Some congruences involving central q-binomial coefficients*

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Abstract. Motivated by recent works of Sun and Tauraso, we prove some variations on the Green-Krammer identity involving central q-binomial coefficients, such as

$$\sum_{k=0}^{n-1}(-1)^k q^{-\binom{k+1}{2}} {2k \brack k}_q \equiv \left(\frac{n}{5}\right) q^{-\lfloor n^4/5 \rfloor} \pmod{\Phi_n(q)},$$

where $\left(\frac{n}{p}\right)$ is the Legendre symbol and $\Phi_n(q)$ is the nth cyclotomic polynomial. As consequences, we deduce that

$$\sum_{k=0}^{3^a m-1} q^k {2k \brack k}_q \equiv 0 \pmod{(1-q^{3^a})/(1-q)},$$

$$\sum_{k=0}^{5^a m-1} (-1)^k q^{-\binom{k+1}{2}} {2k \brack k}_q \equiv 0 \pmod{(1-q^{5^a})/(1-q)},$$

for $a, m \geq 1$, the first one being a partial q-analogue of the Strauss-Shallit-Zagier congruence modulo powers of 3. Several related conjectures are proposed.

Keywords: central binomial coefficients, q-binomial coefficient, congruence, cyclotomic polynomial

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1. Introduction

The p-adic order of several sums involving central binomial coefficients have attracted much attention. For example, among other things, Pan and Sun [13] and Sun and Tauraso

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[18, 19] proved the following congruences modulo a prime p:

$$\sum_{k=0}^{p^a-1} \binom{2k}{k+d} \equiv \left(\frac{p^a-|d|}{3}\right) \pmod{p}, \tag{1.1}$$

$$\sum_{k=0}^{p^a-1} (-1)^k \binom{2k}{k} \equiv \left(\frac{p^a}{5}\right) \pmod{p}, \tag{1.2}$$

and, for $p \geq 3$,

$$\sum_{k=0}^{p^a-1} \binom{2k}{k+d} 2^{-k} \equiv \begin{cases} 0 \pmod{p} & \text{if } p^a \equiv |d| \pmod{2}, \\ 1 \pmod{p} & \text{if } p^a \equiv |d|+1 \pmod{4}, \\ -1 \pmod{p} & \text{if } p^a \equiv |d|-1 \pmod{4}, \end{cases} \tag{1.3}$$

where $\left(\frac{n}{p}\right)$ is the Legendre symbol. It is well known that binomial identities or congruences usually have nice q-analogues (see [2]). Recently Tauraso [20] has noticed that an identity of Greene-Krammer [9] can be served as an inspiration for searching q-analogues of some identities in [18, 19], and, in particular, he has proved the following generalization of (1.1):

$$\sum_{k=0}^{n-1} q^k \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q \equiv \left(\frac{n-|d|}{3}\right) q^{\frac{3}{2}r(r+1)+|d|(2r+1)} \pmod{\Phi_n(q)} \tag{1.4}$$

with $r = \lfloor 2(n-|d|)/3 \rfloor$. Here and in what follows $\Phi_n(q)$ denotes the nth cyclotomic polynomial, and $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficient defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise,} \end{cases}$$

where $(z;q)_n = (1-z)(1-zq)\cdots(1-zq^{n-1})$ is the q-shifted factorial for $n \geq 0$.

The purpose of this paper is to study some q-versions of (1.1)–(1.3) as well as some variations of the same flavor as in [20]. For example, from a q-analogue of (1.2), we will deduce the following two congruences:

$$\sum_{k=0}^{3^a m-1} q^k \begin{bmatrix} 2k \\ k \end{bmatrix}_q \equiv 0 \pmod{(1-q^{3^a})/(1-q)}, \tag{1.5}$$

$$\sum_{k=0}^{5^a m-1} (-1)^k q^{-\binom{k+1}{2}} \begin{bmatrix} 2k \\ k \end{bmatrix}_q \equiv 0 \pmod{(1-q^{5^a})/(1-q)}. \tag{1.6}$$

Note that (1.5) may be deemed to be a partial q-analogue of the Strauss-Shallit-Zagier congruence [15]:

$$\sum_{k=0}^{3^a m-1} \binom{2k}{k} \equiv 0 \pmod{3^{2a}}.$$

The rest of the paper is organized as follows. In Section 2 we will give a q-analogue of (1.2) by using a finite Rogers-Ramanujan identity due to Schur. In Section 3 we will prove (1.5) and (1.6). Some different q-analogues of (1.3) will be given in Section 4 and some open problems will be proposed in the last section.

2. A q-analogue of (1.2)

It was conjectured by Krammer and proved by Greene [9] that

$$1+2\sum_{k=1}^{n-1}(-1)^k q^{-\binom{k}{2}}\begin{bmatrix}2k-1\\k\end{bmatrix}_q=\begin{cases}\left(\dfrac{m}{5}\right)\sqrt{5}, & \text{if } n\equiv 0 \pmod 5,\\ \left(\dfrac{n}{5}\right), & \text{otherwise,}\end{cases} \tag{2.1}$$

where $q=e^{2\pi mi/n}$ with $\gcd(m,n)=1$ (see also [3,6] for some related results). If $n=p^a$, then the left-hand side of (2.1) is a q-analogue of that of (1.2). However, we cannot deduce the Sun-Tauraso congruence (1.2) from (2.1) in the case $n\equiv 0 \pmod 5$. In this section we shall give a new q-series identity which is similar to (2.1) and will imply the Sun-Tauraso congruence (1.2) completely.

