Factors of certain basic hypergeometric sums

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Abstract. We prove that certain truncated basic hypergeometric series contain the factor $\Phi_n(q)^2$, where $\Phi_n(q)$ is the n-th cyclotomic polynomial. This result may be regarded as a generalization of Theorem 1.1 in [J. Math. Anal. Appl. 476 (2019), 851–859].

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1. Introduction

Rodriguez-Villegas [11] numerically found a number of interesting supercongruences related to hypergeometric families of Calabi–Yau manifolds. The simplest one of his finds can be stated as follows: for any odd prime p,

$$\sum_{k=0}^{p-1}\frac{(\frac{1}{2})_k^2}{k!^2}\equiv(-1)^{(p-1)/2}\pmod{p^2},\tag{1.1}$$

where $(a)_n=a(a+1)\cdots(a+n-1)$ denotes the rising factorial. The first proof of this supercongruence was given by Mortenson [9]. The second author and Zeng [7] established the following q-analogue of (1.1):

$$\sum_{k=0}^{p-1}\frac{(q;q^2)_k^2}{(q^2;q^2)_k^2}q^{2k}\equiv(-1)^{(p-1)/2}q^{(p^2-1)/4}\pmod{[p]^2}\quad\text{for any odd prime } p.\tag{1.2}$$

Here and in what follows, for $n\geqslant 0$, $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ is the *q-shifted factorial*, and $[n]=(1-q^n)/(1-q)$ is the *q-integer*. For convenience, we also adopt the abbreviatged notation $(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$. The second author [4] obtained an extension of (1.1): Let $d\geqslant 2$ and $r\leqslant d-2$ be integers subject to

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$\gcd(r,d)=1$. Then, for all positive integers n with $n\equiv -r \pmod{d}$ and $n\geqslant d-r$, there holds

$$\sum_{k=0}^{n-1}\frac{(q^r;q^d)_k^d q^{dk}}{(q^d;q^d)_k^d}\equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.3}$$

Here $\Phi_n(q)$ denotes the n-th *cyclotomic polynomial* in q, which can be factorized as

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity. On the other hand, Deines et al. [2] gave another generalization of (1.1): for any integer $d>1$ and prime $p\equiv 1 \pmod{d}$,

$$\sum_{k=0}^{p-1}\frac{(\frac{d-1}{d})_k^d}{k!^d}\equiv -\Gamma_p(\tfrac{1}{d})^d \pmod{p^2}, \tag{1.4}$$

where $\Gamma_p(x)$ denotes the p-adic Gamma function. Wang and Pan [13] further proved that, for $d\geqslant 3$, the supercongruence (1.4) also holds modulo p^3, confirming a conjecture in [2]. A q-analogue of (1.4) was given by the second author [5]. For more recent q-supercongruences, we refer the reader to [3, 6, 8, 10, 14].

The objective of this paper is to give a generalization of (1.3). Our first result is a generalization of (1.3) for $d=2$.

Theorem 1.1. *Let r be a negative odd integer. Let n and s be integers satisfying $n\equiv -r \pmod{2}$, $n\geqslant 2-r$, and $0\leqslant s\leqslant (n-2+r)/4$. Then*

$$\sum_{k=s}^{n-s-1}\frac{(q^r;q^2)_{k-s}(q^r;q^2)_{k+s}}{(q^2;q^2)_{k-s}(q^2;q^2)_{k+s}}q^{2k}\equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.5}$$

In particular, if p is a prime satisfying $p\equiv -r \pmod{2}$, $p\geqslant 2-r$ and $s\leqslant (p-2+r)/4$, then

$$\sum_{k=s}^{p-s-1}\frac{(\frac{r}{2})_{k-s}(\frac{r}{2})_{k+s}}{(k-s)!(k+s)!}\equiv 0 \pmod{p^2}. \tag{1.6}$$

Our second result is the following generalization of (1.3) for $d=3$.

