A QUADRATIC FORMULA FOR BASIC HYPERGEOMETRIC SERIES RELATED TO ASKEY-WILSON POLYNOMIALS

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Abstract. We prove a general quadratic formula for basic hypergeometric series, from which simple proofs of several recent determinant and Pfaffian formulas are obtained. A special case of the quadratic formula is actually related to a Gram determinant formula for Askey-Wilson polynomials. We also show how to derive a recent double-sum formula for the moments of Askey-Wilson polynomials from Newton’s interpolation formula.

1. Introduction

Throughout this paper we assume that \( q \) is a fixed number in \((0, 1)\). A \( q \)-shifted factorial is defined by

\[
(a; q)_\infty = \prod_{k=0}^{\infty} (1 - a q^k), \quad \text{and} \quad (a; q)_n = (a; q)_\infty (aq^n; q)_\infty, \quad \text{for} \quad n \in \mathbb{Z}.
\]

Following Gasper and Rhaman [7] we shall use the abbreviated notation

\[
(a_1, a_2, \ldots, a_r; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n, \quad \text{for} \quad n \in \mathbb{Z}.
\]

A basic hypergeometric series with \( r \) numerators and \( s \) denominators is then defined by

\[
_{r+1} \phi_s \left[ \begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{array} ; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} \frac{(-1)^n q^{(\frac{n}{2})}}{n!} z^n.
\]

The Askey-Wilson polynomials \( p_n(x; a, b, c, d; q) \) \((n \in \mathbb{N})\) are the \(4\phi_3\) polynomials [3, 7]

\[
\frac{(ab, ac, ad; q)_n}{a^n} \begin{array}{c} q^{-n}, abcdq^{n-1}, ae^{10}, ae^{-10} \\ ab, ac, ad \end{array} ; q, q
\]

where \( x = \cos \theta \) and \( I \) is a complex number such that \( I^2 = -1 \).
Our main result is the following quadratic formula for basic hypergeometric series, which was discovered by applying Desnanot–Jacobi adjoint matrix theorem to compute some determinants of Mehta-Wang type [16, 18, 14, 8] or deformed Gram determinants of orthogonal polynomials [20].

**Theorem 1.1.** Let \( r, s \geq 0, \) \( a, b, c, d, q \in \mathbb{C}, \) \( E_r = (e_1, e_2, \ldots, e_r) \in \mathbb{C}^r, \) \( F_s = (f_1, f_2, \ldots, f_s) \in \mathbb{C}^s. \) Then we have

\[
(a-b)(a-c)(bc-d)(1-d) \\
\times r+4\phi_{s+3} \left[ a^{-1}bc, bq^{-2}, c, dq^{-1}, E_r; q, z \right] \\
= (a-d)(1-b)(1-c)(bc-ad) \\
\times r+4\phi_{s+3} \left[ a^{-1}bc, bq^{-2}, cq^{-1}, d, E_r; q, z \right] \\
= (a-b)(b-c)(c-d) \times r+4\phi_{s+3} \left[ a^{-1}bc, bcq^{-1}, d, E_r; q, z \right].
\]

Let \( s = r, \) \( d = a_0, \) \( c = a_1, \) \( b = 0, \) \( a = b_1, \) \( e_1 = f_1 = 0, \) \( e_i = a_i, \) and \( f_j = b_j \) \((2 \leq j \leq r)\) we get the following result by shifting \( r \) to \( r-3.\)

**Corollary 1.2.** For \( r \geq 1, \) there holds

\[
(a_0 - 1)(a_1 - b_1) \\
\times r+1 \phi_r \left[ a_0/q, a_1, a_2, \ldots, a_r; q, z \right] \\
= (a_0 - a_1)(1-b_1) \\
\times r+1 \phi_r \left[ a_0/a, a_1/a, a_2/a, \ldots, a_r/a; b_1/q, b_2/q, \ldots, b_r/q; q, z \right] \\
= (a_0 - a_1)(a_0 - b_1) \\
\times r+1 \phi_r \left[ a_0/a, a_1/a, a_2/a, \ldots, a_r/a; q, z \right].
\]

Taking \( r = 3, \) \( z = q, \) \( a_0 = q^{-n+1}, \) \( a_1 = abcdq^{-1}, \) \( a_2 = ae^{i\theta}, \) \( a_3 = ae^{-i\theta}, \) \( b_1 = abq, \) \( b_2 = ac \) and \( b_3 = ad \) in Corollary 1.2, we obtain the following quadratic relation for Askey-Wilson polynomials.

