# Combinatorial proofs of a kind of binomial and $q$-binomial coefficient identities*


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**Abstract.** We give combinatorial proofs of some binomial and $q$-binomial identities in the literature, such as

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{(9k^2+3k)/2}\begin{bmatrix}2n\\ n+3k\end{bmatrix}=(1+q^n)\prod_{k=1}^{n-1}(1+q^k+q^{2k})\quad(n\geq 1),$$

and

$$\sum_{k=0}^{\infty}\binom{3n}{2k}(-3)^k=(-8)^n.$$

Two related conjectures are proposed at the end of this paper.


## 1 Introduction

There are many different $q$-analogues of the following binomial coefficient identity

$$\sum_{k=-\infty}^{\infty}(-1)^k\binom{2n}{n+2k}=2^n, \tag{1.1}$$


*This work was supported in part by Shanghai "Chenguang" Project (#2007CG29) and the National Science Foundation of China (#10801054).

in the literature. Here is a list of such identities:

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{2k^2}\begin{bmatrix}2n\\n+2k\end{bmatrix}=(-q;q^2)_n, \tag{1.2}$$

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{2k^2+k}\begin{bmatrix}2n\\n+2k\end{bmatrix}=(1+q^n)(-q^2;q^2)_{n-1}, \tag{1.3}$$

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{2k^2+2k}\begin{bmatrix}2n\\n+2k\end{bmatrix}=(1+q)(-q;q^2)_{n-1}q^{n-1}, \tag{1.4}$$

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{3k^2+k}\begin{bmatrix}2n\\n+2k\end{bmatrix}=(-q;q)_n, \tag{1.5}$$

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{(5k^2+k)/2}\begin{bmatrix}2n\\n+2k\end{bmatrix}=\sum_{k=0}^{\infty}q^{k^2}\begin{bmatrix}n\\k\end{bmatrix}, \tag{1.6}$$

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{(3k^2+k)/2}\begin{bmatrix}2n\\n+2k\end{bmatrix}=\sum_{k=0}^{\infty}q^{nk}\begin{bmatrix}n\\k\end{bmatrix}, \tag{1.7}$$

where the $q$-shifted factorials are defined by $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ and the $q$-binomial coefficients are defined as

$$\begin{bmatrix}n\\k\end{bmatrix}=\begin{cases}\dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{if } 0\le k\le n,\\ 0, & \text{otherwise.}\end{cases}$$

Identities (1.2)–(1.4) can be proved by using the $q$-binomial theorem and $\mathrm{i}^2=-1$ or other methods. For (1.2), see Ismail, Kim and Stanton [5, Proposition 2(2)], Berkovich and Warnaar [2, §7], and Sills [6, (3.3)]. For (1.3), see [5, Proposition 2(3)]. The identity (1.5) corresponds to Slater's Bailey pair $C(1)$. Identities (1.6) and (1.7) were discovered by Bressoud [3, (1.1) and (1.5)], and the former is usually known as a finite form of the first Rogers-Ramanujan identity.

For each of the identities (1.2)–(1.7), one can change $q$ to $q^{-1}$ to find a new identity of the same type. The identities (1.2)–(1.4) are "self-dual", (1.6) and (1.7) are dual, and the dual of (1.5) is as follows:

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{k^2+k}\begin{bmatrix}2n\\n+2k\end{bmatrix}=q^{\binom{n}{2}}(-q;q)_n.$$

This identity is known as the Bailey pair $C(5)$ in Slater's list.

An identity similar to (1.1) is

$$\sum_{k=-\infty}^{\infty}(-1)^k\binom{2n}{n+3k}=\begin{cases}1, & \text{if } n=0,\\ 2\cdot 3^{n-1}, & \text{if } n\ge 1,\end{cases} \tag{1.8}$$

which also has two different $q$-analogues as follows:

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{(9k^2+3k)/2}\begin{bmatrix}2n\\ n+3k\end{bmatrix}=\begin{cases}1, & \text{if } n=0,\\ (1+q^n)\dfrac{(q^3;q^3)_{n-1}}{(q;q)_{n-1}}, & \text{if } n\geq 1,\end{cases}\tag{1.9}$$

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{(9k^2+9k)/2}\begin{bmatrix}2n\\ n+3k\end{bmatrix}=\begin{cases}1, & \text{if } n=0,\\ 1+q, & \text{if } n=1,\\ (1+q+q^2)(1+q^n)\dfrac{(q^3;q^3)_{n-2}}{(q;q)_{n-2}}q^{n-2}, & \text{if } n\geq 2.\end{cases}\tag{1.10}$$

Like (1.2)–(1.4), Identities (1.9) and (1.10) can be proved by the $q$-binomial theorem. Identity (1.9) is equivalent to the Bailey pair $J(2)$ in [8], and can also be found in [5, Proposition 2(5)]. This identity was utilized by Berkovich and Warnaar [2] to prove a 'perfect' Rogers-Ramanujan identity.

