]^2\frac{(n^2-1)(1-q)^2}{24}\right\} \\
&\quad\quad\times\sum_{k=0}^{(n-1)/2}\frac{(q^{1-2n};q^2)_k^3(q^{1+2n};q^2)_k}{(q^2;q^2)_k^3(q^{2-4n};q^2)_k}q^{2k} \pmod{[n]\Phi_n(q)^3},
\end{aligned}$$

where $M=(n-1)/2$ or $n-1$.

Let $d=8$, $r=1$, $x=y=1$ in Theorem 1.1. This means that $n\equiv\lambda \pmod 8$ and we arrive at the following result.

Corollary 2.3. *Let $n\equiv\lambda \pmod 8$ be a positive integer with $\lambda\in\{1,3,5,7\}$. Then, modulo $[n]\Phi_n(q)^3$,*

$$\begin{aligned}
&\sum_{k=0}^{M}[16k+1]\frac{(q;q^8)_k^6}{(q^8;q^8)_k^6}q^{13k} \\
&\quad\equiv [\lambda n]q^{(1-\lambda n)/8}\frac{(q^2;q^8)_{(\lambda n-1)/8}}{(q^8;q^8)_{(\lambda n-1)/8}}\left(1-[\lambda n]^2\sum_{j=1}^{(\lambda n-1)/8}\frac{q^{8j}}{[8j]^2}\right)\sum_{k=0}^{(\lambda n-1)/8}\frac{(q;q^8)_k^3(q^7;q^8)_k}{(q^8;q^8)_k^3(q^2;q^8)_k}q^{8k},
\end{aligned}$$

where $M=(\lambda n-1)/8$ or $n-1$.

It is easy to see that Theorem 1.1 reduces to the following conclusion when $xy=q^{d-r}$.

Corollary 2.4. *Let $d\geqslant 2$ be an integer and r an arbitrary integer. Let $n>1$ be an integer with $\gcd(d,n)=1$. Let λ be the unique integer satisfying $\lambda n\equiv r \pmod d$ and $0\leqslant(\lambda n-r)\leqslant dn-d$. Then*

$$\begin{aligned}
&\sum_{k=0}^{M}[2dk+r]\frac{(q^r;q^d)_k^4}{(q^d;q^d)_k^4}q^{(d-2r)k} \\
&\quad\equiv [\lambda n]q^{r(r-\lambda n)/d}\frac{(q^{2r};q^d)_{(\lambda n-r)/d}}{(q^d;q^d)_{(\lambda n-r)/d}}\left(1-[\lambda n]^2\sum_{j=1}^{(\lambda n-r)/d}\frac{q^{dj}}{[dj]^2}\right) \pmod{[n]\Phi_n(q)^3}, \qquad (2.1)
\end{aligned}$$

where $M=(\lambda n-r)/d$ or $n-1$.

The $n\equiv r \pmod d$ case (or equivalently, $\lambda=1$) case of (2.1) was recently obtained by the author [6]. It may also be treated as a q-analogue of Barman and Saikia's supercongruence (1.6). If $d/2<\lambda\leqslant d-1$, then $(q^{2r};q^d)_{(\lambda n-r)/d}$ contains the factor $1-q^{(2\lambda-d)n}$

and is therefore congruent to 0 modulo $\Phi_n(q)$. In this case, the right-hand side of (2.1) reduces to

$$[\lambda n]q^{r(r-\lambda n)/d}\frac{(q^{2r};q^d)_{(\lambda n-r)/d}}{(q^d;q^d)_{(\lambda n-r)/d}} \pmod{[n]\Phi_n(q)^3}.$$

This result was recently obtained by Chen and Deng [3, Theorem 1.1]. No doubt that we can derive the following three corollaries similar to Corollaries 2.1–2.3, respectively.