Theorem 2.1. *For $n\geq 0$, there holds*

$$\sum_{k=0}^{n-1}(-1)^k q^{-\binom{k+1}{2}}\begin{bmatrix}2k\\k\end{bmatrix}_q\equiv\left(\frac{n}{5}\right)q^{-\lfloor n^4/5\rfloor} \pmod{\Phi_n(q)}. \tag{2.2}$$

In other words, letting $\omega=e^{2\pi mi/n}$ with $\gcd(m,n)=1$, we have

$$\sum_{k=0}^{n-1}(-1)^k\omega^{-\binom{k+1}{2}}\begin{bmatrix}2k\\k\end{bmatrix}_\omega=\begin{cases}0, & \text{if } n\equiv 0 \pmod 5,\\ \omega^{-\lfloor n/5\rfloor}, & \text{if } n\equiv 1 \pmod 5,\\ -\omega^{-\lfloor 3n/5\rfloor}, & \text{if } n\equiv 2 \pmod 5,\\ -\omega^{-\lfloor 2n/5\rfloor}, & \text{if } n\equiv 3 \pmod 5,\\ \omega^{-\lfloor 4n/5\rfloor}, & \text{if } n\equiv 4 \pmod 5.\end{cases} \tag{2.3}$$

Proof. Since $\omega^k\neq 1$ for $1\leq k\leq n-1$ and $\omega^n=1$, we can write

$$\begin{aligned}\omega^{-\binom{k+1}{2}}\begin{bmatrix}2k\\k\end{bmatrix}_\omega&=\omega^{-\binom{k+1}{2}}\prod_{j=1}^{k}\frac{1-\omega^{2k+1-j}}{1-\omega^j}\\&=\omega^{-\binom{k+1}{2}}(-1)^k\omega^{(3k^2+k)/2}\prod_{j=1}^{k}\frac{1-\omega^{n-(2k+1-j)}}{1-\omega^j}\\&=(-1)^k\omega^{k^2}\begin{bmatrix}n-k-1\\k\end{bmatrix}_\omega.\end{aligned} \tag{2.4}$$

Therefore, we derive from Schur's identity (see, for example, [4, p. 50]) that

$$\begin{aligned}\sum_{k=0}^{n-1}(-1)^k\omega^{-\binom{k+1}{2}}\begin{bmatrix}2k\\k\end{bmatrix}_\omega &= \sum_{k=0}^{n-1}\omega^{k^2}\begin{bmatrix}n-k-1\\k\end{bmatrix}_\omega \\ &= \sum_{j=-\infty}^{\infty}(-1)^j\omega^{\frac{j(5j+1)}{2}}\begin{bmatrix}n-1\\ \lfloor\frac{n-5j-1}{2}\rfloor\end{bmatrix}_\omega. \end{aligned} \tag{2.5}$$

Since $\omega^n = 1$, we have

$$\begin{bmatrix}n-1\\k\end{bmatrix}_\omega = \prod_{i=1}^{k}\frac{1-\omega^{n-i}}{1-\omega^i} = \prod_{i=1}^{k}\frac{1-\omega^{-i}}{1-\omega^i} = (-1)^k\omega^{-\binom{k+1}{2}}$$

for $0 \le k \le n-1$, and the identity (2.3) is easily deduced. For example, if $n = 5m$, then there are $2m$ non-zero terms in the right-hand side of (2.5). But the terms indexed $j = -m + 2k$ and $j = -m + 2k + 1$ cancel each other for $k = 0, \dots, m-1$. □

Replacing q by q^{-1}, one sees that (2.2) is equivalent to

$$\sum_{k=0}^{n-1}(-1)^k q^{-\binom{k}{2}}\begin{bmatrix}2k\\k\end{bmatrix}_q \equiv \left(\frac{n}{5}\right) q^{\lfloor n^4/5\rfloor} \pmod{\Phi_n(q)}.$$

If $n = p^a$ is a prime power, letting $q = 1$ in (2.2), one immediately gets the Sun-Tauraso congruence (1.2) by the formula

$$\Phi_n(1) = \begin{cases} p, & \text{if } n = p^a \text{ is a prime power,} \\ 1, & \text{otherwise.}\end{cases} \tag{2.6}$$

(Eq. (2.6) follows from the identity $q^n - 1 = \prod_{d|n}\Phi_d(q)$ by induction.)

Remark. The first part of the proof of Theorem 2.1 can be generalized as follows. We define the q-Fibonacci polynomials (see [5]) by $F_0^q(t) = 0$, $F_1^q(t) = 1$, and

$$F_n^q(t) = F_{n-1}^q(t) + q^{n-2}tF_{n-2}^q(t), \quad n \ge 2.$$

The following is an explicit formula for the q-Fibonacci polynomials:

$$F_n^q(t) = \sum_{k\ge 0} q^{k^2}\begin{bmatrix}n-k-1\\k\end{bmatrix}_q t^k. \tag{2.7}$$

Let $n > d \ge 0$ and let ω be as in Theorem 2.1. Similarly to (2.4), we have

$$\begin{aligned}\omega^{-\binom{k-d}{2}}\begin{bmatrix}2k\\k+d\end{bmatrix}_\omega &= \omega^{-\binom{k-d}{2}}(-1)^{k-d}\omega^{(3k+d+1)(k-d)/2}\prod_{j=1}^{k-d}\frac{1-\omega^{2n-(2k+1-j)}}{1-\omega^j} \\ &= (-1)^{k-d}\omega^{(k+d+1)(k-d)}\begin{bmatrix}2n-k-d-1\\k-d\end{bmatrix}_\omega,\end{aligned}$$

which yields the following congruence

$$\sum_{k=0}^{n-1} q^{-\binom{k-d}{2}} \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q t^k \equiv t^d F^q_{2(n-d)}(-tq^{2d+1}) \pmod{\Phi_n(q)}$$

by applying (2.7).