Theorem 1.2. *Let $r\leqslant 1$ be an integer coprime with 3. Let n and s be integers satisfying $n\equiv -r \pmod{3}$, $n\geqslant 3-r$, and $0\leqslant s\leqslant (n-3+r)/6$. Then*

$$\sum_{k=s}^{n-s-1}\frac{(q^r;q^3)_{k-s}(q^r;q^3)_{k+s}(q^r;q^3)_k}{(q^3;q^3)_{k-s}(q^3;q^3)_{k+s}(q^3;q^3)_k}q^{3k}\equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.7}$$

In particular, if p is a prime satisfying $p \equiv -r \pmod 3$, $p \geqslant 3-r$ and $s \leqslant (p-3+r)/6$, then

$$\sum_{k=s}^{p-s-1} \frac{(\frac{r}{3})_{k-s}(\frac{r}{3})_{k+s}(\frac{r}{3})_k}{(k-s)!(k+s)!k!} \equiv 0 \pmod{p^2}. \tag{1.8}$$

The last result of ours is the following generalization of (1.3) for $d \geqslant 4$.

Theorem 1.3. *Let $d \geqslant 4$ be an integer. Let $r \leqslant d-2$ be an integer such that $\gcd(r,d)=1$. Then, for all integers n and s with $n \equiv -r \pmod d$, $n \geqslant d-r$ and $0 \leqslant s \leqslant (n-d+r)/d$,*

$$\sum_{k=s}^{n-s-1} \frac{(q^r;q^d)_{k-s}(q^r;q^d)_{k+s}(q^r;q^d)_k^{d-2}}{(q^d;q^d)_{k-s}(q^d;q^d)_{k+s}(q^d;q^d)_k^{d-2}} q^{dk} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.9}$$

In particular, if p is a prime satisfying $p \equiv -r \pmod d$, $p \geqslant d-r$ and $s \leqslant (p-d+r)/d$, then

$$\sum_{k=s}^{p-s-1} \frac{(\frac{r}{d})_{k-s}(\frac{r}{d})_{k+s}(\frac{r}{d})_k^{d-2}}{(k-s)!(k+s)!k!^{d-2}} \equiv 0 \pmod{p^2}. \tag{1.10}$$

We shall prove Theorems 1.1–1.3 by using the *creative microscoping* method devised by the second author and Zudilin [8]. More precisely, in order to prove the q-supercongruences (1.5), (1.7) and (1.9), we shall first establish their generalizations with an extra parameter a so that the generalized q-congruences hold modulo $(1-aq^n)(a-q^n)$, then the desired q-supercongruences follow by letting $a=1$ in these parametric q-congruences.

2. Proof of Theorem 1.1

We first establish the following parametric generalization of Theorem 1.1.

Theorem 2.1. *Let r be a negative odd integer. Let n and s be integers satisfying $n \equiv -r \pmod 2$, $n \geqslant 2-r$, and $0 \leqslant s \leqslant (n-2+r)/4$. Then*

$$\sum_{k=s}^{n-s-1} \frac{(aq^r;q^2)_{k-s}(q^r/a;q^2)_{k+s}}{(q^2;q^2)_{k-s}(q^2;q^2)_{k+s}} q^{2k} \equiv 0 \pmod{(1-aq^n)(a-q^n)}. \tag{2.1}$$

Proof. For $a=q^{-n}$, the left-hand side of (2.1) can be written as

$$\begin{aligned}
&\sum_{k=0}^{n-2s-1} \frac{(q^{r-n};q^2)_k(q^{r+n};q^2)_{k+2s}}{(q^2;q^2)_k(q^2;q^2)_{k+2s}} q^{2k+2s} \\
&\quad = \frac{(q^{r+n};q^2)_{2s}}{(q^2;q^2)_{2s}} q^{2s} \sum_{k=0}^{(n-r)/2} \frac{(q^{r-n};q^2)_k(q^{4s+r+n};q^2)_k}{(q^2;q^2)_k(q^{4s+2};q^2)_k} q^{2k},
\end{aligned}$$

where we have realized the fact that $(q^{r-n};q^2)_k=0$ for $k>(n-r)/2$, and $0<(n-r)/2\leqslant n-2s-1$. Let

$$\begin{bmatrix} n\\ k \end{bmatrix} = \begin{bmatrix} n\\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

be the *q-binomial coefficient.* Then

$$\frac{(q^{r-n};q^2)_k q^{2k}}{(q^2;q^2)_k} = (-1)^k \begin{bmatrix} (n-r)/2\\ k \end{bmatrix}_{q^2} q^{2\binom{(n-r)/2-k}{2}-2\binom{(n-r)/2}{2}}.$$