**Corollary 1.3.** Let \( n \) be a positive integer. There holds

\[
ab(1-q^{n-1})(1-cdq^{-n-2})p_n(x; a, b, c, d; q)p_{n-2}(x; aq, bq, c, d; q) \\
= (1-qbq^{-1})(1-abdq^{-n-1})p_{n-1}(x; a, b, c, d; q)p_{n-1}(x; aq, bq, c, d; q) \\
- (1-ab)(1-abd)q^{n-2}p_{n-1}(x; aq, b, c, d; q)p_{n-1}(x; a, b, c, d; q).
\]

We shall prove Theorem 1.1 in the next section. In Section 3, we use the Desnanot-Jacobi adjoint matrix theorem and Corollary 1.3 to derive a determinant formula for Askey-Wilson polynomials (cf. Theorem 3.1), which turns out to be a generalization of several recent determinant and Pfaffian evaluations in [16, 6, 14, 10, 11]. In Section 4, we connect the determinant formula in Theorem 3.1 to a Gram determinant formula of Askey-Wilson polynomials and show how to derive a
recent double–sum formula for the moments of Askey-Wilson polynomials in [4, 9] from Newton’s interpolation formula.

2. Proof of Theorem 1.1

For $0 \leq k \leq n$, we write

$$A_k = (a - b)(a - c)(b - d)(1 - d)\alpha_k \alpha_{n-k+1}$$

$$\times \left( a^{-1}bc, bcy^{-2}, c, dq^{-1}, E_r; q \right)_k (a^{-1}bc, bc, c, dq, E_r; q; q)_{n-k}$$

$$\times (q, aq^{-1}, bq^{-1}, bcd^{-1}, F_s; q)_{k} (q, aq, bq, bcd^{-1}, F_s; q; q)_{n-k},$$

$$B_k = (a - d)(1 - d)(b - c)(bc - ad)\alpha_k \alpha_{n-k+1}$$

$$\times \left( a^{-1}bc, bcy^{-2}, cq^{-1}, d, E_r; q \right)_k (a^{-1}bc, bc, cq, d, E_r; q; q)_{n-k}$$

$$\times (q, aq^{-1}, b, bcd^{-1}q^{-1}, F_s; q)_{k} (q, aq, b, bcd^{-1}q, F_s; q; q)_{n-k},$$

$$C_k = (1 - a)(b - d)(c - d)(a - bc)\alpha_k \alpha_{n-k+1}$$

$$\times \left( a^{-1}bcp^{-1}, bcy^{-2}, c, d, E_r; q \right)_k (a^{-1}bcp, bc, c, d, E_r; q; q)_{n-k}$$

$$\times (q, a, bq^{-1}, bcd^{-1}q^{-1}, F_s; q)_{k} (q, a, bq, bcd^{-1}q, F_s; q; q)_{n-k},$$

where $\alpha_k = \left\{ (-1)^k q^{k(k-1)} \right\}^{n-r}$ for $k \geq 0$. Equating the coefficients of $z^n$ on both sides of (1.1) yields the equivalent identity

$$\sum_{k=0}^{n} (A_k - B_k + C_k) = 0.$$  

The key point to prove (2.1) is the observation that

$$A_k - B_k + C_k + (A_{n-k+1} - B_{n-k+1} + C_{n-k+1}) = 0$$

for $0 \leq k \leq n + 1$, where $A_{n+1} = B_{n+1} = C_{n+1} = 0$. Indeed, summing (2.2) over $k$ from 0 to $n + 1$ on both sides yields immediately (2.1).

To prove (2.2) we start from the identity

$$\left( a - b)(a - c)(d - x)(bc - dx)(x - ay) \right.$$

$$\times (x - by)(x - cy)(y - dz)(ax - bcy)(dy - bcz)$$

$$- (a - d)(b - x)(c - x)(ad - bc)(x - ay)$$

$$\times (y - bz)(y - cz)(x - dy)(ax - bcy)(dx - bcy)$$

$$+ (b - d)(c - d)(a - x)(ax - bcy)(y - az)$$

$$\times (x - by)(x - cy)(x - dy)(ay - bcz)(dx - bcy)$$

$$= xy(a - b)(a - c)(a - d)(b - d)(c - d)$$

$$\times (1 - y)(ad - bc)(x - bcz)(x^2 - bcy)(y^2 - xz),$$

which can be easily checked either by hands or by Maple. Replacing $(x, y, z)$ by $(q, q^k, q^n)$ in (2.3) and multiplying both sides of the resulting identity by

$$\frac{(c, d, a^{-1}bc; q)_{k-1} (bcy^{-2}, E_r; q)_k (c, d, a^{-1}bc, bc, E_r; q; q)_{n-k}}{(aq^{-1}, bq^{-1}, bcd^{-1}q^{-1}; q)_{k+1}(q, F_s; q)_{k}(aq, bq, bcd^{-1}q, F_s; q; q)_{n-k}},$$