There exists another not-so-famous binomial coefficient identity similar to (1.1) and (1.8) as follows:

$$\sum_{k=0}^{\infty}\binom{n}{2k}(-3)^k=\begin{cases}(-2)^n, & \text{if } n\equiv 0 \pmod 3,\\ (-2)^{n-1}, & \text{if } n\equiv 1 \pmod 3,\\ (-2)^{n-1}, & \text{if } n\equiv 2 \pmod 3.\end{cases}\tag{1.11}$$

The main purpose of this paper is to give combinatorial proofs of the identities (1.1)–(1.4), (1.8)–(1.11), and some of their companions which appeared in the literature, such as

$$\sum_{k=-\infty}^{\infty}(-1)^k\binom{2n+1}{n+3k}=3^n.\tag{1.12}$$

However, we are unable to give combinatorial proofs of (1.5)–(1.7).

## 2 Proofs of (1.1)–(1.4)

*Proof of* (1.1). Let $S=\{a_1,\dots,a_{2n}\}$ be a set of $2n$ elements, and let

$$\begin{aligned}\mathscr{F}&=\{A\subseteq S\colon \#A\equiv n \pmod 2\},\\ \mathscr{G}&=\{A\subseteq S\colon \#(A\cap\{a_{2i-1},a_{2i}\})=1 \text{ for all } i=1,\dots,n\}.\end{aligned}$$

It is easy to see that $\mathscr{G}\subseteq\mathscr{F}$ and $\#\mathscr{G}=2^n$. For any $A\in\mathscr{F}$, we associate $A$ with a *sign* $\operatorname{sgn}(A)=(-1)^{(\#A-n)/2}$. It is clear that

$$\sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^k\binom{2n}{n+2k}=\sum_{A\in\mathscr{F}}\operatorname{sgn}(A)=\sum_{A\in\mathscr{F}\setminus\mathscr{G}}\operatorname{sgn}(A)+\sum_{A\in\mathscr{G}}\operatorname{sgn}(A).$$

Clearly, $\operatorname{sgn}(A) = 1$ for $A \in \mathscr{G}$. What remains is to construct a sign-reversing involution on the set $\mathscr{F} \setminus \mathscr{G}$.

For any $A \in \mathscr{F} \setminus \mathscr{G}$, choose the first number $i$ such that $\#(A \cap \{a_{2i-1}, a_{2i}\}) \neq 1$, i.e., $A$ contains both $a_{2i-1}$ and $a_{2i}$ or none of them. Let $A'$ be a subset of $S$ obtained from $A$ as follows:

$$A' = \begin{cases} A \cup \{a_{2i-1}, a_{2i}\}, & \text{if } \{a_{2i-1}, a_{2i}\} \cap A = \emptyset, \\ A \setminus \{a_{2i-1}, a_{2i}\}, & \text{if } \{a_{2i-1}, a_{2i}\} \subseteq A. \end{cases} \tag{2.1}$$

It is obvious that $A' \in \mathscr{F} \setminus \mathscr{G}$, and $A \mapsto A'$ is the desired involution. □

For $A \in S$, we associate it with a *weight* $||A|| = \sum_{a \in A} a$. By the $q$-binomial theorem (cf. Andrews [1, Theorem 3.3])

$$(z;q)_N = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix} (-1)^j z^j q^{\binom{j}{2}},$$

we have

$$\sum_{\substack{A \subseteq [n] \\ \#A = k}} q^{||A||} = \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k+1}{2}}. \tag{2.2}$$

Here and in what follows $[n] := \{1, \dots, n\}$. Now we can give proofs of (1.2)–(1.4).