Moreover, we have

$$\frac{(q^{4s+n+r};q^2)_k}{(q^{4s+2};q^2)_k} = \frac{(q^{4s+2+2k};q^2)_{(n+r-2)/2}}{(q^{4s+2};q^2)_{(n+r-2)/2}}. \tag{2.2}$$

It follows that

$$\sum_{k=s}^{n-s-1} \frac{(q^{r-n};q^2)_{k-s}(q^{r+n};q^2)_{k+s}}{(q^2;q^2)_{k-s}(q^2;q^2)_{k+s}} q^{2k}$$
$$= \frac{(q^{r+n};q^2)_{2s}}{(q^2;q^2)_{2s}} q^{2s} \sum_{k=0}^{(n-r)/2} (-1)^k \begin{bmatrix} (n-r)/2\\ k \end{bmatrix}_{q^2} q^{2\binom{(n-r)/2-k}{2}-2\binom{(n-r)/2}{2}} \frac{(q^{4s+2+2k};q^2)_{(n+r-2)/2}}{(q^{4s+2};q^2)_{(n+r-2)/2}}. \tag{2.3}$$

Recall that the finite form of the q-binomial theorem (see, for instance, [1, p. 36]) can be written as

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n\\ k \end{bmatrix} q^{\binom{k}{2}} z^k = (z;q)_n.$$

Taking $z=q^{-j}$ and replacing k by $n-k$ in the above equation, we get

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n\\ k \end{bmatrix} q^{\binom{n-k}{2}+jk} = 0 \quad \text{for } 0\leqslant j\leqslant n-1. \tag{2.4}$$

Since $(q^{4s+2+2k};q^2)_{(n+r-2)/2}$ is a polynomial in q^{2k} of degree $(n+r-2)/2=(n-r)/2-(1-r)\leqslant (n-r)/2-1$, we conclude that the right-hand side of (2.3) vanishes. This proves that the q-congruence (2.1) holds modulo $1-aq^n$.

For $a=q^n$, the left-hand side of (2.1) can be written as

$$\sum_{k=0}^{n-2s-1} \frac{(q^{r+n};q^2)_k(q^{r-n};q^2)_{k+2s}}{(q^2;q^2)_k(q^2;q^2)_{k+2s}} q^{2k+2s}$$
$$= \frac{(q^{r-n};q^2)_{2s}}{(q^2;q^2)_{2s}} q^{2s} \sum_{k=0}^{(n-r)/2-2s} \frac{(q^{r+n};q^2)_k(q^{4s+r-n};q^2)_k}{(q^2;q^2)_k(q^{4s+2};q^2)_k} q^{2k},$$

where we have used the fact that $(q^{4s+r-n};q^2)_k=0$ for $k>(n-r)/2-2s$, and $0<(n-r)/2-2s\leqslant n-2s-1$. Noticing that

$$\frac{(q^{4s+r-n};q^2)_k q^{2k}}{(q^2;q^2)_k}=(-1)^k\begin{bmatrix}(n-r)/2-2s\\ k\end{bmatrix}_{q^2} q^{2\binom{(n-r)/2-2s-k}{2}-2\binom{(n-r)/2-2s}{2}},$$

and

$$\frac{(q^{r+n};q^2)_k}{(q^{4s+2};q^2)_k}=\frac{(q^{4s+2+2k};q^2)_{(n+r-2)/2-2s}}{(q^{4s+2};q^2)_{(n+r-2)/2-2s}},$$