we obtain

$$A_k - B_k + C_k = (q^{n-k+1} - q^k)G_k G_{n-k+1},$$

where $G_k$ is a quadratic formula for basic hypergeometric series.
where
\[ G_k = \frac{(1 - q^k)(1 - bcq^{k-2})(a^{-1}bc, c; q)_{k-1}(bc; q)_{k-2}(E_r; q)_k\alpha_k}{(a, b, bc^{-1}, q; q)_k(F_s; q)_k}, \]
and
\[ \Xi = (a - b)(a - c)(a - d)(b - d)(c - d)(ad - bc)(1 - bcq^{n-1}) \]
\[ \times \frac{(1 - a)(1 - b)(1 - bcq^{-1})(1 - bcq^{-2})(1 - bcq^{-3}):(F_s; q)_1}{adq(1 - aq^{-1})(1 - bq^{-1})(1 - bcq^{-1})(E_r; q)_1}, \]
which is independent of \( k \). Clearly (2.4) implies (2.2).

3. Application to determinant and Pfaffian evaluation

Given a matrix \( M \), if \( i_1, \ldots, i_r \) (resp. \( j_1, \ldots, j_r \)) are row (resp. column) indices, we denote by \( M_{i_1,\ldots,i_r}^{j_1,\ldots,j_r} \) the matrix that remains when the rows \( i_1, \ldots, i_r \) and columns \( j_1, \ldots, j_r \) are deleted. Let \( n \geq 2 \) and \( M \) be an \( n \times n \) matrix. Then the Desnanot-Jacobi adjoint matrix theorem [1, Lemma 7.7] reads
\[
\text{det} M = \text{det} M_1^n = \text{det} M_2^n - \text{det} M_1^n, \quad \text{det} M_1^n = 1 \quad \text{if} \quad n = 2.
\]

**Theorem 3.1.** For \( n \geq 1 \), \( 0 \leq i \leq n - 1 \) and \( 1 \leq j \leq n \), let
\[
B_{i,j} = \frac{(ab; q)_{i+j-1}(aq^{i+j})}{(abcd; q)_{i+j}} \times [c + d - 2x + (1 - cd)(aq^i + bq^{i-1}) - ab(c + d - 2cdx)q^{i+j-1}].
\]
Then
\[
\text{det}(B_{i,j})_{0 \leq i \leq n-1, 1 \leq j \leq n} = D_n(a, b) \cdot p_n(x; a, b, c, d; q),
\]
where
\[
D_n(a, b) = a^{n(n-1)/2}b^{n(n+1)/2}q^{n(n-1)(2n-1)/6} \prod_{i=0}^{n-1} \frac{(ab, cd, q; q)_i}{(abcd; q)_{n+i}}.
\]

**Proof.** By (3.4) we have
\[
D_n(a, b)/D_{n-1}(a, b) = a^{n-1}b^nq^{(n-1)/2} \frac{(ab, cd, q)_{n-1}(q; q)_{n-1}}{(abcdq^{n-1}; q)_n(abcd; q)_{2n-2}},
\]
\[
D_n(aq, b)/D_n(a, b) = q^{n(n-1)/2} \frac{(abq; q)_{n-1}(1 - abcd)^n}{(abcdq^n; q)_n(1 - ab)^{n-1}}.
\]
Therefore
\[
D_n(a, b)/D_{n-1}(a, b) = \frac{ab(ab; q)_2(1 - cdq^{n-2})(1 - q^{n-1})}{(1 - abq^{n-1})(1 - abcdq^{n-1})(abcd; q)_2},
\]
and
\[
D_{n-1}(aq, b)/D_{n-1}(a, b) = \frac{1 - abq}{1 - abq^{n-1}} \frac{1 - abcdq^{2n-2}}{1 - abd} q^{n-2} \frac{(1 - abcd)^n}{(1 - abd)^{n-1}}.
\]
Let \( M_n(a, b) := \det(B_{i,j})_{0 \leq i \leq n-1, 1 \leq j \leq n} \). For \( n = 1 \), formula (3.3) is obvious. Assume that \( n \geq 2 \). Applying (3.1) to the determinant in (3.3) we obtain

\[
M_n(a, b) M_{n-2}(aq, bq) = \frac{(ab; q)_2}{(abcd; q)_2} M_{n-1}(a, b) M_{n-1}(aq, bq) - \left( \frac{1 - ab}{1 - abcd} \right)^n \frac{1 - abcd}{1 - ab} \frac{(1 - abcdq) - \frac{n-2}{M_{n-1}(aq, b) M_{n-1}(a, bq)}.
\]