*Proof of* (1.2). Let $\{a_{2i-1}, a_{2i}\} = \{-(2i-1)/2, (2i-1)/2\}$ for $i = 1, \dots, n$. Since $a_{2i-1} + a_{2i} = 0$, the involution in the proof of (1.1) is indeed weight-preserving and sign-reversing. It follows that

$$\sum_{A \in \mathscr{F}} \operatorname{sgn}(A) q^{||A||} = \sum_{A \in \mathscr{F} \setminus \mathscr{G}} \operatorname{sgn}(A) q^{||A||} + \sum_{A \in \mathscr{G}} \operatorname{sgn}(A) q^{||A||} = \sum_{A \in \mathscr{G}} \operatorname{sgn}(A) q^{||A||}. \tag{2.3}$$

It is easy to see that $S$ is obtained from $[2n]$ by a shift $-(2n+1)/2$. By (2.2), the left-hand of (2.3) equals

$$\sum_{k=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} \sum_{\substack{A \subseteq S \\ \#A = n+2k}} \operatorname{sgn}(A) q^{||A||} = \sum_{k=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} 2n \\ n+2k \end{bmatrix} q^{\binom{n+2k+1}{2}} q^{-(n+2k)(2n+1)/2}.$$

On the other hand, the right-hand side of (2.3) is given by

$$\prod_{i=1}^{n} (q^{-(2i-1)/2} + q^{(2i-1)/2}) = (-q;q^2)_n q^{-n^2/2}.$$

After simplification, we obtain (1.2). □

*Proof of* (1.3). Note that the index $i$ in (2.1) is always less than $n$. Otherwise, $\#(A\cap\{a_{2i-1},a_{2i}\})=1$ for $i=1,\dots,n-1$ and $\#(A\cap\{a_{2n-1},a_{2n}\})\neq 1$, which is contradictory to the condition $\#A\equiv n \pmod 2$. Thus, if we take $\{a_{2i-1},a_{2i}\}=\{-i,i\}$ for $i=1,\dots,n-1$ and $\{a_{2n-1},a_{2n}\}=\{0,n\}$, then the involution in the proof of (1.1) is also weight-preserving and sign-reversing, and (2.3) still holds. Similarly as before, we obtain

$$\sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^k\begin{bmatrix}2n\\ n+2k\end{bmatrix}q^{\binom{n+2k+1}{2}}q^{-n(n+2k)}=(q^0+q^n)\prod_{i=1}^{n-1}(q^{-i}+q^i),$$

which is equivalent to (1.3). $\square$

*Proof of* (1.4). Let $\{a_{2i-1},a_{2i}\}=\{-(2i-1)/2,(2i-1)/2\}$ for $i=1,\dots,n-1$ and $\{a_{2n-1},a_{2n}\}=\{(2n-1)/2,(2n+1)/2\}$. Then $S=\{i-(2n-1)/2\colon i\in[2n]\}$ and the previous involution yields

$$\begin{aligned}&\sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^k\begin{bmatrix}2n\\ n+2k\end{bmatrix}q^{\binom{n+2k+1}{2}}q^{-(n+2k)(2n-1)/2}\\ &=(q^{(2n-1)/2}+q^{(2n+1)/2})\prod_{i=1}^{n-1}(q^{-(2i-1)/2}+q^{(2i-1)/2}),\end{aligned}$$

which, after simplification, leads to (1.4). $\square$

Similarly, if we set $S=\{a_1,\dots,a_{2n+1}\}$ be a set of $2n+1$ elements, and again let

$$\begin{aligned}\mathscr{F}&=\{A\subseteq S\colon \#A\equiv n \pmod 2\},\\ \mathscr{G}&=\{A\subseteq S\colon \#(A\cap\{a_{2i-1},a_{2i}\})=1 \text{ for all } i=1,\dots,n\}.\end{aligned}$$

then the same argument implies that

$$\sum_{k=-\infty}^{\infty}(-1)^k\binom{2n+1}{n+2k}=2^n.$$

Furthermore, letting $\{a_{2i-1},a_{2i}\}=\{-(2i-1)/2,(2i-1)/2\}, i=1,\dots,n$, and $a_{2n+1}=(2n+1)/2$, we obtain

$$\sum_{k=-\infty}^{\infty}(-1)^kq^{2k^2}\begin{bmatrix}2n+1\\ n+2k\end{bmatrix}=(-q;q^2)_n \tag{2.4}$$

(see [5, Propositon 2(2)]); while letting $\{a_{2i-1},a_{2i}\}=\{-i,i\}, i=1,\dots,n$, and $a_{2n+1}=0$, we obtain

$$\sum_{k=-\infty}^{\infty}(-1)^kq^{2k^2-k}\begin{bmatrix}2n+1\\ n+2k\end{bmatrix}=(-q^2;q^2)_n.$$