Corollary 2.5. *Let $n\geqslant 3$ be an odd integer. Then, modulo $[n]\Phi_n(q)^3$,*

$$\sum_{k=0}^{M}[4k+2n+1]\frac{(q^{1+2n};q^2)_k^4}{(q^2;q^2)_k^4}q^{-4nk}$$
$$\equiv [3n]q^{(2n+1)(1-n)/2}\frac{(q^{2+4n};q^2)_{(n-1)/2}}{(q^2;q^2)_{(n-1)/2}}\left\{1+[3n]^2\frac{(n^2-1)(1-q)^2}{24}\right\},$$

where $M=(n-1)/2$ or $n-1$.

Corollary 2.6. *Let $n\geqslant 3$ be an odd integer. Then, modulo $[n]\Phi_n(q)^3$,*

$$\sum_{k=0}^{M}[4k-2n+1]\frac{(q^{1-2n};q^2)_k^4}{(q^2;q^2)_k^4}q^{4nk}$$
$$\equiv [-n]q^{(2n-1)(n-1)/2}\frac{(q^{2-4n};q^2)_{(n-1)/2}}{(q^2;q^2)_{(n-1)/2}}\left\{1+[-n]^2\frac{(n^2-1)(1-q)^2}{24}\right\},$$

where $M=(n-1)/2$ or $n-1$.

Corollary 2.7. *Let $n\equiv\lambda \pmod 8$ be a positive integer with $\lambda\in\{1,3,5,7\}$. Then, modulo $[n]\Phi_n(q)^3$,*

$$\sum_{k=0}^{M}[16k+1]\frac{(q;q^8)_k^4}{(q^8;q^8)_k^4}q^{6k}\equiv[\lambda n]q^{(1-\lambda n)/8}\frac{(q^2;q^8)_{(\lambda n-1)/8}}{(q^8;q^8)_{(\lambda n-1)/8}}\left(1-[\lambda n]^2\sum_{j=1}^{(\lambda n-1)/8}\frac{q^{8j}}{[8j]^2}\right),$$

where $M=(\lambda n-1)/8$ or $n-1$.

3. Some lemmas

Recall that the basic hypergeometric ${}_{r+1}\phi_r$ series (see [4]) is defined by

$${}_{r+1}\phi_r\left[\begin{matrix}a_1,a_2,\dots,a_{r+1}\\ b_1,\dots,b_r\end{matrix};q,z\right]:=\sum_{k=0}^{\infty}\frac{(a_0,a_1,\dots,a_r;q)_k}{(q,b_1,\dots,b_r;q)_k}z^k.$$

Then Watson's ${}_8\phi_7$ transformation formula (see [4, Appendix (III.18)]) can be stated as follows:

$${}_8\phi_7\left[\begin{matrix}a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n}\\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1}\end{matrix};q,\frac{a^2q^{n+2}}{bcde}\right]$$

$$= \frac{(aq, aq/de;q)_n}{(aq/d, aq/e;q)_n} {}_4\phi_3\left[\begin{matrix} aq/bc,\ d,\ e,\ q^{-n} \\ aq/b,\ aq/c,\ deq^{-n}/a \end{matrix}; q,\ q\right]. \tag{3.1}$$

We require the following result, which is due to Liu and Wang [16, Lemma 1] (the proof is similar to that of [8, Lemma 2.2]).

Lemma 3.1. *Let d, n be positive integers with $\gcd(d, n) = 1$. Let r be an integer and let a, b, x, y be indeterminates. Then*

$$\sum_{k=0}^{m} [2dk + r] \frac{(aq^r, q^r/a, q^r/b, xq^r, yq^r, q^r; q^d)_k}{(aq^d, q^d/a, bq^d, q^d/x, q^d/y, q^d; q^d)_k} \left(\frac{bq^{2d-3r}}{xy}\right)^k \equiv 0 \pmod{[n]}, \tag{3.2}$$

$$\sum_{k=0}^{n-1} [2dk + r] \frac{(aq^r, q^r/a, q^r/b, xq^r, yq^r, q^r; q^d)_k}{(aq^d, q^d/a, bq^d, q^d/x, q^d/y, q^d; q^d)_k} \left(\frac{bq^{2d-3r}}{xy}\right)^k \equiv 0 \pmod{[n]}, \tag{3.3}$$

where $0 \leqslant m \leqslant n-1$ and $dm \equiv -r \pmod{n}$.