3. Congruences modulo $\Phi_{3^j}(q)$ and $\Phi_{5^j}(q)$

In this section we give a proof of (1.5) and (1.6). It is well known that

$$\frac{1-q^{p^a}}{1-q} = \prod_{j=1}^{a} \Phi_{p^j}(q)$$

for any prime p. We need the following two lemmas.

Lemma 3.1. *For $n \geq 0$, there holds*

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q = (-1)^n \left(\frac{n+1}{3}\right) q^{\frac{n(n-1)}{6}}.$$

Lemma 3.2. *Let m, k, d be positive integers, and write $m = ad + b$ and $k = rd + s$, where $0 \leq b, s \leq d-1$. Let ω be a primitive d-th root of unity. Then*

$$\begin{bmatrix} m \\ k \end{bmatrix}_\omega = \binom{a}{r} \begin{bmatrix} b \\ s \end{bmatrix}_\omega.$$

Remark. Lemma 3.1 has appeared in the literature from different origins (see [7]). A proof using mathematical induction is given in [20] and a multiple extension is proposed in [10]. Lemma 3.2 is equivalent to the q-Lucas theorem (see [12] and [8, Proposition 2.2]).

We first establish the following theorem.

Theorem 3.3. *Let $m, n \geq 1$. Then*

$$\sum_{k=0}^{mn-1} q^k \begin{bmatrix} 2k \\ k \end{bmatrix}_q \equiv \sum_{j=0}^{m-1} \binom{2j}{j} \sum_{k=0}^{n-1} q^k \begin{bmatrix} 2k \\ k \end{bmatrix}_q \pmod{\Phi_n(q)}, \tag{3.1}$$

$$\sum_{k=0}^{mn-1} (-1)^k q^{-\binom{k+1}{2}} \begin{bmatrix} 2k \\ k \end{bmatrix}_q \equiv \sum_{j=0}^{m-1} (-1)^j \binom{2j}{j} \sum_{k=0}^{n-1} (-1)^k q^{-\binom{k+1}{2}} \begin{bmatrix} 2k \\ k \end{bmatrix}_q \pmod{\Phi_n(q)}. \tag{3.2}$$

Proof. Let $q=\omega$ be a primitive nth root of unity. Then $\omega^n=1$ and

$$\sum_{k=jn}^{jn+n-1}\omega^k\begin{bmatrix}2k\\k\end{bmatrix}_\omega=\sum_{k=0}^{n-1}\omega^k\begin{bmatrix}2jn+2k\\jn+k\end{bmatrix}_\omega, \tag{3.3}$$

$$\sum_{k=jn}^{jn+n-1}(-1)^k\omega^{-\binom{k+1}{2}}\begin{bmatrix}2k\\k\end{bmatrix}_\omega=(-1)^{jn}\sum_{k=0}^{n-1}(-1)^k\omega^{-\binom{jn+k+1}{2}}\begin{bmatrix}2jn+2k\\jn+k\end{bmatrix}_\omega. \tag{3.4}$$

By Lemma 3.2 we have

$$\begin{bmatrix}2jn+2k\\jn+k\end{bmatrix}_\omega=\binom{2j}{j}\begin{bmatrix}2k\\k\end{bmatrix}_\omega,$$

which is equal to 0 if $2k\geq n$. Noticing that

$$\omega^{-\binom{jn+k+1}{2}}=\omega^{-\binom{jn+1}{2}}\cdot\omega^{-\binom{k+1}{2}}$$

and

$$(-1)^{jn}\omega^{-\binom{jn+1}{2}}=(-1)^j,$$

we can write Eqs. (3.3) and (3.4) as

$$\sum_{k=jn}^{jn+n-1}\omega^k\begin{bmatrix}2k\\k\end{bmatrix}_\omega=\binom{2j}{j}\sum_{k=0}^{n-1}\omega^k\begin{bmatrix}2k\\k\end{bmatrix}_\omega, \tag{3.5}$$

$$\sum_{k=jn}^{jn+n-1}(-1)^k\omega^{-\binom{k+1}{2}}\begin{bmatrix}2k\\k\end{bmatrix}_\omega=(-1)^j\binom{2j}{j}\sum_{k=0}^{n-1}(-1)^k\omega^{-\binom{k+1}{2}}\begin{bmatrix}2k\\k\end{bmatrix}_\omega. \tag{3.6}$$

Summing (3.5) and (3.6) over j from 0 to $m-1$, we complete the proof. □

We now state our main theorem in this section.