Moreover, replacing $q$ by $q^{-1}$ in (2.4) and using the relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}$$

yields

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{2k^2-2k} \begin{bmatrix} 2n+1 \\ n+2k \end{bmatrix} = (-q;q^2)_n q^n.$$

# 3 Proofs (1.8)–(1.10)

Recall that the *symmetric difference* of two sets $A$ and $B$, denoted by $A\Delta B$, is the set of elements belonging to one but not both of $A$ and $B$ (cf. [4, p. 3]). In other words,

$$A\Delta B := A \cup B \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A).$$

It is easy to see that $(A\Delta B)\Delta B = A$. Here we shall use the notation $A\Delta B$ to polish our description of certain involution.

*Proof of* (1.8). Let $S = \{a_1, \dots, a_{2n}\}$ $(n \geq 1)$, and let

$$\mathscr{P} := \{A \subseteq S \colon \#A \equiv n \pmod 3\}. \tag{3.1}$$

For any $A \in \mathscr{P}$, we associate $A$ with a *sign* $\operatorname{sgn}(A) = (-1)^{(\#A-n)/3}$. Then

$$\sum_{k=-\lfloor n/3 \rfloor}^{\lfloor n/3 \rfloor} (-1)^k \binom{2n}{n+3k} = \sum_{A \in \mathscr{P}} \operatorname{sgn}(A).$$

We define a subset of $\mathscr{Q} \subseteq \mathscr{P}$ as follows:

$$\mathscr{Q} := \{A \in \mathscr{P} \colon \#(A \cap \{a_1, \dots, a_{2i+1}\}) \notin \{i-1, i+2\} \text{ for } i = 1, \dots, n-1\}. \tag{3.2}$$

We will show that the elements of $\mathscr{P} \setminus \mathscr{Q}$ cancel pairwise, i.e.,

$$\sum_{A \in \mathscr{P} \setminus \mathscr{Q}} \operatorname{sgn}(A) = 0. \tag{3.3}$$

For any $A \in \mathscr{P} \setminus \mathscr{Q}$, there exist some numbers $i \leq n-1$ such that $\#(A \cap \{a_1, \dots, a_{2i+1}\}) \in \{i-1, i+2\}$. Choose the smallest such $i$ and let

$$A' = A\Delta\{a_1, \dots, a_{2i+1}\}. \tag{3.4}$$

Then $\#A' = \#A \pm 3$ and $A' \in \mathscr{P} \setminus \mathscr{Q}$. It is easy to see that $A \mapsto A'$ is a sign-reversing involution, and therefore (3.3) holds. It remains to evaluate the following summation

$$\sum_{A \in \mathscr{Q}} \operatorname{sgn}(A).$$

For any $A \in \mathscr{Q}$, we claim that

$$\#(A \cap \{a_1, \dots, a_{2i+1}\} \in \{i, i+1\}, \text{ for all } i = 1, \dots, n-1. \tag{3.5}$$

Indeed, by definition, the statement (3.5) is obviously true for $i = 1$. Suppose it holds for $i - 1$, i.e.,

$$\#(A \cap \{a_1, \dots, a_{2i-1}\}) \in \{i-1, i\},$$

Then

$$\#(A \cap \{a_1, \dots, a_{2i+1}\}) \in \{i-1, i, i+1, i+2\}.$$

By (3.2), we confirm our claim. In particular,

$$\#(A \cap \{a_1, \dots, a_{2n-1}\}) \in \{n-1, n\}. \tag{3.6}$$

Thus by (3.1), we must have $\#A = n$ and so $\operatorname{sgn}(A) = 1$. Note that we have 2 possible choices for $A \cap \{a_1\}$. By (3.5), we have 3 possible choices for each $A \cap \{a_{2i}, a_{2i+1}\}$, $i = 1, \dots, n-1$. Finally, we only have one choice for $A \cap \{a_{2n}\}$ according to (3.6) and $\#A = n$. This proves that $\#\mathscr{Q} = 2 \cdot 3^n$ and therefore completes the proof of (1.8). $\square$