we have

$$\begin{aligned}&\sum_{k=s}^{n-s-1}\frac{(q^{r+n};q^2)_{k-s}(q^{r-n};q^2)_{k+s}}{(q^2;q^2)_{k-s}(q^2;q^2)_{k+s}}q^{2k}\\ &\quad=\frac{(q^{r-n};q^2)_{2s}}{(q^2;q^2)_{2s}}q^{2s}\sum_{k=0}^{(n-r)/2-2s}\frac{(q^{4s+2+2k};q^2)_{(n+r-2)/2-2s}}{(q^{4s+2};q^2)_{(n+r-2)/2-2s}}\\ &\qquad\times(-1)^k\begin{bmatrix}(n-r)/2-2s\\ k\end{bmatrix}_{q^2} q^{2\binom{(n-r)/2-2s-k}{2}-2\binom{(n-r)/2-2s}{2}}. \qquad (2.5)\end{aligned}$$

Since $(q^{4s+2+2k};q^2)_{(n+r-2)/2-2s}$ is a polynomial in q^{2k} of degree $(n+r-2)/2-2s\leqslant(n-r)/2-2s-1$, by (2.4), we deduce that the right-hand side of (2.5) vanishes. This proves that (2.1) holds modulo $a-q^n$.

Noticing that $1-aq^n$ and $a-q^n$ are coprime polyinomials in q, we complete the proof of (2.1). □

Proof of Theorem 1.1. When $a=1$, the polynomial $(1-aq^n)(a-q^n)=(1-q^n)^2$ has the factor $\Phi_n(q)^2$. Furthermore, the denominators of the left-hand side of (2.1) are all coprime with $\Phi_n(q)$. The proof of (1.5) then follows from taking $a=1$ in (2.1). Finally, letting $n=p$ be a prime, taking the limits as $q\to 1$ in both sides of (1.5), we arrive at (1.6). □

3. Proof of Theorem 1.2

Similarly, we first give a parametric generalization of Theorem 1.2.

Theorem 3.1. *Let $r\leqslant 1$ be an integer coprime with 3. Let n and s be integers satisfying $n\equiv -r \pmod 3$, $n\geqslant 3-r$, and $0\leqslant s\leqslant (n-3+r)/6$. Then*

$$\sum_{k=s}^{n-s-1}\frac{(a^2q^r;q^3)_{k-s}(q^r;q^3)_{k+s}(q^r/a^2;q^3)_k}{(q^3;q^3)_{k-s}(aq^3;q^3)_{k+s}(q^3/a;q^3)_k}q^{3k}\equiv 0 \pmod{(1-aq^n)(a-q^n)}. \qquad (3.1)$$

Proof. For $a = q^{-n}$, the left-hand side of (3.1) can be written as

$$\sum_{k=0}^{n-2s-1} \frac{(q^{r-2n};q^3)_k (q^r;q^3)_{k+2s} (q^{r+2n};q^3)_{k+s}}{(q^3;q^3)_k (q^{3-n};q^3)_{k+2s} (q^{3+n};q^3)_{k+s}} q^{3k+3s}$$
$$= \frac{(q^r;q^3)_{2s}(q^{r+2n};q^3)_s}{(q^{3-n};q^3)_{2s}(q^{3+n};q^3)_s} q^{3s} \sum_{k=0}^{(2n-r)/3} \frac{(q^{r-2n};q^3)_k (q^{6s+r};q^3)_k (q^{3s+r+2n};q^3)_k}{(q^3;q^3)_k (q^{6s+3-n};q^3)_k (q^{3s+3+n};q^3)_k} q^{3k},$$

where we have used the fact that $(q^{r-2n};q^3)_k = 0$ for $k > (2n-r)/3$, and $0 < (2n-r)/3 \leqslant n-2s-1$. Using

$$\frac{(q^{r-2n};q^3)_k q^{3k}}{(q^3;q^3)_k} = (-1)^k \begin{bmatrix} (2n-r)/3 \\ k \end{bmatrix}_{q^3} q^{3\binom{(2n-r)/3-k}{2} - 3\binom{(2n-r)/3}{2}},$$