Theorem 3.4. *Let $a\geq 1$ and $m\geq 1$. Then*

$$\sum_{k=0}^{3^am-1}q^k\begin{bmatrix}2k\\k\end{bmatrix}_q\equiv 0\pmod{\prod_{j=1}^{a}\Phi_{3^j}(q)}, \tag{3.7}$$

$$\sum_{k=0}^{5^am-1}(-1)^kq^{-\binom{k+1}{2}}\begin{bmatrix}2k\\k\end{bmatrix}_q\equiv 0\pmod{\prod_{j=1}^{a}\Phi_{5^j}(q)}. \tag{3.8}$$

Proof. Let ω be a primitive nth root of unity. Then

$$\omega^k\begin{bmatrix}2k\\k\end{bmatrix}_\omega=\omega^{k^2+k}\begin{bmatrix}2k\\k\end{bmatrix}_{\omega^{-1}}=\mathrm{conj}\left(\omega^{-k^2-k}\begin{bmatrix}2k\\k\end{bmatrix}_\omega\right),$$

where $\mathrm{conj}(z)$ denotes the complex conjugate of $z\in\mathbb{C}$. From (2.4) we deduce that

$$\omega^{-k^2-k}\begin{bmatrix}2k\\k\end{bmatrix}_\omega=(-1)^k\omega^{\binom{k}{2}}\begin{bmatrix}n-k-1\\k\end{bmatrix}_\omega.$$

Therefore, by Lemma 3.1, we have

$$\sum_{k=0}^{n-1}\omega^k\begin{bmatrix}2k\\k\end{bmatrix}_\omega=\mathrm{conj}\left(\sum_{k=0}^{n-1}(-1)^k\omega^{\binom{k}{2}}\begin{bmatrix}n-k-1\\k\end{bmatrix}_\omega\right)=(-1)^{n-1}\left(\frac{n}{3}\right)\omega^{-\frac{(n-1)(n-2)}{6}}.$$

This implies that

$$\sum_{k=0}^{n-1}q^k\begin{bmatrix}2k\\k\end{bmatrix}_q\equiv 0\pmod{\Phi_n(q)}\quad\text{if}\quad 3|n,\tag{3.9}$$

which also follows directly from Tauraso's congruence (1.4).

Now, letting $n=3^j$ with $1\le j\le a$ in (3.9) and letting $n=5^j$ with $1\le j\le a$ in (2.2), we get

$$\sum_{k=0}^{3^j-1}q^k\begin{bmatrix}2k\\k\end{bmatrix}_q\equiv 0\pmod{\Phi_{3^j}(q)},$$
$$\sum_{k=0}^{5^j-1}(-1)^kq^{-\binom{k+1}{2}}\begin{bmatrix}2k\\k\end{bmatrix}_q\equiv 0\pmod{\Phi_{5^j}(q)}.$$

Letting $m\to 3^{a-j}m$, $n\to 3^j$ in (3.1) and $m\to 5^{a-j}m$, $n\to 5^j$ in (3.2) respectively, we obtain

$$\sum_{k=0}^{3^am-1}q^k\begin{bmatrix}2k\\k\end{bmatrix}_q\equiv 0\pmod{\Phi_{3^j}(q)}\quad(1\le j\le a),$$
$$\sum_{k=0}^{5^am-1}(-1)^kq^{-\binom{k+1}{2}}\begin{bmatrix}2k\\k\end{bmatrix}_q\equiv 0\pmod{\Phi_{5^j}(q)}\quad(1\le j\le a).$$

Since the cyclotomic polynomials are pairwise relatively prime, we complete the proof of (3.7) and (3.8). □

We have the following conjecture.

Conjecture 3.5. *Let $a\ge 1$ and $m\ge 1$. Then*

$$\sum_{k=0}^{3^am-1}q^k\begin{bmatrix}2k\\k\end{bmatrix}_q\equiv 0\pmod{\prod_{j=1}^{a}\Phi_{3^j}^2(q)},$$
$$\sum_{k=0}^{5^a-1}(-1)^k\binom{2k}{k}\equiv 5^a\pmod{5^{a+1}}.$$

We now give a dual form of Theorem 2.1. The reader is encouraged to compare it with [20, Theorem 5.1].

Theorem 3.6. *Let $q = e^{2\pi mi/n}$ with $\gcd(m,n)=1$. Then*

$$\sum_{k=0}^{n-1} q^{2k+1} \begin{bmatrix} 2k \\ k \end{bmatrix}_q = \begin{cases} \left(\frac{m}{3}\right) i\sqrt{3}, & \text{if } 3|n, \\ \left(\frac{n}{3}\right), & \text{otherwise.} \end{cases}$$

Proof. First note that

$$q^{2k+1} \begin{bmatrix} 2k \\ k \end{bmatrix}_q = q^{(k+1)^2} \begin{bmatrix} 2k \\ k \end{bmatrix}_{q^{-1}} = \operatorname{conj}\left(q^{-(k+1)^2} \begin{bmatrix} 2k \\ k \end{bmatrix}_q \right)$$

and $\Phi_n(q) = 0$. From (2.4) we deduce that

$$q^{-(k+1)^2} \begin{bmatrix} 2k \\ k \end{bmatrix}_q = (-1)^k q^{\binom{k-1}{2}-2} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q .$$

Therefore,

$$\sum_{k=0}^{n-1} q^{2k+1} \begin{bmatrix} 2k \\ k \end{bmatrix}_q = \operatorname{conj}\left(\sum_{k=0}^{n-1} (-1)^k q^{\binom{k-1}{2}-2} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q \right).$$