For $A \in S$, recall that its weight is defined by $||A|| = \sum_{a \in A} a$. In order to prove (1.9) and (1.10), we need to consider the following weighted sum

$$\sum_{A \in \mathscr{P}} \operatorname{sgn}(A) q^{||A||}$$

on a particular $\mathscr{P}$. As one might have seen, the involution $A \mapsto A'$ in (3.4) is in general not weight-preserving. Nevertheless, a little modification will fix this problem. For any $A \in \mathscr{P} \setminus \mathscr{Q}$, choose the same $i$ as in (3.4), and let $A''$ be constructed as follows:

- $a_1 \in A''$ if and only if $a_1 \notin A$;
- $a_{2j}, a_{2j+1} \in A''$ if $a_{2j}, a_{2j+1} \notin A$ $(j = 1, \dots, i-1)$;
- $a_{2j}, a_{2j+1} \notin A''$ if $a_{2j}, a_{2j+1} \in A$ $(j = 1, \dots, i-1)$;
- $a_{2j} \in A''$ and $a_{2j+1} \notin A''$ if $a_{2j} \in A$ and $a_{2j+1} \notin A$ $(j = 1, \dots, i-1)$;
- $a_{2j} \notin A''$ and $a_{2j+1} \in A''$ if $a_{2j} \notin A$ and $a_{2j+1} \in A$ $(j = 1, \dots, i-1)$;
- $a_k \in A''$ if and only if $a_k \in A$ $(2i + 2 \le k \le 2n)$.

It is clear that $\#A'' = \#A' = \#A \pm 3$. Furthermore, if we putting $a_1 = a_{2j} + a_{2j+1} = 0$ then $A \mapsto A''$ is a weight-preserving and sign-reversing involution. Now we can give proofs of (1.9) and (1.10) by selecting the set $\{a_1, \dots, a_{2n}\}$ properly.

*Proof of* (1.9). Let $a_1 = 0$, $a_{2n} = n$ and $\{a_{2i}, a_{2i+1}\} = \{-i, i\}$ for $i = 1, \dots, n-1$. Then the above involution $A \mapsto A''$ gives

$$\sum_{A \in \mathscr{P} \setminus \mathscr{Q}} \operatorname{sgn}(A) q^{||A||} = 0,$$

or

$$\sum_{A\in\mathscr{P}} \operatorname{sgn}(A)q^{||A||} = \sum_{A\in\mathscr{Q}} \operatorname{sgn}(A)q^{||A||}. \tag{3.7}$$

By (2.2), the left-hand of (3.7) may be written as

$$\sum_{k=-\lfloor n/3\rfloor}^{\lfloor n/3\rfloor} \sum_{\substack{A\subseteq S\\ \#A=n+3k}} \operatorname{sgn}(A)q^{||A||} = \sum_{k=-\lfloor n/3\rfloor}^{\lfloor n/3\rfloor} (-1)^k \begin{bmatrix} 2n \\ n+3k \end{bmatrix} q^{\binom{n+3k+1}{2}} q^{-(n+3k)n}.$$

Let

$$\mathscr{Q}^* := \{A \subseteq \{a_1,\dots,a_{2n-1}\} \colon \#(A\cap\{a_1,\dots,a_{2i+1}\}) \notin \{i-1, i+2\},\ i=1,\dots,n-1\}.$$

Then (3.5) also holds for $A\in\mathscr{Q}^*$. Moreover, for $i=1,\dots,n-1$, we have three choices for each $A\cap\{a_{2i},a_{2i+1}\}$, namely, $\{a_{2i}\},\{a_{2i+1}\},\{a_{2i},a_{2i+1}\}$ if $\#(A\cap\{a_1,\dots,a_{2i-1}\})=i-1$, and $\emptyset,\{a_{2i}\},\{a_{2i+1}\}$ if $\#(A\cap\{a_1,\dots,a_{2i-1}\})=i$. Noticing that $a_{2i}+a_{2i+1}=0$, we have

$$\sum_{A\in\mathscr{Q}^*} q^{||A||} = \sum_{\substack{A\in\mathscr{Q}^*\\ \#A=n-1}} q^{||A||} + \sum_{\substack{A\in\mathscr{Q}^*\\ \#A=n}} q^{||A||} = 2\prod_{i=1}^{n-1}(q^i+q^{-i}+q^0).$$