It suffices to show that if we substitute \( M_n(a, b) \) by \( D_n(a, b)p_n(x; a, b, c, d; q) \) the above identity still holds, i.e., for \( n \geq 2 \),

\[
D_n(a, b) D_{n-2}(aq, bq)p_n(x; a, b, c, d; q)p_{n-2}(x; aq, bq, c, d; q)
\]

\[
= \frac{(ab; q)_2}{(abcd; q)_2} D_{n-1}(a, b) D_{n-1}(aq, bq)p_{n-1}(x; a, b, c, d; q)p_{n-1}(x; aq, bq, c, d; q)
\]

\[
- \left( \frac{1 - ab}{1 - abcd} \right)^n \frac{1 - abcd}{1 - ab} \frac{(1 - abcdq) - \frac{n-2}{D_{n-1}(aq, b) D_{n-1}(a, bq)} \times p_{n-1}(x; aq, bq, c, d; q)p_{n-1}(x; aq, b, c, d; q).
\]

Dividing the two sides of (3.7) by \( D_{n-1}(a, b) D_{n-1}(aq, bq) \) and applying the two identities (3.5) and (3.6), we see that (3.7) is exactly the quadratic formula (1.3). The result then follows by induction on \( n \).

The following formula is an extension of Nishizawa’s \(q\)-analogue of Mehta-Wang’s formula [16, 18, 8].

**Corollary 3.2.** For \( n \geq 1 \), there holds

\[
\det \left( \frac{q^{i-1} - cq^{i-1}}{(abq^2; q)_k} \right)_{1 \leq i, j \leq n} = (-1)^n a^{\frac{n(n-3)}{2}} q^{\frac{n(n+1)(2n-5)}{6}} (abcq; q^2)_n \prod_{k=1}^{n} \frac{(aq; q)_{k-1}(aq; q)_{k}(aq; q)_{k-2}}{(abq; q)_{k+n-2}}
\]

\[
\times \phi_3 \left[ q^{-n}, abq^n, (acq)^\frac{1}{2}, -(qacq)^\frac{1}{2}, aq, (abq)^\frac{1}{2}, -(abq)^\frac{1}{2}; q, q \right].
\]

**Proof.** Since the Askey-Wilson polynomials are symmetric on \( a \) and \( b \), in (3.3) replacing \( p_n(x; a, b, c, d; q) \) by \( p_n(x; b, a, c, d; q) \) and making the following substitution:

\[
x \leftarrow 0, \quad a \leftarrow (aq/c)^{1/2}I, \quad b \leftarrow -(acq)^{1/2}I, \quad c \leftarrow b^{1/2}I, \quad d \leftarrow -b^{1/2}I
\]

where \( I^2 = -1 \), gives (3.8) as \( (aq; q)_{-1} = 1 / (1 - b) \).

If \( c = 1 \), we can sum the \( \phi_3 \) in (3.8) by Andrews’ terminating \(q\)-analogue of Watson’s formula [7, II.17]

\[
\phi_3 \left[ q^{-n}, a^2q^{n+1}, b, -b; aq, -aq, b^2; q, q \right] = \begin{cases} 
0, & \text{if } n \text{ is odd,} \\
\frac{b^n(aq, a^2q^2/b^2, q^2)_{n/2}}{(a^2q^2, b^2q^2, q^2)_{n/2}}, & \text{if } n \text{ is even,}
\end{cases}
\]

and deduce the following result, which was first proved in [11], and is also a \(q\)-analogue of [14, Theorem 6].
Corollary 3.3. For \( m \geq 1 \), there holds
\[
\det \left( (q^{i-1} - q^{j-1}) \frac{(aq; q)_{i+j-2}}{(abq^2; q)_{i+j-2}} \right)_{1 \leq i,j \leq 2m} = a^{2m(m-1)} q^{\frac{m(m-1)(4m+1)}{3}} \prod_{k=1}^{m} \left( \frac{(q, aq; q)_{2k-1} - (bq; q)_{2k-2}}{(abq^2; q)_{2(k+m)-3}} \right)^2.
\]

Recall [1] that the Pfaffian of a skew-symmetric matrix \( A = (A_{i,j})_{1 \leq i,j \leq 2m} \) is defined by
\[
\text{Pf} A = \sum_{\pi \in \mathcal{M}[1,\ldots,2m]} \text{sgn} \pi \prod_{i<j \atop i,j \text{ matched in } \pi} A_{i,j}.
\]
Here \( \mathcal{M}[a,\ldots,b] \) is the set of perfect matchings of the complete graph on \( \{a,\ldots,b\} \) for any nonnegative integers \( a \) and \( b \) such that \( a < b \), and \( \text{sgn} \pi = (-1)^{cr \pi} \), where \( cr \pi \) is the number of matched pairs \( (i, j) \) and \( (i', j') \) in \( \pi \) such that \( i < i' < j < j' \). The following result was first proved by Ishikawa et al. [11].