We need to build the following result, which is a generalization of [25, Theorem 3.4] and [16, Lemma 2].

Lemma 3.2. *Let $d \geqslant 2$ be an integer and r an arbitrary integer. Let $n > 1$ be an integer with $\gcd(d, n) = 1$. Let λ be the unique integer satisfying $\lambda n \equiv r \pmod{d}$ and $0 \leqslant (\lambda n - r) \leqslant dn - d$, and let x and y be indeterminates. Then, modulo $\Phi_n(q)(1 - aq^{\lambda n})(a - q^{\lambda n})$,*

$$\sum_{k=0}^{(\lambda n-r)/d} [2dk + r] \frac{(aq^r, q^r/a, q^r/b, xq^r, yq^r, q^r; q^d)_k}{(aq^d, q^d/a, bq^d, q^d/x, q^d/y, q^d; q^d)_k} \left(\frac{bq^{2d-3r}}{xy}\right)^k$$

$$\equiv [\lambda n] \left(\frac{b}{q^r}\right)^{(\lambda n-r)/d} \frac{(q^{2r}/b; q^d)_{(\lambda n-r)/d}}{(bq^d; q^d)_{(\lambda n-r)/d}} \sum_{k=0}^{(\lambda n-r)/d} \frac{(aq^r, q^r/a, q^r/b, q^{d-r}/xy; q^d)_k}{(q^d, q^d/x, q^d/y, q^{2r}/b; q^d)_k} q^{dk}. \tag{3.4}$$

Proof. For $a = q^{\lambda n}$ or $a = q^{-\lambda n}$, the left-hand side of (3.4) can be written as

$$\sum_{k=0}^{(\lambda n-r)/d} [2dk + r] \frac{(q^{r+\lambda n}, q^{r-\lambda n}, q^r/b, xq^r, yq^r, q^r; q^d)_k}{(q^{d+\lambda n}, q^{d-\lambda n}, bq^d, q^d/x, q^d/y, q^d; q^d)_k} \left(\frac{bq^{2d-3r}}{xy}\right)^k$$

$$= [r]\, {}_8\phi_7\left[\begin{matrix} q^r,\ q^{d+\frac{r}{2}},\ -q^{d+\frac{r}{2}},\ xq^r,\ yq^r,\ q^r/b,\ q^{r+\lambda n},\ q^{r-\lambda n} \\ q^{\frac{r}{2}},\ -q^{\frac{r}{2}},\ q^d/x,\ q^d/y,\ bq^d,\ q^{d-\lambda n},\ q^{d+\lambda n} \end{matrix}; q,\ \frac{bq^{2d-3r}}{xy}\right]. \tag{3.5}$$

By Watson's ${}_8\phi_7$ transformation (3.1), the right-hand side of (3.5) is equal to

$$[r] \frac{(q^{d+r}, bq^{d-\lambda n-r}; q^d)_{(\lambda n-r)/d}}{(bq^d, q^{d-\lambda n}; q^d)_{(\lambda n-r)/d}} {}_4\phi_3\left[\begin{matrix} q^{d-r}/xy,\ q^r/b,\ q^{r+\lambda n},\ q^{r-\lambda n} \\ q^d/x,\ q^d/y,\ q^{2r}/b \end{matrix}; q^d,\ q^d\right]$$

$$= [\lambda n] \left(\frac{b}{q^r}\right)^{(\lambda n-r)/d} \frac{(q^{2r}/b; q^d)_{(\lambda n-r)/d}}{(bq^d; q^d)_{(\lambda n-r)/d}} \sum_{k=0}^{(\lambda n-r)/d} \frac{(q^{r+\lambda n}, q^{r-\lambda n}, q^r/b, q^{d-r}/xy; q^d)_k}{(q^d, q^d/x, q^d/y, q^{2r}/b; q^d)_k} q^{dk},$$

