Proof of a supercongruence conjectured by Z.-H. Sun

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Abstract. The Franel numbers are defined by $f_n=\sum_{k=0}^n\binom{n}{k}^3$. Motivated by the recent work of Z.-W. Sun on Franel numbers, we prove that

$$\sum_{k=0}^{n-1}(3k+1)(-16)^{n-k-1}\binom{2k}{k}f_k\equiv 0 \pmod{n\binom{2n}{n}},$$

$$\sum_{k=0}^{p-1}\frac{3k+1}{(-16)^k}\binom{2k}{k}f_k\equiv p(-1)^{\frac{p-1}{2}} \pmod{p^3}.$$

where $n>1$ and p is an odd prime. The second congruence modulo p^2 confirms a recent conjecture of Z.-H. Sun. We also show that, if p is a prime of the form $4k+3$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k}{(-16)^k}\equiv 0 \pmod{p},$$

which confirms a special case of another conjecture of Z.-H. Sun.

Keywords: Franel numbers, Babbage's congruence, Morley's congruence

AMS Subject Classifications: 11A07, 11B65, 05A10, 05A19

1 Introduction

The numbers f_n, defined by

$$f_n=\sum_{k=0}^n\binom{n}{k}^3,$$

were first studied by Franel [1, 2], who obtained the following recurrence relation:

$$(n+1)^2f_{n+1}=(7n^2+7n+2)f_n+8n^2f_{n-1},\ n=1,2,\dots.$$

MacMahon [3, p. 122] gave the following identity related to Franel numbers:

$$\sum_{k=0}^n\binom{n}{k}^3x^k=\sum_{k=0}^n\binom{n+k}{3k}\binom{3k}{2k}\binom{2k}{k}x^k(1+x)^{n-2k} \tag{1.1}$$

(see also Foata [4] or Riordan [5, p. 41]). Strehl [6] gave another expression for Franel numbers:

$$f_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}.$$

Jarvis and Verrill [7] obtained the following congruence:

$$f_n \equiv (-8)^n f_{p-1-n} \pmod{p},$$

where p is a prime and $0 \le n \le p-1$. Recently, Z.-W. Sun [8,9] proved many interesting congruences involving Franel numbers. On the other hand, during his working on congruences for Legendre polynomials [10–12], Z.-H. Sun [13] proposed a lot of conjectures on supercongruences concerning Franel numbers , such as

Conjecture 1.1. (see [13, Conjecture 4.23])*Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{3k+1}{(-16)^k} \binom{2k}{k} f_k \equiv p(-1)^{\frac{p-1}{2}} \pmod{p^2}.$$

Conjecture 1.2. [13, Conjecture 4.14]*Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} f_k}{(-16)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2}, & \text{if } p = x^2 + y^2 \equiv 1 \bmod 12 \text{ with } 6 \mid y, \\ 2p - 4x^2 \pmod{p^2}, & \text{if } p = x^2 + y^2 \equiv 1 \bmod 12 \text{ with } 6 \mid x-3, \\ 4\left(\frac{xy}{3}\right) xy \pmod{p^2}, & \text{if } p = x^2 + y^2 \equiv 5 \bmod 12, \\ 0 \pmod{p^2}, & \text{if } p \equiv 3 \bmod 4, \end{cases}$$

where $\left(\frac{a}{3}\right)$ is the Legendre symbol.

In this paper, we shall prove the following results.

Theorem 1.3. *Let $n > 1$ be a positive integer. Then*

$$\sum_{k=0}^{n-1} (3k+1)(-16)^{n-k-1} \binom{2k}{k} f_k \equiv 0 \pmod{n\binom{2n}{n}}. \tag{1.2}$$

Theorem 1.4. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{3k+1}{(-16)^k} \binom{2k}{k} f_k \equiv p(-1)^{\frac{p-1}{2}} \pmod{p^3}. \tag{1.3}$$

Theorem 1.5. *Let p be a prime of the form $4k+3$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} f_k}{(-16)^k} \equiv 0 \pmod{p}. \tag{1.4}$$

It is obvious that the congruence (1.3) modulo p^2 confirms Conjecture 1.1, while the congruence (1.4) is a special case of Conjecture 1.2.