It is not hard to image that there should exist a bijection from $\{A\in\mathscr{Q}^*\colon \#A=n-1\}$ to $\{A\in\mathscr{Q}^*\colon \#A=n\}$ which preserves the weight. Indeed, our definition of the involution $A\mapsto A''$ on $\mathscr{P}\setminus\mathscr{Q}$ can be simultaneously applied to $\mathscr{Q}^*$, which yields the desired bijection. It follows that

$$\sum_{\substack{A\in\mathscr{Q}^*\\ \#A=n-1}} q^{||A||} = \sum_{\substack{A\in\mathscr{Q}^*\\ \#A=n}} q^{||A||} = \prod_{i=1}^{n-1}(q^i+q^{-i}+q^0).$$

Since

$$\mathscr{Q} = \{A\in\mathscr{Q}^*\colon \#A=n\} \biguplus \{A\cup\{a_{2n}\}\colon A\in\mathscr{Q}^*, \#A=n-1\}$$

($a_{2n}=n$ in this proof), the right-hand of (3.7) equals

$$\sum_{A\in\mathscr{Q}} q^{||A||} = \sum_{\substack{A\in\mathscr{Q}^*\\ \#A=n}} q^{||A||} + q^n \sum_{\substack{A\in\mathscr{Q}^*\\ \#A=n-1}} q^{||A||} = (1+q^n)\prod_{i=1}^{n-1}(q^i+q^{-i}+q^0).$$

The proof then follows after simplification. □

*Proof of* (1.10). Suppose $n\geq 3$. Let $a_1=0$, $a_{2n-2}=n-1$, $a_{2n-1}=n$, $a_{2n}=n+1$ and $\{a_{2i},a_{2i+1}\}=\{-i,i\}$ for $i=1,\dots,n-2$. For any $A\in\mathscr{P}\setminus\mathscr{Q}$, we claim that

$$\#(A\cap\{a_1,\dots,a_{2n-1}\}) \notin \{n-2, n+1\}.$$

Otherwise, we have

$$\#A \in \{n-2, n-1, n+1, n+2\},$$

which is contrary to the definition (3.1). Therefore, the index $i$ we choose for (3.4) is indeed less than $n-1$. Since $a_{2i} + a_{2i+1} = 0$ $(1 \le i \le n-2)$ here, the previous involution $A \mapsto A''$ is still weight-preserving and sign-reversing, and thus (3.7) holds again. In this case, the left-hand of (3.7) equals

$$\sum_{k=-\lfloor n/3 \rfloor}^{\lfloor n/3 \rfloor} (-1)^k \begin{bmatrix} 2n \\ n+3k \end{bmatrix} q^{\binom{n+3k+1}{2}} q^{-(n+3k)(n-1)}.$$

To evaluate the right-hand side of (3.7), we introduce

$$\mathscr{Q}^\star := \{A \subseteq \{a_1, \dots, a_{2n-3}\} \colon \#(A \cap \{a_1, \dots, a_{2i+1}\}) \notin \{i-1, i+2\},\ i = 1, \dots, n-2\}.$$

Then the same argument as $\mathscr{Q}^*$ implies that

$$\sum_{A \in \mathscr{Q}^\star} q^{||A||} = \sum_{\substack{A \in \mathscr{Q}^\star \\ \#A = n-1}} q^{||A||} + \sum_{\substack{A \in \mathscr{Q}^\star \\ \#A = n-2}} q^{||A||} = 2 \prod_{i=1}^{n-2} (q^i + q^{-i} + q^0). \tag{3.8}$$

Moreover, our definition for the involution $A \mapsto A''$ on $\mathscr{P} \setminus \mathscr{Q}$ can also be applied to $\mathscr{Q}^\star$, and we have

$$\sum_{\substack{A \in \mathscr{Q}^\star \\ \#A = n-1}} q^{||A||} = \sum_{\substack{A \in \mathscr{Q}^\star \\ \#A = n-2}} q^{||A||} = \prod_{i=1}^{n-2} (q^i + q^{-i} + q^0). \tag{3.9}$$

It is easy to see that the right-hand of (3.7) equals

$$\begin{aligned} \sum_{A \in \mathscr{Q}} q^{||A||} = &\sum_{\substack{A \in \mathscr{Q}^\star \\ \#A = n-1}} q^{||A||} \left(q^{a_{2n-2}} + q^{a_{2n-1}} + q^{a_{2n}}\right) \\ &+ \sum_{\substack{A \in \mathscr{Q}^\star \\ \#A = n-2}} q^{||A||} \left(q^{a_{2n-2} + a_{2n-1}} + q^{a_{2n-2} + a_{2n}} + q^{a_{2n-1} + a_{2n}}\right). \end{aligned}$$