which is just the $a=q^{\lambda n}$ or $a=q^{-\lambda n}$ case of the right-hand side of (3.4). This proves that the q-congruence (3.4) holds modulo $1-aq^{\lambda n}$ and $a-q^{\lambda n}$. It is clear that the right-hand side of (3.4) is congruent to 0 modulo $\Phi_n(q)$. Since $[n]$, $1-aq^n$, and $a-q^n$ are pairwise coprime polynomials in q, applying Lemma 3.1, we accomplish the proof of the lemma. □

We also need another q-congruence modulo $b-q^{\lambda n}$ as follows.

Lemma 3.3. *Let $d\geqslant 2$ be an integer and r an arbitrary integer. Let $n>1$ be an integer with $\gcd(d,n)=1$. Let λ be the unique integer satisfying $\lambda n\equiv r \pmod d$ and $0\leqslant(\lambda n-r)\leqslant dn-d$, and let x and y be indeterminates. Then, modulo $b-q^{\lambda n}$,*

$$\sum_{k=0}^{(\lambda n-r)/d}[2dk+r]\frac{(aq^r,q^r/a,q^r/b,xq^r,yq^r,q^r;q^d)_k}{(aq^d,q^d/a,bq^d,q^d/x,q^d/y,q^d;q^d)_k}\left(\frac{bq^{2d-3r}}{xy}\right)^k$$
$$\equiv[\lambda n]\frac{(q^r,q^{d-r};q^d)_{(\lambda n-r)/d}}{(aq^d,aq^d/a;q^d)_{(\lambda n-r)/d}}\sum_{k=0}^{(\lambda n-r)/d}\frac{(aq^r,q^r/a,q^r/b,q^{d-r}/xy,q^d)_k}{(q^d,q^d/x,q^d/y,q^{2r}/b;q^d)_k}q^{dk}. \tag{3.6}$$

Proof. For $b=q^{\lambda n}$, the left-hand side of (3.6) can be written as

$$\sum_{k=0}^{(\lambda n-r)/d}[2dk+r]\frac{(aq^r,q^r/a,q^{r-\lambda n},xq^r,yq^r,q^r;q^d)_k}{(aq^d,q^d/a,q^{d+\lambda n},q^d/x,q^d/y,q^d;q^d)_k}\left(\frac{q^{\lambda n+2d-3r}}{xy}\right)^k$$
$$=[r]\,{}_8\phi_7\left[\begin{matrix}q^r, & q^{d+\frac r2}, & -q^{d+\frac r2}, & xq^r, & yq^r, & aq^r, & q^r/a, & q^{r-\lambda n}\\ & q^{\frac r2}, & -q^{\frac r2}, & q^d/x, & q^d/y, & q^d/a, & aq^d, & q^{d+\lambda n}\end{matrix};q,\ \frac{q^{\lambda n+2d-3r}}{xy}\right]. \tag{3.7}$$

In view of Watson's transformation (3.1), the right-hand side of (3.7) is equal to

$$[r]\frac{(q^{d+r},q^{d-r};q^d)_{(\lambda n-r)/d}}{(aq^d,q^d/a;q^d)_{(\lambda n-r)/d}}\,{}_4\phi_3\left[\begin{matrix}q^{d-r}/xy,\ aq^r,\ q^r/a,q^{r-\lambda n}\\ q^d/x,\ q^d/y,\ q^{2r-\lambda n}\end{matrix};q^d,\ q^d\right]$$
$$=[\lambda n]\frac{(q^r,q^{d-r};q^d)_{(\lambda n-r)/d}}{(aq^d,q^d/a;q^d)_{(\lambda n-r)/d}}\sum_{k=0}^{(\lambda n-r)/d}\frac{(aq^r,q^r/a,q^{r-\lambda n},q^{d-r}/xy;q^d)_k}{(q^d,q^d/x,q^d/y,q^{2r-\lambda n};q^d)_k}q^{dk},$$

which is equal to the $b=q^{\lambda n}$ case of the right-hand side of (3.7). This means that the q-congruence (3.6) holds modulo $b-q^{\lambda n}$. □