2 Proof of Theorem 1.3

We need the following identity due to Z.-W. Sun [8, (2.3)]:

$$f_n = \sum_{k=0}^{n} \binom{n+2k}{3k}\binom{3k}{k}\binom{2k}{k}(-4)^{n-k}, \tag{2.1}$$

which can be proved by the Zeilberger algorithm (see [14, 15]). Moreover, by a straightforward induction on n, we can prove that, for all $0 \leqslant k \leqslant n$,

$$\sum_{m=k}^{n-1}(3m+1)(-16)^{n-m-1}\binom{2m}{m}\binom{m+2k}{3k}(-4)^{m-k} = \binom{2n}{n}\binom{n+2k}{3k}\frac{n(k-n)(-4)^{n-k}}{8(2k+1)}. \tag{2.2}$$

By (2.1) and (2.2), we have

$$\begin{aligned}
&\sum_{m=0}^{n-1}(3m+1)(-16)^{n-m-1}\binom{2m}{m}f_m \\
&\quad = \sum_{m=0}^{n-1}(3m+1)(-16)^{n-m-1}\binom{2m}{m}\sum_{k=0}^{m}\binom{m+2k}{3k}\binom{3k}{k}\binom{2k}{k}(-4)^{m-k}. \\
&\quad = \sum_{k=0}^{n-1}\binom{2n}{n}\binom{n+2k}{3k}\binom{3k}{k}\binom{2k}{k}\frac{n(k-n)(-4)^{n-k}}{8(2k+1)}.
\end{aligned} \tag{2.3}$$

Note that, for $n \geqslant 2$ and $0 \leqslant k < n$, both

$$\frac{1}{2k+1}\binom{3k}{k} = \binom{3k}{k} - 2\binom{3k}{k-1}$$

and $\binom{2k}{k}\frac{(-4)^{n-k}}{8}$ are integers. From (2.3) we deduce that

$$\frac{1}{n\binom{2n}{n}}\sum_{k=0}^{n-1}(3k+1)(-16)^{n-k-1}\binom{2k}{k}f_k = \sum_{k=0}^{n-1}\binom{n+2k}{3k}\binom{3k}{k}\binom{2k}{k}\frac{(k-n)(-4)^{n-k}}{8(2k+1)} \tag{2.4}$$

is an integer. This proves (1.2).

3 Proof of Theorem 1.4

It follows from (2.4) that

$$\frac{1}{p}\sum_{k=0}^{p-1}\frac{3k+1}{(-16)^k}\binom{2k}{k}f_k = -\,4^{1-p}\binom{2p-1}{p-1}\sum_{k=0}^{p-1}\binom{p+2k}{3k}\binom{3k}{k}\binom{2k}{k}\frac{k-p}{(2k+1)(-4)^k}. \tag{3.1}$$

Note that Babbage [16] proved the following congruence:

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}. \tag{3.2}$$

Moreover, we have

$$\binom{p+2k}{3k}\binom{3k}{k} = \frac{(p+2k)!}{(2k)!k!(p-k)!} \equiv \begin{cases} (-1)^{k-1}\frac{p}{k}, & \text{if } 1 \leqslant k < \frac{p-1}{2} \\ (-1)^{k-1}\frac{2p}{k}, & \text{if } \frac{p+1}{2} \leqslant k < p \end{cases} \pmod{p^2}, \tag{3.3}$$

and, when $k = \frac{p-1}{2}$,

$$\binom{p+2k}{3k}\binom{3k}{k}\binom{2k}{k}\frac{k-p}{2k+1} = -\binom{2p-1}{p-1}\binom{p-1}{\frac{p-1}{2}}^2 \equiv -16^{p-1} \pmod{p^2}, \tag{3.4}$$

where we have used Babbage's congruence (3.2) and Morley's congruence [17]:

$$\binom{p-1}{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}} 4^{p-1} \pmod{p^3} \quad \text{for } p > 3.$$