and relations similar to (2.2), we have

$$\sum_{k=s}^{n-s-1} \frac{(q^{r-2n};q^3)_{k-s} (q^r;q^3)_{k+s} (q^{r+2n};q^3)_k}{(q^3;q^3)_{k-s} (q^{3-n};q^3)_{k+s} (q^{3+n};q^3)_k} q^{3k}$$
$$= \frac{(q^r;q^3)_{2s}(q^{r+2n};q^3)_s}{(q^{3-n};q^3)_{2s}(q^{3+n};q^3)_s} q^{3s} \sum_{k=0}^{(2n-r)/3} (-1)^k \begin{bmatrix} (2n-r)/3 \\ k \end{bmatrix}_{q^3} q^{3\binom{(2n-r)/3-k}{2} - 3\binom{(2n-r)/3}{2}}$$
$$\times \frac{(q^{3s+3+n+3k};q^3)_{(n+r-3)/3} (q^{6s+3-n+3k};q^3)_{(n+r-3)/3}}{(q^{3s+3+n};q^3)_{(n+r-3)/3} (q^{6s+3-n};q^3)_{(n+r-3)/3}}. \qquad (3.2)$$

Since $(q^{3s+3+n+3k};q^3)_{(n+r-3)/3}(q^{6s+3-n+3k};q^3)_{(n+r-3)/3}$ is a polynomial in q^{3k} of degree $2(n+r-3)/3 \leqslant (2n-r)/3-1$. In view of (2.4), the right-hand side of (3.2) vanishes. This proves that the q-congruence (3.1) holds modulo $1-aq^n$.

Similarly, for $a = q^n$, the left-hand side of (3.1) can be written as

$$\sum_{k=0}^{n-2s-1} \frac{(q^{r+2n};q^3)_k (q^r;q^3)_{k+2s} (q^{r-2n};q^3)_{k+s}}{(q^3;q^3)_k (q^{3+n};q^3)_{k+2s} (q^{3-n};q^3)_{k+s}} q^{3k+3s}$$
$$= \frac{(q^r;q^3)_{2s}(q^{r-2n};q^3)_s}{(q^{3+n};q^3)_{2s}(q^{3-n};q^3)_s} q^{3s} \sum_{k=0}^{(2n-r)/3-s} \frac{(q^{r+2n};q^3)_k (q^{6s+r};q^3)_k (q^{3s+r-2n};q^3)_k}{(q^3;q^3)_k (q^{6s+3+n};q^3)_k (q^{3s+3-n};q^3)_k} q^{3k},$$
$$= \frac{(q^r;q^3)_{2s}(q^{r-2n};q^3)_s}{(q^{3+n};q^3)_{2s}(q^{3-n};q^3)_s} q^{3s} \sum_{k=0}^{(2n-r)/3-s} (-1)^k \begin{bmatrix} (2n-r)/3-s \\ k \end{bmatrix}_{q^3} q^{3\binom{(2n-r)/3-s-k}{2} - 3\binom{(2n-r)/3-s}{2}}$$
$$\times \frac{(q^{6s+3+n+3k};q^3)_{(n+r-3)/3-2s} (q^{3s+3-n+3k};q^3)_{(n+r-3)/3+s}}{(q^{6s+3+n};q^3)_{(n+r-3)/3-2s} (q^{3s+3-n};q^3)_{(n+r-3)/3+s}}. \qquad (3.3)$$

Note that $(q^{6s+3+n+3k};q^3)_{(n+r-3)/3-2s}(q^{3s+3-n+3k};q^3)_{(n+r-3)/3+s}$ is a polynomial in q^{3k} of degree $2(n+r-3)/3-s \leqslant (2n-r)/3-s-1$. In view of (2.4), the right-hand side of (3.3) vanishes. This proves that the q-congruence (3.1) holds modulo $1-aq^n$. □

Proof of Theorem 1.2. Letting $a = 1$ in (3.1), we immediately obtain (1.7). Moreover, for $n = p$ a prime, taking the limits as $q \to 1$ in both sides of (1.7), we are led to (1.8). □

4. Proof of Theorem 1.3

Like before, we need to establish a parametric generalization of Theorem 1.3. However, this time the parametric form is more complicated.