Corollary 3.4. For \( m \geq 1 \), there holds
\[
\text{Pf} \left( (q^{j-1} - q^{i-1}) \frac{(aq; q)_{i+j-2}}{(abq^2; q)_{i+j-2}} \right)_{1 \leq i,j \leq 2m} = \varepsilon_m a^{m(m-1)} q^{\frac{m(m-1)(4m+1)}{3}} \prod_{k=1}^{m} \left( \frac{(q, aq; q)_{2k-1} - (bq; q)_{2k-2}}{(abq^2; q)_{2(k+m)-3}} \right),
\]
where \( \varepsilon_m^2 = 1 \). By (3.10), the factor \( \varepsilon_m \) is a rational function of \( a \), \( b \) and \( q \), and only takes values \( 1 \) or \( -1 \). Hence, for fixed \( m \), we must have \( \varepsilon_m = 1 \) or \( \varepsilon_m = -1 \) regardless the values of \( a \), \( b \) and \( q \). It remains to show that \( \varepsilon_m = 1 \) for all \( m \geq 1 \). Obviously we have \( \varepsilon_1 = 1 \). Suppose that \( m \geq 2 \). Taking \( b = 0 \) and replacing \( a \) by \( q^{a-1} \) the identity (3.10) reduces to
\[
\text{Pf} \left( (q^i - q^j)(q^a; q)_{i+j} \right)_{0 \leq i,j \leq 2m-1} = \varepsilon_m q^{m(m-1)(a-1) + \frac{m(m-1)(4m+1)}{3}} \prod_{k=1}^{m} (q, qa; q)_{2k-1}.
\]
Using the \( q \)-gamma function [7, p. 20]
\[
\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty (1-q)^{1-x}}, \quad 0 < q < 1,
\]
we can rewrite (3.11) as

\begin{equation}
\text{Pf} \left( q^{[j - i]} \Gamma_q(a + i + j) \right)_{0 \leq i,j \leq 2m-1} = \varepsilon_m q^{m(m-1)(a-1) + \frac{m(m+1)(m+2)}{2}} \prod_{k=1}^{m} [2k - 1]_q! \Gamma_q(a + 2k - 1),
\end{equation}

where \([n]_q! = [1]_q[2]_q \cdots [n]_q\) and \([k]_q = (1 - q^k)/(1 - q)\). If we multiply both sides of (3.12) by \([a + 1]_q\) and let \(a\) tend to \(-1\), then, the right-hand side becomes

\begin{equation}
\varepsilon_m q^{\frac{m(m-1)(4m-5)}{3}} \prod_{k=1}^{m} [2k - 1]_q! \prod_{k=1}^{m-1} \Gamma_q(2k).
\end{equation}

On the other hand, by (3.9), the left-hand side can be written as

\begin{equation}
\sum_{\pi \in \mathcal{M}[0, \ldots, 2m-1]} \text{sgn} \, \pi \lim_{a \to -1} [a + 1]_q \prod_{i<j, i \text{, \text{m}atched \ in} \pi} q^{[j - i]} \Gamma_q(a + i + j).
\end{equation}

In this sum, matchings \(\pi\) for which all matched pairs \(i, j\) satisfy \(i + j > 1\) will not contribute, because the corresponding summands vanish. Therefore, the survival matchings must match 0 and 1 and the sum in (3.14) reduces to

\begin{equation}
\sum_{\pi' \in \mathcal{M}[2, \ldots, 2m-1]} \text{sgn} \, \pi' \prod_{i<j, i \text{, \text{matched \ in} \pi'}} q^{[j - i]} \Gamma_q(i + j - 1)
\end{equation}

\begin{align*}
&= \text{Pf} \left( q^{[j - i]} \Gamma_q(i + j - 1) \right)_{2 \leq i,j \leq 2m-1} \\
&= \text{Pf} \left( q^{i+2[ j - i]} \Gamma_q(i + j + 3) \right)_{0 \leq i,j \leq 2m-3} \\
&= \varepsilon_{m-1} q^{\frac{m(m-1)(4m-5)}{3}} \prod_{k=1}^{m} [2k - 1]_q! \prod_{k=1}^{m-1} \Gamma_q(2k).
\end{align*}

Comparing with (3.13) we see that \(\varepsilon_m = \varepsilon_{m-1} = \cdots = \varepsilon_1 = 1\). □