Substituting (3.2)–(3.4) into (3.1), and observing that $\binom{2k}{k} \equiv 0 \pmod{p}$ for $\frac{p-1}{2} < k < p$, we obtain

$$\begin{aligned} \frac{1}{p}\sum_{k=0}^{p-1}\frac{3k+1}{(-16)^k}\binom{2k}{k}f_k &\equiv p4^{1-p} + (-4)^{\frac{p-1}{2}} + 4^{1-p}\sum_{k=1}^{\frac{p-3}{2}}\binom{2k}{k}\frac{p-p^2/k}{(2k+1)4^k} \\ &\equiv (-4)^{\frac{p-1}{2}} + 4^{1-p}\sum_{k=0}^{\frac{p-3}{2}}\binom{2k}{k}\frac{p}{(2k+1)4^k} \pmod{p^2}. \end{aligned} \tag{3.5}$$

Since

$$\binom{2k}{k}4^{-k} \equiv (-1)^k\binom{\frac{p-1}{2}}{k} \pmod{p}, \tag{3.6}$$

we may rewrite (3.5) as

$$\frac{1}{p}\sum_{k=0}^{p-1}\frac{3k+1}{(-16)^k}\binom{2k}{k}f_k \equiv (-4)^{\frac{p-1}{2}} + 4^{1-p}\sum_{k=0}^{\frac{p-3}{2}}(-1)^k\binom{\frac{p-1}{2}}{k}\frac{p}{2k+1} \pmod{p^2}, \tag{3.7}$$

Applying the famous identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{x+k} = \frac{n!}{x(x+1)\cdots(x+n)}$$

with $x=\frac{1}{2}$ and $n=\frac{p-1}{2}$, we may simplify (3.7) as

$$\frac{1}{p}\sum_{k=0}^{p-1}\frac{3k+1}{(-16)^k}\binom{2k}{k}f_k \equiv (-4)^{\frac{p-1}{2}}+2^{1-p}\binom{p-1}{\frac{p-1}{2}}^{-1}-(-1)^{\frac{p-1}{2}}4^{1-p}$$
$$\equiv (-1)^{\frac{p-1}{2}}\left(2^{p-1}+8^{1-p}-4^{1-p}\right) \pmod{p^2}.$$

By Fermat's little theorem, we have $2^{p-1}-1\equiv 0 \pmod{p}$ and so

$$2^{p-1}+8^{1-p}-4^{1-p}-1=(2^{p-1}-1)^2(2^{1-p}+4^{1-p}+8^{1-p})\equiv 0 \pmod{p^2}.$$

This completes the proof. □

4 Proof of Theorem 1.5

We first give a binomial coefficient identity.

Lemma 4.1. *Let n and k be nonnegative integers with $k\leqslant n$. Then*

$$\sum_{m=k}^{n}\binom{n}{m}\binom{m+2k}{3k}(-1)^{m-k}=\binom{2k}{n-k}(-1)^{n-k}. \tag{4.1}$$

Proof. The identity (4.1) is a special case of the Chu-Vandermonde summation formula (see, for example, [14, p. 84]):

$${}_2F_1\left[\begin{matrix}a, -n\\ c\end{matrix};1\right]=\frac{(c-a)_n}{(c)_n}$$

with $a=3k+1$, $c=k+1$, and $n\to n-k$.

□

By (3.6) and (2.1), we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k}{(-16)^k}\equiv\sum_{m=0}^{p-1}\frac{\binom{\frac{p-1}{2}}{m}f_m}{4^m}$$
$$=\sum_{m=0}^{p-1}\frac{\binom{\frac{p-1}{2}}{m}}{4^m}\sum_{k=0}^{m}\binom{m+2k}{3k}\binom{3k}{k}\binom{2k}{k}(-4)^{m-k} \pmod{p}. \tag{4.2}$$

Exchanging the summation order in the right-hand side of (4.2), and applying (4.1) with $n=\frac{p-1}{2}$, we obtain