Since

$$q^{\binom{k-1}{2}-2} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q = q^{-n} \left(q^{\binom{k+1}{2}} \begin{bmatrix} n-k \\ k+1 \end{bmatrix}_q - q^{\binom{k+1}{2}} \begin{bmatrix} n-k-1 \\ k+1 \end{bmatrix}_q \right),$$

by Lemma 3.1 we have

$$\sum_{k=0}^{n-1} (-1)^k q^{\binom{k-1}{2}} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q = (-1)^n \left(\left(\frac{n+2}{3}\right) q^{\frac{n(n-5)}{6}} + \left(\frac{n+1}{3}\right) q^{\frac{n(n-7)}{6}} \right).$$

The result then follows easily. □

Corollary 3.7. *For any positive integer n with $\gcd(n,3)=1$, there holds*

$$\sum_{k=0}^{n-1} q^{2k+1} \begin{bmatrix} 2k \\ k \end{bmatrix}_q \equiv \left(\frac{n}{3}\right) \pmod{\Phi_n(q)}.$$

For the following remarkable congruence of Sun and Tauraso [18, (1.1) with $d=0$]:

$$\sum_{k=0}^{p^a-1} \binom{2k}{k} \equiv \left(\frac{p^a}{3}\right) \pmod{p^2},$$

we have two interesting q-versions to offer:

Conjecture 3.8. *Let p be a prime and $a \geq 1$. Then*

$$\sum_{k=0}^{p^a-1} q^k \begin{bmatrix} 2k \\ k \end{bmatrix}_q \equiv \left(\frac{p^a}{3}\right) q^{\left\lfloor \frac{p^a}{2} - \left(\frac{p^a}{3}\right)\frac{p^a}{6}\right\rfloor + \left\lfloor \frac{p^a}{3}\right\rfloor p^a} \pmod{\Phi_{p^a}^2(q)},$$

and, for $p \neq 3$,

$$\sum_{k=0}^{p^a-1} q^{2k+1} \begin{bmatrix} 2k \\ k \end{bmatrix}_q \equiv \left(\frac{p^a}{3}\right) q^{\left(\left\lfloor \frac{p^a+1}{3}\right\rfloor + \left(\frac{p^a}{3}\right)\right) p^a} \pmod{\Phi_{p^a}^2(q)}.$$

4. Some q-analogues of (1.3)

To give q-analogues of (1.3), we need to establish the following q-series identities:

Theorem 4.1. *Let $n \geq 1$ and $d = 0, 1, \ldots, n$. Then*

$$\sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q (-q^{k+1}; q)_{n-k} = \begin{cases} q^{\frac{n^2-d^2}{2}} \begin{bmatrix} n \\ \frac{n-d}{2} \end{bmatrix}_{q^2}, & \text{if } n-d \text{ is even} \\ 0, & \text{if } n-d \text{ is odd,} \end{cases} \tag{4.1}$$

$$\sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q (-q^{k}; q)_{n-k} = \begin{cases} q^{\frac{n^2-d^2}{2}} \begin{bmatrix} n \\ \frac{n-d}{2} \end{bmatrix}_{q^2}, & \text{if } n-d \text{ is even} \\ q^{\frac{n^2-d^2-1}{2}} (q^n - 1) \begin{bmatrix} n-1 \\ \frac{n-d-1}{2} \end{bmatrix}_{q^2}, & \text{if } n-d \text{ is odd.} \end{cases} \tag{4.2}$$

Proof. The $d = 0$ case of (4.1) was found by Andrews [2, Theorem 5.5]. Both (4.1) and (4.2) can be proved similarly by using Andrews's q-analogue of Gauss's second theorem [1,2]:

$$\sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k q^{\binom{k+1}{2}}}{(q;q)_k (abq;q^2)_k} = \frac{(-q;q)_\infty (aq;q^2)_\infty (bq;q^2)_\infty}{(abq;q^2)_\infty}, \tag{4.3}$$

where $(z;q)_\infty = \lim_{n\to\infty} (z;q)_n$. We first sketch the proof of (4.1).

Recall that $(q;q)_{2n} = (q;q^2)_n (q^2;q^2)_n$, $(a;q)_n(-a;q)_n = (a^2;q^2)_n$ and

$$(a;q)_{n-k} = \frac{(a;q)_n}{(q^{1-n}/a;q)_k} \left(-\frac{q}{a}\right)^k q^{\binom{k}{2} - nk}.$$

Replacing k by $n-k$, we can write the left-hand side of (4.1) as

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} 2n-2k \\ n-k+d \end{bmatrix}_q (-q^{n-k+1};q)_k \\
&= \begin{bmatrix} 2n \\ n+d \end{bmatrix}_q \sum_{k=0}^{n} \frac{(q^{-n-d};q)_k (q^{-n+d};q)_k q^{k(k+1)/2}}{(q;q)_k (q^{-2n+1};q^2)_k} \\
&= \begin{bmatrix} 2n \\ n+d \end{bmatrix}_q \frac{(-q;q)_\infty (q^{-n-d+1};q^2)_\infty (q^{-n+d+1};q^2)_\infty}{(q^{-2n+1};q^2)_\infty} \qquad \text{(by (4.3))} \\
&= \begin{cases} q^{\frac{n^2-d^2}{2}} \begin{bmatrix} n \\ \frac{n-d}{2} \end{bmatrix}_{q^2}, & \text{if } n-d \text{ is even} \\ 0, & \text{if } n-d \text{ is odd.} \end{cases}
\end{aligned}$$

This proves (4.1).