Substituting (3.8) and $\{a_{2n-2}, a_{2n-1}, a_{2n}\} = \{n-1, n, n+1\}$ into the above equation, we complete the proof of (1.10). □

No doubt that we may define the involution $A \mapsto A''$ on the set $\{a_1, \dots, a_{2n+1}\}$. Let $\{a_1, \dots, a_{2n}\}$ be as in the proof of (1.9). Then putting $a_{2n+1} = -n$ we obtain

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{(9k^2-3k)/2} \begin{bmatrix} 2n+1 \\ n+3k \end{bmatrix} = \frac{(q^3; q^3)_n}{(q; q)_n}, \tag{3.10}$$

while putting $a_{2n+1}=n+1$ we get

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{(9k^2+3k)/2}\begin{bmatrix}2n+1\\ n+3k\end{bmatrix}=\frac{(q^3;q^3)_{n-1}}{(q;q)_{n-1}}(1+q^n+q^{n+1})\quad(n\geq 1). \tag{3.11}$$

Both (3.10) and (3.11) are $q$-analogues of (1.12). Finally, we point out that the following two identities:

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{(9k^2+9k)/2}\begin{bmatrix}2n\\ n+3k+1\end{bmatrix}=\frac{(q^3;q^3)_{n-1}}{(q;q)_{n-1}}q^{n-1}\chi(n>0),$$
$$\sum_{k=-\infty}^{\infty}(-1)^k q^{(9k^2+3k)/2}\begin{bmatrix}2n+1\\ n+3k+1\end{bmatrix}=\frac{(q^3;q^3)_{n}}{(q;q)_{n}}$$

appearing in [5] can also be proved in the same way.

# 4 Proofs of (1.11)

*First Proof.* By the binomial theorem, we have

$$\begin{aligned}\left(\sqrt{3}+\mathrm{i}\right)^n&=\sum_{k=0}^{\infty}\binom{n}{k}3^{k/2}\mathrm{i}^{n-k}\\ &=\mathrm{i}^n\sum_{k=0}^{\infty}\binom{n}{2k}(-3)^k+\mathrm{i}^{n-1}\sqrt{3}\sum_{k=0}^{\infty}\binom{n}{2k+1}(-3)^k.\end{aligned} \tag{4.1}$$

On the other hand, there holds

$$\left(\sqrt{3}+\mathrm{i}\right)^n=2^n\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)^n=2^n\left(\cos\frac{n\pi}{6}+\mathrm{i}\sin\frac{n\pi}{6}\right). \tag{4.2}$$

Comparing (4.1) and (4.2), we immediately get (1.11) and its companion

$$\sum_{k=0}^{\infty}\binom{n}{2k+1}(-3)^k=\begin{cases}0, & \text{if } n\equiv 0 \pmod 3,\\ (-2)^{n-1}, & \text{if } n\equiv 1 \pmod 3,\\ (-1)^n 2^{n-1}, & \text{if } n\equiv 2 \pmod 3.\end{cases} \tag{4.3}$$

□

*Second Proof.* Let $\Gamma=\{a,b,c,d,e\}$ denote an alphabet. For a word $w=w_1\cdots w_n\in\Gamma^*$, its *length* $n$ is denoted by $|w|$. For any $x\in\Gamma$, let $|w|_x$ be the number of $x$'s appearing in the word $w$. Let $W_n$ denote the set of words $w=w_1\cdots w_n\in\Gamma^*$ satisfying the following conditions:

(i) $|w|_a+|w|_b+|w|_c=|w|_d$.

(ii) If we remove all $e$'s from $w$, then each $d$ is in the even position.

It is easy to see that there are $\binom{n}{2k}3^k$ words $w \in W_n$ such that $|w|_d = k$, and so

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}(-3)^k = \sum_{w \in W_n} (-1)^{|w|_d}.$$

We call $(-1)^{|w|_d}$ the *sign* of the word $w$. In what follows, we shall construct an involution on $W_n$ which is sign-reversing for all non-fixed points.