Theorem 4.1. *Let d, r, n be given as in the conditions of Theorem 1.3. Then, modulo $(1-aq^n)(a-q^n)$,*

$$\sum_{k=s}^{n-s-1} \frac{(q^r;q^d)_{k-s}(a^{d-1}q^r;q^d)_{k+s}(a^{d-3}q^r,\dots,a^2q^r;q^d)_k(a^{1-d}q^r,a^{3-d}q^r,\dots,a^{-2}q^r;q^d)_k}{(q^d;q^d)_{k-s}(a^{d-2}q^d;q^d)_{k+s}(a^{d-4}q^d,\dots,aq^d;q^d)_k(a^{2-d}q^d,a^{4-d}q^d,\dots,a^{-1}q^d;q^d)_k} q^{dk} \equiv 0, \tag{4.1}$$

if d is odd, and

$$\sum_{k=s}^{n-s-1} \frac{(aq^r;q^d)_{k-s}(a^3q^r;q^d)_{k+s}(a^5q^r,\dots,a^{d-1}q^r;q^d)_k(a^{1-d}q^r,a^{3-d}q^r,\dots,a^{-1}q^r;q^d)_k}{(q^d;q^d)_{k-s}(a^{-2}q^d;q^d)_{k+s}(a^{2-d}q^d,\dots,a^{-4}q^d;q^d)_k(a^{d-2}q^d,a^{d-4}q^d,\dots,q^d;q^d)_k} q^{dk} \equiv 0, \tag{4.2}$$

if d is even.

Proof. Since $\gcd(r,d) = 1$ and $n \equiv -r \pmod d$, we know that $\gcd(d,n) = 1$. Thus, none of the numbers $d, 2d, \dots (n-1)d$ are divisible by n, and so the denominators of the left-hand sides of (4.1) and (4.2) do not have the factor $1 - aq^n$ nor $1 - a^{-1}q^n$. Hence, for $a = q^{-n}$, the left-hand side of (4.1) can be written as

$$\begin{aligned}
&\frac{(q^{r-(d-1)n};q^d)_{2s}(q^{r-(d-3)n},\dots,q^{r-2n};q^d)_s(q^{r-(1-d)n},q^{r-(3-d)n},\dots,q^{r+2n};q^d)_s}{(q^{d-(d-2)n};q^d)_{2s}(q^{d-(d-4)n},\dots,q^{d-n};q^d)_s(q^{d-(2-d)n},q^{d-(4-d)n},\dots,q^{d+n};q^d)_s} q^{ds}\\
&\quad\times \sum_{k=0}^{\frac{dn-n-r}{d}-2s} \frac{(q^r;q^d)_k(q^{2ds+r-(d-1)n};q^d)_k(q^{ds+r-(d-3)n},\dots,q^{ds+r-2n};q^d)_k}{(q^d;q^d)_k(q^{2ds+d-(d-2)n};q^d)_k(q^{ds+d-(d-4)n},\dots,q^{ds+d-n};q^d)_k} q^{dk}\\
&\quad\times \frac{(q^{ds+r-(1-d)n},q^{ds+r-(3-d)n},\dots,q^{ds+r+2n};q^d)_k}{(q^{ds+d-(2-d)n},q^{ds+d-(4-d)n},\dots,q^{ds+d+n};q^d)_k},
\end{aligned} \tag{4.3}$$

where we have used the fact that $(q^{2ds+r-(d-1)n};q^d)_k = 0$ for $k > (dn-n-r)/d - 2s$, and $0 < (dn-n-r)/d - 2s \leqslant n - 1 - 2s$. It is easy to see that

$$\frac{(q^{2ds+r-(d-1)n};q^d)_k q^{dk}}{(q^d;q^d)_k} = (-1)^k \begin{bmatrix} (dn-n-r)/d-2s \\ k \end{bmatrix}_{q^d} q^{d\binom{k}{2}+(2ds+n+r-dn+d)k}, \tag{4.4}$$