Observing that

$$\frac{(a;q)_k (b;q)_k q^{\binom{k}{2}}}{(q;q)_k (abq;q^2)_k} - \frac{(a;q)_k (b;q)_k q^{\binom{k+1}{2}}}{(q;q)_k (abq;q^2)_k} = \frac{(1-a)(1-b)(aq;q)_{k-1} (bq;q)_{k-1} q^{\binom{k}{2}}}{(1-abq)(q;q)_{k-1} (abq^3;q^2)_{k-1}},$$

we derive the following q-series identity from (4.3):

$$\sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k q^{\binom{k}{2}}}{(q;q)_k (abq;q^2)_k} = \frac{(-q;q)_\infty (aq;q^2)_\infty (bq;q^2)_\infty}{(abq;q^2)_\infty} + \frac{(-q;q)_\infty (a;q^2)_\infty (b;q^2)_\infty}{(abq;q^2)_\infty}. \tag{4.4}$$

Replacing k by $n-k$, we can write the left-hand side of (4.2) as

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} 2n-2k \\ n-k+d \end{bmatrix} (-q^{n-k};q)_k \\
&= \begin{bmatrix} 2n \\ n+d \end{bmatrix}_q \sum_{k=0}^{n} \frac{(q^{-n-d};q)_k (q^{-n+d};q)_k q^{\binom{k+1}{2}}}{(q;q)_k (q^{-2n+1};q^2)_k} \frac{(1+q^{n-k})}{(1+q^n)} \\
&= \begin{bmatrix} 2n \\ n+d \end{bmatrix}_q \frac{(-q;q)_\infty (q^{-n-d+1};q^2)_\infty (q^{-n+d+1};q^2)_\infty}{(q^{-2n+1};q^2)_\infty} \\
&\quad + \frac{q^n}{1+q^n} \begin{bmatrix} 2n \\ n+d \end{bmatrix}_q \frac{(-q;q)_\infty (q^{-n-d};q^2)_\infty (q^{-n+d};q^2)_\infty}{(q^{-2n+1};q^2)_\infty} \qquad \text{(by (4.3) and (4.4))} \\
&= \begin{cases} q^{\frac{n^2-d^2}{2}} \begin{bmatrix} n \\ \frac{n-d}{2} \end{bmatrix}_{q^2}, & \text{if } n-d \text{ is even} \\ q^{\frac{n^2-d^2-1}{2}} (q^n-1) \begin{bmatrix} n-1 \\ \frac{n-d-1}{2} \end{bmatrix}_{q^2}, & \text{if } n-d \text{ is odd.} \end{cases}
\end{aligned}$$

This proves (4.2). □

Since $q^n \equiv 1 \pmod{\Phi_n(q)}$ and

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_q = \prod_{j=1}^{k} \frac{1-q^{n-j}}{1-q^j} \equiv (-1)^k q^{-\binom{k+1}{2}} \pmod{\Phi_n(q)},$$

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} = \prod_{j=1}^{k} \frac{1-q^{2n-2j}}{1-q^{2j}} \equiv (-1)^k q^{-k(k+1)} \pmod{\Phi_n(q)},$$

$$\begin{bmatrix} n-2 \\ k \end{bmatrix}_{q^2} = \prod_{j=1}^{k} \frac{1-q^{2n-2j-2}}{1-q^{2j}} \equiv (-1)^k q^{-k(k+3)} \frac{1-q^{2k+2}}{1-q^2} \pmod{\Phi_n(q)},$$

we obtain the following result by substituting n with $n-1$ in (4.1) and (4.2).

Corollary 4.2. *Let $n \geq 1$ and $d = 0, 1, \ldots, n-1$. Then*

$$\sum_{k=0}^{n-1} q^k \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q (-q^{k+1}; q)_{n-k-1} \equiv \begin{cases} 0, & \text{if } n-d \text{ is even} \\ (-1)^{\frac{n+d-1}{2}} q^{\frac{d(2n-3d)-(n+1)^2}{4}}, & \text{if } n-d \text{ is odd} \end{cases} \pmod{\Phi_n(q)}, \tag{4.5}$$

$$\sum_{k=0}^{n-1} q^k \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q (-q^{k}; q)_{n-k-1} \equiv \begin{cases} (-1)^{\frac{n+d}{2}} q^{\frac{d(2n-3d)-n^2+2d}{4}} \dfrac{1-q^{n-d}}{1+q}, & \text{if } n-d \text{ is even} \\ (-1)^{\frac{n+d-1}{2}} q^{\frac{d(2n-3d)-(n+1)^2}{4}}, & \text{if } n-d \text{ is odd} \end{cases} \pmod{\Phi_n(q)}. \tag{4.6}$$

Replacing q by q^{-1} in (4.5), we get

$$\sum_{k=0}^{n-1} q^{-\binom{k+1}{2}} \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q (-q^{k+1}; q)_{n-k-1} \equiv \begin{cases} 0, & \text{if } n-d \text{ is even} \\ (-1)^{\frac{n+d-1}{2}} q^{\frac{1-(n-d)^2}{4}}, & \text{if } n-d \text{ is odd} \end{cases} \pmod{\Phi_n(q)}.$$

We also have the following variant of Theorem 4.1.