**Remark 3.5.** Except the trivial but crucial point that \(\varepsilon_m\) is independent of \(a, b\) and \(q\), the above proof is a \(q\)-adaptation of Ciucu and Krattenthaler’s proof [6] for the \(q \to 1\) case of (3.12):

\begin{equation}
Pf((j - i) \Gamma(a + i + j))_{0 \leq i,j \leq 2m-1} = \prod_{k=1}^{m} (2k - 1)! \Gamma(a + 2k - 1).
\end{equation}

As we have shown that \(\varepsilon_m\) is actually independent of \(q\), we could also reduce the proof directly to (3.15) by taking the limit \(q \to 1\) in (3.12).

### 4. Link to Gram determinants of Askey-Wilson polynomials

In this section we show that the determinant formula in Theorem 3.1 is actually related to a Gram determinant for the Askey-Wilson orthogonal polynomials (see [20]). This permits to enlighten the origin of the peculiar matrix coefficients (3.2).

Let \(\{p_n(x)\}\) be a sequence of orthogonal polynomials with respect to a measure \(d\mu\), and \(\{\phi_k\}\) and \(\{\psi_k\}\) be two sequences of polynomials such that \(\phi_k\) and \(\psi_k\) are
of exact degree $k$. Then

$$
\begin{vmatrix}
\mu_{0,0} & \mu_{0,1} & \cdots & \mu_{0,n} \\
\mu_{1,0} & \mu_{1,1} & \cdots & \mu_{1,n} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{n-1,0} & \mu_{n-1,1} & \cdots & \mu_{n-1,n} \\
\phi_0(x) & \phi_1(x) & \cdots & \phi_n(x)
\end{vmatrix} = C \cdot p_n(x),
$$

(4.1)

where $\mu_{i,j} = \int \psi_i(x)\phi_j(x) d\mu$ and $C$ represents a factor of normalization.

Note that the three variables $x, \theta, z$ are related each other as follows:

$$
z = e^{i\theta}, \quad x = \cos \theta, \quad x = \frac{z + z^{-1}}{2}.
$$

For $|a|, |b|, |c|, |d| < 1$ and $n \geq 0$, let $h_n := h_n(a, b, c, d, q)$ with

$$
h_0 = \frac{(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty},
$$

$$
h_n = h_0 \frac{(1 - q^{n-1}abcd)(q, ab, ac, ad, bc, bd, cd; q)_n}{(1 - q^{2n-1}abcd)(abcd; q)_n}.
$$

Then, the orthogonality of Askey-Wilson polynomials reads

$$
\int_C p_m \cos \theta; a, b, c, d; q) p_n \cos \theta; a, b, c, d; q) w(\cos \theta) \frac{dz}{4\pi i z} = \frac{h_n}{h_0} \delta_{mn},
$$

(4.2)

where the contour $C$ is the unit circle with suitable deformations (see [3, 4]) and the weight function $w(x) := w(x, a, b, c, d; q)$ is defined by

$$
w(\cos \theta) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{h_0 \cdot (ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_\infty}.
$$

Except the constant factor $C$, the following representation for the Askey-Wilson polynomials in terms of moments was already given by Atakishiyev and Suslov [2]. Their proof was based on a generalization of Hahn’s approach and Wilson [20] suggested a method to evaluate this determinant directly.

**Theorem 4.1.** For $n \geq 1$, $0 \leq i \leq n - 1$ and $0 \leq j \leq n$ let

$$
A_{i,j} = (ac, ad; q)_i (bc, bd; q)_j \frac{(ab; q)_{i+j}}{(abcd; q)_{i+j}}
$$

(4.3)

and $A_{n,j} = (bz, b/z; q)_j$. Then

$$
\det(A_{i,j})_{0 \leq i, j \leq n} = C \cdot p_n(x; a, b, c, d; q),
$$

(4.4)

where $x = (z + z^{-1})/2$ and

$$
C = (-1)^n a^{n(n-1)/2} b^{n(n+1)/2} q^{n(n-1)(2n-1)/6} \prod_{i=0}^{n-1} \frac{(ab, ac, ad, bc, bd, cd, q; q)_i}{(abcd; q)_{n+i}}.
$$
Proposition 4.3. The two determinant formulae (4.4) are (3.3) equivalent.