$$\frac{(q^{ds+r-(d-2j-1)n};q^d)_k}{(q^{ds+d-(d-2j)n};q^d)_k} = \frac{(q^{ds+d-(d-2j)n+dk};q^d)_{(n+r-d)/d}}{(q^{ds+d-(d-2j)n};q^d)_{(n+r-d)/d}} \quad \text{for } 2 \leqslant j \leqslant \frac{d-3}{2}, \tag{4.5}$$

$$\frac{(q^{ds+(d-2j+1)n+r};q^d)_k}{(q^{ds+(d-2j)n+d};q^d)_k}=\frac{(q^{ds+(d-2j)n+dk+d};q^d)_{(n+r-d)/d}}{(q^{ds+(d-2j)n+d};q^d)_{(n+r-d)/d}}\quad\text{for } 1\leqslant j\leqslant\frac{d-1}{2},\tag{4.6}$$

and

$$\frac{(q^{ds+r-(d-3)n};q^d)_k}{(q^{2ds+d-(d-2)n};q^d)_k}=\frac{(q^{2ds+d-(d-2)n+dk};q^d)_{(n+r-d)/d-s}}{(q^{2ds+d-(d-2)n};q^d)_{(n+r-d)/d-s}},\tag{4.7}$$

$$\frac{(q^r;q^d)_k}{(q^{ds+d-n};q^d)_k}=\frac{(q^{ds+d-n+dk};q^d)_{(n+r-d)/d-s}}{(q^{ds+d-n};q^d)_{(n+r-d)/d-s}},\tag{4.8}$$

Note that the right-hand sides of (4.5) and (4.6) are polynomials in q^{dk} of degree $(n+r-d)/d$, and the right-hand sides of (4.7) and (4.8) are polynomials in q^{dk} of degree $(n+r-d)/d-s$, and

$$d\binom{k}{2}+(2ds+n+r-dn+d)k=d\binom{(dn-n-r)/d-2s-k}{2}-d\binom{(dn-n-r)/d-2s}{2}.$$

We can write the summation in (4.3) as

$$\sum_{k=0}^{(dn-n-r)/d-2s}(-1)^k q^{d\binom{(dn-n-r)/d-2s-k}{2}-d\binom{(dn-n-r)/d-2s}{2}}\begin{bmatrix}(dn-n-r)/d-2s\\k\end{bmatrix}_{q^d}P(q^{dk}),\tag{4.9}$$

where $P(q^{dk})$ is a polynomial in q^{dk} of degree $(n+r-d)(d-3)/d+2(n+r-d-ds)/d=(dn-n-r)/d-2s-(d-r-1)\leqslant(dn-n-r)/d-2s-1$. In light of (2.4), we conclude that (4.9) is equal to 0 and so is (4.3). This means that (4.1) is true modulo $1-aq^n$.

For $a=q^n$, the left-hand side of (4.1) can be written as

$$\begin{aligned}&\frac{q^{ds}(q^{r+(d-1)n};q^d)_{2s}(q^{r+(d-3)n},\ldots,q^{r+2n};q^d)_s(q^{r+(1-d)n},q^{r+(3-d)n},\ldots,q^{r-2n};q^d)_s}{(q^{d+(d-2)n};q^d)_{2s}(q^{d+(d-4)n},\ldots,q^{d+n};q^d)_s(q^{d+(2-d)n},q^{d+(4-d)n},\ldots,q^{d-n};q^d)_s}\\&\quad\times\sum_{k=0}^{\frac{dn-n-r}{d}-s}\frac{q^{dk}(q^r;q^d)_k(q^{2ds+r+(d-1)n};q^d)_k(q^{ds+r+(d-3)n},\ldots,q^{ds+r+2n};q^d)_k}{(q^d;q^d)_k(q^{2ds+d+(d-2)n};q^d)_k(q^{ds+d+(d-4)n},\ldots,q^{ds+d+n};q^d)_k}\\&\quad\times\frac{q^{ds+r+(1-d)n},q^{ds+r+(3-d)n},\ldots,q^{ds+r-2n};q^d)_k}{(q^{ds+d+(2-d)n},q^{ds+d+(4-d)n},\ldots,q^{ds+d-n};q^d)_k},\end{aligned}\tag{4.10}$$