Theorem 4.3. *Let $n \geq 1$ and $d = 0, 1, \ldots, n$. Then*

$$\sum_{k=0}^{n} (-q)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q (-q^{k+1}; q)_{n-k} = \begin{cases} \begin{bmatrix} n \\ \frac{n-d}{2} \end{bmatrix}_{q^2}, & \text{if } n-d \text{ is even} \\ (1-q^{2n}) \begin{bmatrix} n-1 \\ \frac{n-d-1}{2} \end{bmatrix}_{q^2}, & \text{if } n-d \text{ is odd.} \end{cases} \tag{4.7}$$

$$\sum_{k=0}^{n} (-q)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q (-q^{k}; q)_{n-k} = \begin{cases} \begin{bmatrix} n \\ \frac{n-d}{2} \end{bmatrix}_{q^2}, & \text{if } n-d \text{ is even} \\ (1-q^{n}) \begin{bmatrix} n-1 \\ \frac{n-d-1}{2} \end{bmatrix}_{q^2}, & \text{if } n-d \text{ is odd.} \end{cases} \tag{4.8}$$

Proof. We would only prove (4.7), since the proof of (4.8) is similar. Replacing q by q^{-1} and multiplying by $q^{n^2-d^2+1}$, one sees that (4.7) is equivalent to the following identity:

$$\sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k-1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q (-q^{k+1}; q)_{n-k}$$
$$= \begin{cases} q^{\frac{n^2-d^2+2}{2}} \begin{bmatrix} n \\ \frac{n-d}{2} \end{bmatrix}_{q^2}, & \text{if } n-d \text{ is even} \\ q^{\frac{(n-1)^2-d^2}{2}} (q^{2n}-1) \begin{bmatrix} n-1 \\ \frac{n-d-1}{2} \end{bmatrix}_{q^2}, & \text{if } n-d \text{ is odd.} \end{cases} \tag{4.9}$$

Replacing k by $n-k$, we can write the left-hand side of (4.9) as

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k-1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} 2n-2k \\ n-k+d \end{bmatrix}_q (-q^{n-k+1}; q)_k$$
$$= \begin{bmatrix} 2n \\ n+d \end{bmatrix}_q \sum_{k=0}^{n} \frac{(q^{-n-d}; q)_k (q^{-n+d}; q)_k q^{\binom{k}{2}+1}}{(q; q)_k (q^{-2n+1}; q^2)_k}$$
$$= q \begin{bmatrix} 2n \\ n+d \end{bmatrix}_q \frac{(q^{-n-d+1}; q^2)_\infty (q^{-n+d+1}; q^2)_\infty + (q^{-n-d}; q^2)_\infty (q^{-n+d}; q^2)_\infty}{(q; q^2)_\infty (q^{-2n+1}; q^2)_\infty} \quad \text{(by (4.4))},$$

which is equal to the right-hand side of (4.9). □

Remark. Whenever they are discovered, both Theorem 4.1 and Theorem 4.3 can be proved by the q-Zeilberger algorithm (see, for example, [11, p. 113]).

As before, we have the following consequences.

Corollary 4.4. *Let $n \geq 1$ and $d = 0, 1, \ldots, n-1$. Then*

$$\sum_{k=0}^{n-1} q^{-k(k+3)/2} \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q (-q^{k+1};q)_{n-k-1}$$
$$\equiv \begin{cases} (-1)^{\frac{n+d-2}{2}} q^{\frac{5-(n-d+1)^2}{4}} (1-q^{n-d}), & \text{if } n-d \text{ is even} \\ (-1)^{\frac{n+d-1}{2}} q^{\frac{5-(n-d)^2}{4}}, & \text{if } n-d \text{ is odd} \end{cases} \pmod{\Phi_n(q)}, \tag{4.10}$$

$$\sum_{k=0}^{n-1} q^{-k(k+3)/2} \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q (-q^{k};q)_{n-k-1}$$
$$\equiv \begin{cases} (-1)^{\frac{n+d-2}{2}} q^{\frac{9-(n-d+1)^2}{4}} \dfrac{1-q^{n-d}}{1+q}, & \text{if } n-d \text{ is even} \\ (-1)^{\frac{n+d-1}{2}} q^{\frac{5-(n-d)^2}{4}}, & \text{if } n-d \text{ is odd} \end{cases} \pmod{\Phi_n(q)}. \tag{4.11}$$

If we change q to q^{-1}, then the congruence (4.10) may be rewritten as

$$\sum_{k=0}^{n-1} q^{2k} \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q (-q^{k+1};q)_{n-k-1}$$
$$\equiv \begin{cases} (-1)^{\frac{n+d}{2}} q^{\frac{d(2n-3d)-n^2+2d-4}{4}} (1-q^{n-d}), & \text{if } n-d \text{ is even} \\ (-1)^{\frac{n+d-1}{2}} q^{\frac{d(2n-3d)-(n+1)^2-4}{4}}, & \text{if } n-d \text{ is odd} \end{cases} \pmod{\Phi_n(q)},$$

while the congruences (4.6) and (4.11) exchange each other.

5. Open problems

Inspired by the $q=1$ case of congruences (1.5)–(1.6) and the work of Sun [16], we would like to make the following conjectures:

Conjecture 5.1. *Let p be a prime factor of $4m-1$ with $m \in \mathbb{Z}$ and let $a, n \geq 1$. Then*

$$\sum_{k=0}^{p^a n-1} \binom{2k}{k} m^k \equiv 0 \pmod{p^a}.$$

Conjecture 5.2. *Let m be a positive integer. Then*

$$\sum_{k=0}^{4m-2} \binom{2k}{k} m^k \equiv 0 \pmod{(4m-1)},$$
$$\sum_{k=0}^{4m} \binom{2k}{k} (-m)^k \equiv 0 \pmod{(4m+1)}.$$

It is easy to see that Conjecture 5.1 implies Conjecture 5.2 but not vice versa.