4. Proof of Theorem 1.1

It is easy to see that $\Phi_n(q)(1-aq^{\lambda n})(a-q^{\lambda n})$ and $b-q^{\lambda n}$ are coprime polynomials (or Laurent polynomials if $\lambda<0$) in q. By the Chinese remainder theorem for polynomials, we are able to determine the remainder of the left-hand side of (3.4) modulo $\Phi_n(q)(1-aq^{\lambda n})(a-q^{\lambda n})(b-q^{\lambda n})$ from (3.4) and (3.6). In fact, using the q-congruences:

$$\frac{(b-q^{\lambda n})(ab-1-a^2+aq^{\lambda n})}{(a-b)(1-ab)}\equiv 1 \pmod{(1-aq^{\lambda n})(a-q^{\lambda n})},$$

$$\frac{(1-aq^{\lambda n})(a-q^{\lambda n})}{(a-b)(1-ab)} \equiv 1 \pmod{b-q^{\lambda n}},$$

(3.4) and (3.6), we deduce that, modulo $\Phi_n(q)(1-aq^{\lambda n})(a-q^{\lambda n})(b-q^{\lambda n})$,

$$\begin{aligned}
&\sum_{k=0}^{(\lambda n-r)/d} [2dk+r]\frac{(aq^r, q^r/a, q^r/b, xq^r, yq^r, q^r; q^d)_k}{(aq^d, q^d/a, bq^d, q^d/x, q^d/y, q^d; q^d)_k}\left(\frac{bq^{2d-3r}}{xy}\right)^k \\
&\equiv \frac{(b-q^{\lambda n})(ab-1-a^2+aq^{\lambda n})}{(a-b)(1-ab)}[\lambda n]\left(\frac{b}{q^r}\right)^{(\lambda n-r)/d}\frac{(q^{2r}/b;q^d)_{(\lambda n-r)/d}}{(bq^d;q^d)_{(\lambda n-r)/d}} \\
&\quad\times\sum_{k=0}^{(\lambda n-r)/d}\frac{(aq^r, q^r/a, q^r/b, q^{d-r}/xy; q^d)_k}{(q^d, q^d/x, q^d/y, q^{2r}/b; q^d)_k}q^{dk} \\
&\quad+\frac{(1-aq^{\lambda n})(a-q^{\lambda n})}{(a-b)(1-ab)}[\lambda n]\frac{(q^r, q^{d-r}; q^d)_{(\lambda n-r)/d}}{(aq^d, aq^d/a; q^d)_{(\lambda n-r)/d}} \\
&\quad\times\sum_{k=0}^{(\lambda n-r)/d}\frac{(aq^r, q^r/a, q^r/b, q^{d-r}/xy; q^d)_k}{(q^d, q^d/x, q^d/y, q^{2r}/b; q^d)_k}q^{dk}. \qquad (4.1)
\end{aligned}$$

Note that $1-q^{\lambda n}$ has the factor $\Phi_n(q)$. Thereby, putting $b=1$ in (4.1) and noticing that

$$(1-q^{\lambda n})(1+a^2-a-aq^{\lambda n}) = (1-a)^2+(1-aq^{\lambda n})(a-q^{\lambda n}),$$

we are led to the following q-congruence: modulo $\Phi_n(q)^2(1-aq^{\lambda n})(a-q^{\lambda n})$,