Proof. By (4.2) we have \( \oint_C w(\cos \theta) \frac{d\theta}{4\pi iz} = 1 \). It follows that

\[
\oint_C w(x, a, b, c, d; q)(az, a/z; q)_i(bz, b/z; q)_j \frac{dz}{4\pi iz} = \frac{h_0(aq^j, bq^j, c, d; q)}{h_0(a, b, c, d, q)} \oint_C w(x, aq^j, bq^j, c, d; q) \frac{dz}{4\pi iz} = (ac, ad; q)_i(bc, bd; q)_j \frac{(ab; q)_{i+j}}{(abcd; q)_{i+j}},
\]

which coincides with \( A_{i,j} \) in (4.3) for \( i, j = 0, \ldots, n \). Thus, by choosing \( \psi_i = (az, a/z; q)_i \) and \( \phi_j = (bz, b/z; q)_j \) in (4.1) we obtain the determinant in (4.4). It remains to compute the factor \( C \). As

\[
p_n(x; a, b, c, d; q) = 2^n(abcdq^{-n}; q)_n x^n + \text{lower terms},
\]

comparing the coefficients of \( x^n \) in (4.1) we get

\[
C = (-1)^n h^n q^{n(n-1)/2} \frac{abq^n(abcdq^{-n}; q)_n}{(abcdq^n; q)_n} \det(A_{i,j})_{0 \leq i, j \leq n-1}.
\]

The determinant in (4.6) is essentially the Hankel determinant associated to the moments of little \( q \)-Jacobi polynomials (see [10])

\[
\det \left( \frac{(ab; q)_{i+j}}{(abcd; q)_{i+j}} \right)_{0 \leq i, j \leq n-1} = (ab)^{n(n-1)/2} q^{n(n-1)(n-2)/3} \prod_{k=0}^{n-1} \frac{(q, ab, cd; q)_k}{(abcd; q)_{k+n-1}},
\]

which is also a special case of (3.8). It follows from (4.7) that

\[
\det \left( \frac{(ac, ad; q)_i(bc, bd; q)_j (ab; q)_{i+j}}{(abcd; q)_{i+j}} \right)_{0 \leq i, j \leq n-1} = (ab)^{n(n-1)/2} q^{n(n-1)(n-2)/3} \prod_{j=1}^{n-1} (ac, ad, bc, bd; q)_j \prod_{k=0}^{n-1} \frac{(q, ab, cd; q)_k}{(abcd; q)_{k+n-1}}.
\]

Substituting this in (4.6) we recover the factor \( C \) in formula (4.4). \( \square \)

Remark 4.2. Using (4.8), formula (4.4) can also be proven from the Desnanot-Jacobi adjoint matrix theorem and a special case of the known contiguous relation (see [13, (3.3)])

\[
r_C \left[ aq, A, B, q, z \right] - n_C \left[ a, b, B, q, z \right] = \frac{(-1)^{1+s-r}z(a-b)}{(1-b)(1-bq)} \prod_{i=1}^{n-1}(1-A_i) \prod_{i=1}^{n-1}(1-B_i) r_C \left[ aq, Aq, b, Bq, q, q^{1+s-r}z \right],
\]

where \( A = (A_1, \ldots, A_{r-1}) \) and \( B = (B_1, \ldots, B_{s-1}) \) for \( r, s \geq 2 \).

Actually it is not hard to see that Theorems 3.1 and 4.1 are equivalent.
Proof. Let \( z = e^{i\theta} \) and \( x = \cos \theta \). Consider the matrix \( A := (A_{i,j})_{0 \leq i,j \leq n} \), where \( A_{i,j} \) are given in (4.3). Upon multiplying the \((j-1)\)st column of \( A \) by \( 1-2bxq^{j-1} + b^2q^{2j-2} \) and subtracting from the \( j \)th column, for \( j = n, n-1, \ldots, 1 \), the last row of the matrix becomes \((1,0,\ldots,0)\), and

\[
A_{i,j} - (1-2bxq^{j-1} + b^2q^{2j-2})A_{i,j-1} = (ac, ad; q)_i(bc, bd; q)_{j-1}B_{i,j},
\]

where \( B_{i,j} \) is given in (3.2) for \( 0 \leq i \leq n-1 \) and \( 1 \leq j \leq n \). Hence, the two formulae (4.4) and (3.3) are equivalent. \( \square \)

By (4.2) the linear functional \( L : \mathbb{C}[x] \to \mathbb{C} \) associated to the orthogonal measure of the Askey-Wilson polynomials has the explicit integral expression:

\[
L(x^n) = \int_{\mathcal{C}} \left( \frac{z+z^{-1}}{2} \right)^n w(\cos \theta) \frac{dz}{4\pi i z}.
\]

It follows from (4.5) with \( j = 0 \) and \( i = n \) that

\[
L((az, a/z; q)_n) = \frac{(ab, ac, ad; q)_n}{(abcd; q)_n}.
\]

Clearly, if we take \( \psi_i(x) = \phi_i(x) = x^i \) for \( i \geq 0 \) in (4.1), then \( \mu_{i,j} = L(x^{i+j}) \) are the moments of Askey-Wilson polynomials and we obtain another determinant expression for the Askey-Wilson polynomials. Such a formula would be interesting if we have a simple formula for the moments (4.10). Recently Corteel et al. [4] and Ismail-Rahman [5] have published a double-sum formula for the moments of Askey-Wilson polynomials. We would like to point out that their formula does follow straightforwardly from (4.11) and the Newton interpolation formula (see [17, Chapter 1]), that we recall below.