For any word $w = w_1 \cdots w_n \in W_n$, let $u_i = w_{3i-2}w_{3i-1}w_{3i}$, $i = 1, \ldots, \lfloor n/3 \rfloor$. According to the conditions (i) and (ii), the subwords $u_i$ have at most 43 cases. Let us classify them into three types as follows:

$$\begin{aligned} X\colon & ade, bde, cde, aed, bed, ced, ead, ebd; \\ Y\colon & eee, aee, bee, cee, eae, ebe, ece, eea, eeb, eec, dee, ede, eed, \\ & ecd, ada, adb, adc, bda, bdb, bdc, cda, cdb, cdc, dad, dbd, dcd; \\ Z\colon & eda, edb, edc, dea, deb, dec, dae, dbe, dce. \end{aligned}$$

We claim that all the words in $W_n$ with a $u_i$ of type $Y$ cancel pairwise. Indeed, for such a word $w$, choose the smallest number $i$ such that $u_i$ is of type $Y$. Then we obtain a word $w'$ by replacing $u_i$ by $u_i'$, where $u_i \longleftrightarrow u_i'$ is determined by the following table:

| | | | | |
|---|---|---|---|---|
| $eee \longleftrightarrow ecd$ | $aee \longleftrightarrow ada$ | $bee \longleftrightarrow adb$ | $cee \longleftrightarrow adc$ | $eae \longleftrightarrow bda$ |
| $ebe \longleftrightarrow bdb$ | $ece \longleftrightarrow bdc$ | $eea \longleftrightarrow cda$ | $eeb \longleftrightarrow cdb$ | $eec \longleftrightarrow cdc$ |
| $dee \longleftrightarrow dad$ | $ede \longleftrightarrow dbd$ | $eed \longleftrightarrow dcd$ | | |

It is clear that $w' \in W_n$, $|w'|_d = |w|_d \pm 1$, and hence $w \mapsto w'$ is a sign-reversing involution.

On the other hand, for any word $w \in W_n$, we claim that if no $u_i$ in $w$ is of type $Y$, then no $u_i$ in $w$ is of type $Z$. In fact, by the definition of $w$, $u_1$ must be of type $X$ or $Y$. By the condition (ii), none of $dd$, $ded$, $deed$ can appear in $w$ and therefore no $u_i$ of type $X$ in $w$ can be followed by a $u_j$ of type $Z$. This proves the claim. It follows that the remained words in $W_n$ are just those all $u_i$ are of type $X$, and vice versa. Namely,

$$\sum_{w \in W_n} (-1)^{|w|_d} = \sum_{\substack{w \in W_n \\ \text{all } u_i \text{ is of type } X}} (-1)^{|w|_d}. \tag{4.4}$$

Consider the right-hand side of (4.4) ($RHS(4.4)$ for short). Note that each $u_i$ has 8 possible choices. We have the following three cases:

- If $n \equiv 0 \pmod 3$, then $|w|_d = n/3$ and $RHS(4.4) = (-8)^{n/3}$.
- If $n \equiv 1 \pmod 3$, then $w$ must be ended by a letter $e$, $|w|_d = (n-1)/3$, and $RHS(4.4) = (-8)^{(n-1)/3}$.

- If $n \equiv 2 \pmod 3$, then $w$ may be ended by $ee$, $ad$, $bd$, or $cd$, and

$$RHS(4.4) = (-8)^{(n-2)/3} + 3(-1)^{(n+1)/3} 8^{(n-2)/3} = (-2)^{n-1}.$$

This completes the proof. $\square$

The combinatorial proof of (4.3) is exactly analogous. We need only to replace the condition (i) by $|w|_a + |w|_b + |w|_c = |w|_d - 1$, and change "even" to "odd" in the condition (ii).

It is difficult to find $q$-analogues of (1.11) and (4.3). However, the mathematics software Maple hints us to propose the following two interesting conjectures.

**Conjecture 4.1** *Let $l, m, n \geq 0$ and $\epsilon \in \{0, 1\}$. Then*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{2lk^2+2mk} \begin{bmatrix} n \\ 2k+\epsilon \end{bmatrix} (1+q+q^2)^k$$

*is divisible by* $(1+q)^{\lfloor (n+2)/4 \rfloor}(1+q^2)^{\lfloor (n+4)/8 \rfloor}$.

**Conjecture 4.2** *Let $m, n \geq 0$ and $\epsilon \in \{0, 1\}$. Then*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{2k^2+2mk} \begin{bmatrix} n \\ 2k+\epsilon \end{bmatrix} (1+q+q^2)^k$$

*is divisible by* $(1+q)^{\lfloor n/2 \rfloor}(1+q^2)^{\lfloor n/4 \rfloor}$.

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