$$\begin{aligned}
&\sum_{k=0}^{(\lambda n-r)/d} [2dk+r]\frac{(aq^r, q^r/a, q^r, xq^r, yq^r, q^r; q^d)_k}{(aq^d, q^d/a, q^d, q^d/x, q^d/y, q^d; q^d)_k}\left(\frac{q^{2d-3r}}{xy}\right)^k \\
&\equiv \left\{1+\frac{(1-aq^{\lambda n})(a-q^{\lambda n})}{(1-a)^2}\right\}[\lambda n]q^{r(r-\lambda n)/d}\frac{(q^{2r};q^d)_{(\lambda n-r)/d}}{(q^d;q^d)_{(\lambda n-r)/d}} \\
&\quad\times\sum_{k=0}^{(\lambda n-r)/d}\frac{(aq^r, q^r/a, q^r, q^{d-r}/xy; q^d)_k}{(q^d, q^d/x, q^d/y, q^{2r}; q^d)_k}q^{dk} \\
&\quad-\frac{(1-aq^{\lambda n})(a-q^{\lambda n})}{(1-a)^2}[\lambda n]\frac{(q^r, q^{d-r}; q^d)_{(\lambda n-r)/d}}{(aq^d, aq^d/a; q^d)_{(\lambda n-r)/d}} \\
&\quad\times\sum_{k=0}^{(\lambda n-r)/d}\frac{(aq^r, q^r/a, q^r, q^{d-r}/xy; q^d)_k}{(q^d, q^d/x, q^d/y, q^{2r}; q^d)_k}q^{dk}. \qquad (4.2)
\end{aligned}$$

In view of $q^n\equiv 1 \pmod{\Phi_n(q)}$, it is not hard to see that

$$\begin{aligned}
(q^r;q^d)_{(\lambda n-r)/d} &= (1-q^r)(1-q^{d+r})\cdots(1-q^{\lambda n-d}) \\
&\equiv (1-q^{r-\lambda n})(1-q^{d+r-\lambda n})\cdots(1-q^{-d})
\end{aligned}$$

$$= (-1)^{(\lambda n-r)/d}(q^d;q^d)_{(\lambda n-r)/d}q^{-(d+\lambda n-r)(\lambda n-r)/(2d)} \pmod{\Phi_n(q)},$$

and similarly,

$$(q^{d-r};q^d)_{(\lambda n-r)/d} \equiv (-1)^{(\lambda n-r)/d}(q^{2r};q^d)_{(\lambda n-r)/d}q^{-(\lambda n+3r-d)(\lambda n-r)/(2d)} \pmod{\Phi_n(q)}.$$

It follows that

$$(q^r,q^{d-r};q^d)_{(\lambda n-r)/d} \equiv (q^{2r},q^d;q^d)_{(\lambda n-r)/d}q^{r(r-\lambda n)/d} \pmod{\Phi_n(q)},$$

and so the q-congruence (4.2) can be written as follows: modulo $\Phi_n(q)^2(1-aq^{\lambda n})(a-q^{\lambda n})$,

$$\begin{aligned}
&\sum_{k=0}^{(\lambda n-r)/d}[2dk+r]\frac{(aq^r,q^r/a,q^r,xq^r,yq^r,q^r;q^d)_k}{(aq^d,q^d/a,q^d,q^d/x,q^d/y,q^d;q^d)_k}\left(\frac{q^{2d-3r}}{xy}\right)^k\\
&\equiv [\lambda n]q^{r(r-\lambda n)/d}\frac{(q^{2r};q^d)_{(\lambda n-r)/d}}{(q^d;q^d)_{(\lambda n-r)/d}}\sum_{k=0}^{(\lambda n-r)/d}\frac{(aq^r,q^r/a,q^r,q^{d-r}/xy;q^d)_k}{(q^d,q^d/x,q^d/y,q^{2r};q^d)_k}q^{dk}\\
&\quad+[\lambda n]q^{r(r-\lambda n)/d}\frac{(1-aq^{\lambda n})(a-q^{\lambda n})}{(1-a)^2}\left\{\frac{(q^{2r};q^d)_{(\lambda n-r)/d}}{(q^d;q^d)_{(\lambda n-r)/d}}-\frac{(q^{2r},q^d;q^d)_{(\lambda n-r)/d}}{(aq^d,q^d/a;q^d)_{(\lambda n-r)/d}}\right\}\\
&\quad\times\sum_{k=0}^{(\lambda n-r)/d}\frac{(aq^r,q^r/a,q^r,q^{d-r}/xy;q^d)_k}{(q^d,q^d/x,q^d/y,q^{2r};q^d)_k}q^{dk}
\end{aligned}\tag{4.3}$$