**Theorem 4.4** (Newton’s interpolation formula). Let \( b_0, b_1, \ldots, b_{n-1} \) be distinct complex numbers. Then, for any polynomial \( f \) of degree less than or equal to \( n \) we have

\[
f(x) = \sum_{k=0}^{n} \left( \sum_{j=0}^{k} \frac{f(b_j)}{\prod_{r=0,r\neq j}^{k} (b_j - b_r)} \right) (x-b_0) \cdots (x-b_{k-1}).
\]

Indeed, if \( b_j = (q^{-j}/a + aq^j)/2 \) for \( j = 0, \ldots, n-1 \), then

\[
\prod_{r=0}^{j-1} (b_j - b_r) = (-1)^{j-1} 2^{-j} a^j q^{(j\choose 2)} (q, q^{-2j+1}/a^2; q)_j,
\]

\[
\prod_{r=j+1}^{k} (b_j - b_r) = (-1)^{k-j} 2^{-k-j} a^{k-j} q^{-(j+1)(k-j)-(k-j\choose 2)} (q, a^2 q^{2j+1}; q)_{k-j}.
\]

As \( x = (z + 1/z)/2 \) we have

\[
(x-b_0) \cdots (x-b_{k-1}) = (-1)^k 2^{-k} a^{-k} q^{-(k\choose 2)} (az, a/z; q)_k.
\]

Substituting the above into (4.12) we obtain Proposition 3.1 of [4], i.e.,

\[
f(x) = \sum_{k=0}^{n} (az, a/z; q)_k \sum_{j=0}^{k} q^{k-j} a^{-2j} f(q^a + q^{-j}/a) (q, q^{2j+1}/a^2; q)_j (q, q^{2j+1}/a^2; q)_{k-j}.
\]
In particular, applying the linear functional $\mathcal{L}$ to the two sides of (4.13) with $f(x) = (t + x)^n$ we obtain Theorem 1.13 of [4] with a typo corrected, which also appears in [9].

**Theorem 4.5** (Corteel-Stanley-Stanton-Williams, Ismail-Rahman). For any fixed $t \in \mathbb{C}$, we have

\[
\mathcal{L}((t + x)^n) = \sum_{k=0}^{n} \frac{(ac, ab, ad; q)_k}{(abcd; q)_k} \sum_{j=0}^{k} \frac{q^{k-j} a^{-2j} (t + \frac{q^j a + q^{-j} a}{2})^n}{(q, q^{-2j+1}/a^2; q)_j (q, q^{2j+1}/a^2; q)_{k-j}}.
\]

**Remark 4.6.** The proof of the above formula in [4, 9] was based on a result of Ismail and Stanton [4, Theorem 3.3], which is equivalent to (4.13). Since Ismail and Stanton’s formula was originally proved using the Askey-Wilson operator in [5, Theorem 20], our proof seems more accessible for people who are not familiar with Askey-Wilson operator. The $t = 0$ of (4.14) is the starting point of [12].

Finally, it is interesting to note that the following special case of Newton’s formula (4.12) has been rediscovered recently by Mansour et al. [15].

**Corollary 4.7.** We have

\[
(x + a_0) \cdots (x + a_{n-1}) = \sum_{k=0}^{n} u(n, k) (x - b_0) \cdots (x - b_{k-1}),
\]

where

\[
u(n, k) = \sum_{r=0}^{k} \frac{\prod_{j=0}^{r-1} (b_r + a_j)}{\prod_{j=0, j \neq x} (b_r - b_j)} \quad \text{for } k = 0, \ldots, n.
\]

**Remark 4.8.** Clearly the connection coefficients $u(n, k)$ in (4.15) are characterized by the recurrence

\[
u(n, k) = \nu(n-1, k-1) + (a_{n-1} + b_k) \nu(n-1, k)
\]

with boundary conditions $u(n, 0) = \prod_{j=0}^{n-1} (a_i + b_0)$ and $u(0, k) = \delta_{0,k}$ (the Kronecker delta function). Hence we recover the main result of Mansour et al. [15] (see also [19]).

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