where we have used $(q^{ds+r+(1-d)n};q^d)_k=0$ for $k>(dn-n-r)/d-s$, and $0<(dn-n-r)/d-s\leqslant n-1-2s$. Similarly as before, we have

$$\frac{(q^{ds+r+(1-d)n};q^d)_kq^{dk}}{(q^d;q^d)_k}=(-1)^k\begin{bmatrix}(dn-n-r)/d-s\\k\end{bmatrix}_{q^d}q^{d\binom{k}{2}+(ds+n+r-dn+d)k},$$

$$\frac{(q^{ds+r-(d-2j-1)n};q^d)_k}{(q^{ds+d-(d-2j)n};q^d)_k}=\frac{(q^{ds+d-(d-2j)n+dk};q^d)_{(n+r-d)/d}}{(q^{ds+d-(d-2j)n};q^d)_{(n+r-d)/d}}\quad\text{for } 1\leqslant j\leqslant\frac{d-3}{2},$$

$$\frac{(q^{ds+r+(d-2j+1)n};q^d)_k}{(q^{ds+d+(d-2j)n};q^d)_k}=\frac{(q^{ds+d+(d-2j)n+dk};q^d)_{(n+r-d)/d}}{(q^{ds+d+(d-2j)n};q^d)_{(n+r-d)/d}} \quad \text{for } 2\leqslant j\leqslant \frac{d-1}{2},$$

and

$$\frac{(q^{2ds+r+(d-1)n};q^d)_k}{(q^{2ds+d+(d-2)n};q^d)_k}=\frac{(q^{2ds+d+(d-2)n+dk};q^d)_{(n+r-d)/d}}{(q^{2ds+d+(d-2)n};q^d)_{(n+r-d)/d}},$$

$$\frac{(q^r;q^d)_k}{(q^{ds+d-n};q^d)_k}=\frac{(q^{ds+d-n+dk};q^d)_{(n+r-d)/d-s}}{(q^{ds+d-n};q^d)_{(n+r-d)/d-s}},$$

$$d\binom{k}{2}+(ds+n+r-dn+d)k=d\binom{(dn-n-r)/d-s-k}{2}-d\binom{(dn-n-r)/d-s}{2}.$$

Therefore, the summation in (4.10) can be written as

$$\sum_{k=0}^{(dn-n-r)/d-s}(-1)^k q^{d\binom{(dn-n-r)/d-s-k}{2}-d\binom{(dn-n-r)/d-s}{2}}\begin{bmatrix}(dn-n-r)/d-s\\ k\end{bmatrix}_{q^d}P(q^{dk}), \qquad (4.11)$$

where $P(q^{dk})$ is a polynomial in q^{dk} of degree $(n+r-d)(d-2)/d+(n+r-ds-d)/d=(dn-n-r)/d-s-(d-r-1)\leqslant(dn-n-r)/d-s-1$. By (2.4), we conclude that (4.11) is equal to 0 and so is (4.10). This means that (4.1) is true modulo $a-q^n$.

In the same way, we can establish the q-congruence (4.2). □

Proof of Theorem 1.3. It is well known that $\Phi_n(q)$ is a factor of $1-q^m$ if and only if n divides m. Thus, when $a=1$ the denominators of (4.1) and (4.2) are all coprime with $\Phi_n(q)$. On the other hand, when $a=1$, the polynomial $(1-aq^n)(a-q^n)=(1-q^n)^2$ contains the factor $\Phi_n(q)^2$. Therefore, the q-supercongruence (1.9) follows by taking $a=1$ in (4.1) and (4.2). Finally, assume that $n=p$ is a prime, taking the limits as $q\to 1$ in both sides of (1.9), we get (1.10). □

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