By the L'Hôpital rule, we have

$$\begin{aligned}
&\lim_{a\to1}\frac{(1-aq^{\lambda n})(a-q^{\lambda n})}{(1-a)^2}\left\{\frac{(q^{2r};q^d)_{(\lambda n-r)/d}}{(q^d;q^d)_{(\lambda n-r)/d}}-\frac{(q^{2r},q^d;q^d)_{(\lambda n-r)/d}}{(aq^d,q^d/a;q^d)_{(\lambda n-r)/d}}\right\}\\
&\quad=-[\lambda n]^2\frac{(q^{2r};q^d)_{(n-r)/d}}{(q^d;q^d)_{(\lambda n-r)/d}}\sum_{j=1}^{(\lambda n-r)/d}\frac{q^{dj}}{[dj]^2}.
\end{aligned}\tag{4.4}$$

Letting a tend to 1 in (4.3) and utilizing the limit (4.4), we see that the q-congruence (1.14) holds modulo $\Phi_n(q)^4$ for $M=(\lambda n-r)/d$. It is easy to see that, for $(\lambda n-r)/d<k\leqslant n-1$, the q-shifted factorial $(q^r;q^d)$ contains the factor $1-q^{\lambda n}$, and so the k-th summand on the left-hand side of (1.14) is congruent to 0 modulo $\Phi_n(q)^4$. Furthermore, the proof of [16, Lemma 1] indicates that the q-congruences (3.2) and (3.3) also hold for $a=b=1$. This proves that the q-congruence (1.14) holds modulo $[n]$ for $M=(\lambda n-r)/d$ or $n-1$. Since the least common multiple of $\Phi_n(q)^4$ and $[n]$ is $[n]\Phi_n(q)^3$, we complete the proof of the theorem.

5. Concluding remarks and an open problem

Wei [27, Theorem 3.1] gave another stronger generalization of (1.10): for any positive odd integer n, modulo $[n]\Phi_n(q)^4$,

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{(q;q^2)_k^6}{(q^2;q^2)_k^6}q^k \equiv q^{(1-n)/2}[n]\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}q^{2k}\left\{1-[n]^2q^{-n}\sum_{j=1}^{2k}\frac{q^j}{[j]^2}\right\}, \quad (5.1)$$

where we have replaced the original factor $2-q^n$ by q^{-n} for simplicity. However, we cannot deduce (1.11) from (5.1) directly. For n prime, letting $q\to 1$ in (5.1), we are led to the following supercongruence: for odd primes p,

$$\sum_{k=0}^{(p-1)/2}(4k+1)\frac{(\frac{1}{2})_k^6}{k!^6} \equiv p\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{2})_k^4}{k!^4}\left(1-p^2\sum_{j=1}^{2k}\frac{1}{j^2}\right) \pmod{p^5}. \quad (5.2)$$

On the basis of numerical calculation, we believe that the following generalization of (5.2) should be true.

Conjecture 5.1. *For any prime $p>5$, the supercongruence* (5.2) *holds modulo p^6.*

Note that the q-supercongruence (5.1) does not hold modulo $[n]\Phi_n(q)^5$ in general. Nor does Wei's original version. I spent a lot of time trying to build a q-analogue of (5.2) modulo p^6 but did not succeed. It would be very interesting if Conjecture 5.1 can be settled by an reader.

Data availability. All data generated during this study are included in the published article.

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School of Mathematics, Hangzhou Normal University, Hangzhou 311121, People's Republic of China
E-mail address: jwguo@math.ecnu.edu.cn