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k}{(-16)^k}\equiv\sum_{k=0}^{\frac{p-1}{2}}\frac{(-1)^{\frac{p-1}{2}-k}}{4^k}\binom{3k}{k}\binom{2k}{k}\binom{2k}{\frac{p-1}{2}-k} \pmod{p}. \tag{4.3}$$

It is easy to see that

$$\binom{2k}{\frac{p-1}{2}-k} \equiv \binom{\frac{3(p-1)}{2}-3k}{\frac{p-1}{2}-k}(-1)^{\frac{p-1}{2}-k} \pmod p. \tag{4.4}$$

Thus, by (3.6) and (4.4), we may rewrite (4.3) as

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} f_k}{(-16)^k} \equiv \sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{\frac{p-1}{2}}{k}\binom{3k}{k}\binom{\frac{3(p-1)}{2}-3k}{\frac{p-1}{2}-k} \pmod p. \tag{4.5}$$

If p is a prime of the form $4k+3$, then $\frac{p-1}{2}$ is odd, and by the symmetry of binomial coefficients, the right-hand side of (4.5) equals 0. This completes the proof.

5 Concluding remarks and open problems

Inspired by Conjectures 4.22 and 4.23 in [13] and Theorem 1.3 in this paper, we propose the following two conjectures.

Conjecture 5.1. *Let $n > 1$ be a positive integer and let $(a,b,c) \in \{(9,4,5),(5,2,16),(9,2,50),(5,1,96),(6,1,320),(90,13,896),(102,11,10400)\}$. Then*

$$\sum_{k=0}^{n-1} (ak+b)c^{n-k-1}\binom{2k}{k} f_k \equiv 0 \pmod{n\binom{2n}{n}}.$$

Conjecture 5.2. *Let $n > 1$ be a positive integer and let $(a,b,c) \in \{(15,4,-49),(9,2,-112),(99,17,-400),(855,109,-2704),(585,58,-24304)\}$. Then*

$$\sum_{k=0}^{n-1} (ak+b)c^{n-k-1}\binom{2k}{k} f_k \equiv 0 \pmod{n\binom{2n}{n}}.$$

We have verified Conjectures 5.1 and 5.2 for n up to 500 via Maple.

We also make the third conjecture on Franel numbers as follows.

Conjecture 5.3. *Let m,n be positive integers and let $a_1,\ldots,a_m$ be integers. Then*

$$\sum_{k=0}^{n-1} (3k+2)(-1)^{(m-1)k} f_k \prod_{i=1}^{m}\binom{a_i n-1}{k}\binom{a_i n+k}{k} \equiv 0 \pmod{n^2}, \tag{5.1}$$

$$\sum_{k=0}^{n-1} (9k^2+5k)(-1)^{(m-1)k} f_k \prod_{i=1}^{m}\binom{a_i n-1}{k}\binom{a_i n+k}{k} \equiv 0 \pmod{n^2}. \tag{5.2}$$

Note that, for any prime p and nonnegative integer $k \leqslant p-1$, there holds

$$\binom{a_i p - 1}{k}\binom{a_i p + k}{k} = \prod_{j=1}^{k} \frac{(a_i p - j)(a_i p + j)}{j^2} \equiv (-1)^k \pmod{p^2}.$$

Therefore, by [8, Theorem 1.1], the congruences (5.1) and (5.2) are true for any prime n. On the other hand, the author [18] has proved that

$$\sum_{k=0}^{n-1} (3k+2)(-1)^k f_k \equiv 0 \pmod{2n^2},$$

which was conjectured by Z.-W. Sun (see [9, Conjecture 1.3]); and for $m=0$ and $n>1$, the congruence (5.2) can be further strengthened as

$$\sum_{k=0}^{n-1} (9k^2+5k)(-1)^k f_k \equiv 0 \pmod{n^2(n-1)},$$

which was also conjectured by Z.-W. Sun (see Conjecture 5.3 in a previous version of [9]: `http://arxiv.org/pdf/1112.1034v10.pdf`).

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