Conjecture 5.3. *Let a be a positive integer. Then*

$$\sum_{k=0}^{3^a-1}(-2)^k\binom{2k}{k}\equiv 3^a \pmod{3^{a+1}},$$

$$\sum_{k=0}^{3^a-1}(-5)^k\binom{2k}{k}\equiv 2\cdot 3^a \pmod{3^{a+1}},$$

$$\sum_{k=0}^{7^a-1}(-5)^k\binom{2k}{k}\equiv 7^a \pmod{7^{a+1}}.$$

Conjecture 5.4. *Let m be a positive integer. If $4m-1$ is a prime and $m\neq 1$, then*

$$\sum_{k=0}^{(4m-1)^a-1}\binom{2k}{k}m^k\equiv (4m-1)^a \pmod{(4m-1)^{a+1}}.$$

If $4m+1$ is a prime, then

$$\sum_{k=0}^{(4m+1)^a-1}\binom{2k}{k}(-m)^k\equiv (4m+1)^a \pmod{(4m+1)^{a+1}}.$$

Conversely, we make the following conjecture, which gives a sufficient condition for whether $4m-1$ or $4m+1$ is a prime. We have checked the cases $m\leq 1500$ via Maple, not finding any counter examples.

Conjecture 5.5. *Let m be a positive integer. If $m\neq 30$ and*

$$\sum_{k=0}^{4m-2}\binom{2k}{k}m^k\equiv 4m-1 \pmod{(4m-1)^2},$$

then $4m-1$ is a prime. If

$$\sum_{k=0}^{4m}\binom{2k}{k}(-m)^k\equiv 4m+1 \pmod{(4m+1)^2},$$

then $4m+1$ is a prime.

The following conjecture looks a little different but seems also very challenging.

Conjecture 5.6. *Let a and n be positive integers. Then*

$$\sum_{k=0}^{5^a n-1}\binom{4k}{2k}\binom{2k}{k}^2\equiv 0 \pmod{5^a},$$

$$\sum_{k=0}^{5^a-1}\binom{4k}{2k}\binom{2k}{k}^2\equiv (-1)^a 5^a \pmod{5^{a+1}}.$$

Remark. Recently, Pan and Sun [14] have confirmed the first congruence in Conjecture 3.5 and Sun [17] has proved Conjectures 5.1–5.4 (naturally including the second congruence in Conjecture 3.5).

Problem 5.7. *Are there any q-analogues of Conjectures 5.1–5.6?*

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References

[1] G.E. Andrews, On the q-analog of Kummer's theorem and applications, Duke Math. J. 40 (1973) 525–528.

[2] G.E. Andrews, Applications of basic hypergeometric functions, SIAM Rev. 16 (1974) 441–484.

[3] G.E. Andrews, On the Greene-Krammer theorem and related identities, Ganita 43 (1992) 69–73.

[4] G.E. Andrews, The Theory of Partitions, Cambridge University Press, Cambridge, 1998.

[5] L. Carlitz, Fibonacci notes IV: q-Fibonacci polynomials, Fibonacci Quart. 13 (1975) 97–102.

[6] K.J. Chen, H.M. Srivastava, A generalization of two q-identities of Andrews, J. Combin. Theory Ser. A 95 (2001) 381–386.

[7] J. Cigler, A new class of q-Fibonacci polynomials, Electron. J. Combin. 10 (2003) #R19.

[8] J. Désarménien, Un analogue des congruences de Kummer pour les q-nombres d'Euler, European J. Combin. 3 (1982) 19–28.

[9] J. Greene, On a conjecture of Krammer, J. Combin. Theory, Ser. A 56 (1991) 309–311.

[10] V.J.W. Guo, J. Zeng, Multiple extensions of a finite Euler's pentagonal number theorem and the Lucas formulas, Discrete Math. 308 (2008) 4069–4078.

[11] W. Koepf, Hypergeometric Summation, An Algorithmic Approach to Summation and Special Function Identities, Friedr. Vieweg & Sohn, Braunschweig, 1998.

[12] G. Olive, Generalized powers, Amer. Math. Monthly 72 (1965) 619–627.

[13] H. Pan, Z.-W. Sun, A combinatorial identity with application to Catalan numbers, Discrete Math. 306 (2006) 1921–1940.

[14] H. Pan, Z.-W. Sun, Some q-congruences related to 3-adic valuations, arXiv:0910.4170.

[15] N. Strauss, J. Shallit, D. Zagier, Some strange 3-adic identities, Amer. Math. Monthly 99 (1992) 66–69.

[16] Z.-W. Sun, Binomial coefficients, Catalan numbers and Lucas quotients, arXiv:0909.5648.

[17] Z.-W. Sun, p-Adic valuations of some sums of multinomial coefficients, arXiv:0910.3892.

[18] Z.-W. Sun and R. Tauraso, On some new congruences for binomial coefficients, Acta Arith., to appear.

[19] Z.-W. Sun, R. Tauraso, New congruences for central binomial coefficients, Adv. in Appl. Math. (2010), doi:10.1016/j.aam.2010.01.001.

[20] R. Tauraso, q-Analogs of some congruences involving Catalan numbers, arXiv:0905.3816v1.