Proof of Andrews’ conjecture on a $4\phi_3$ summation

Victor J. W. Guo

Department of Mathematics, East China Normal University, Shanghai 200062, People’s Republic of China
jwguo@math.ecnu.edu.cn, http://math.ecnu.edu.cn/~jwguo

Abstract. We give a new proof of a $4\phi_3$ summation due to G.E. Andrews and confirm another $4\phi_3$ summation conjectured by him recently. Some variations of these two $4\phi_3$ summations are also given.

Keywords: basic hypergeometric series; Catalan numbers; $4\phi_3$ summation; Andrews’ conjecture

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1 Introduction

Recall that the basic hypergeometric series $r+1\phi_r$ [3, p. 4] is defined as

$$r+1\phi_r \left[ \frac{a_1, a_2, \ldots, a_{r+1}}{b_1, b_2, \ldots, b_r} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_k z^k}{(q, b_1, b_2, \ldots, b_r; q)_k},$$

where $(a_1, \ldots, a_m; q)_n = \prod_{i=1}^{m} (1 - a_i)(1 - a_i q^{-1})(1 - a_i q^{-2}) \cdots (1 - a_i q^{n-1})$.

Recently, Andrews [1] gave a new $4\phi_3$ summation formula as follows.

Theorem 1.1 (Andrews). For $n \geq 0$, there holds

$$4\phi_3 \left[ \frac{q^{-2n}, a, b, q^2}{q^{2-2n}/a, q^{2-2n}/b, abq : q^2, q^2} \right] = \frac{q^{-n}(a, b, -q; q)_n(ab; q^2)_n}{(ab; q)_n(a, b; q^2)_n}. \quad (1.1)$$

Andrews’ identity (1.1) is a deep extension of Shapilo’s identity (see [6, p. 123, (5.12)] and [10, p. 31, Ex. 6.C.14])

$$\sum_{k=0}^{n} C_{2k}C_{2n-2k} = 4^n C_n,$$

where $C_n = \frac{1}{n+1} \binom{2n}{n}$ are Catalan numbers.

At the end of his paper, Andrews [1] made the following conjecture:

Conjecture 1.2 (Andrews). For $n \geq 0$, there holds

$$4\phi_3 \left[ \frac{q^{-2n}, a, b, q^3-2n/ab}{q^{2-2n}/a, q^{1-2n}/b, abq ; q^2, q^2} \right] = \frac{(a, -q; q)_n(b; q)_{n-1}(ab; q^2)_{n-1}(abq^{2n-2}(b-q^2) + abq^{n-1}(q-1) + q - b)}{q^{n+1}(1 - abq^{2n-1})(ab; q)_n(a, b/q^2; q^2)_n}. \quad (1.2)$$

Andrews proved (1.1) by using the $q$-binomial theorem and two special cases of the $q$-Pfaff-Saalschütz summation formula [3, p. 13, (1.7.2)]. In this paper, we first give a new proof of (1.1) along the lines of the proofs in [4,5]. Then we shall prove (1.2) similarly by using (1.1). Some variations of (1.1) and (1.2) are given in the last section.
2 A new proof of Theorem 1.1

For $a = q^2$, the identity (1.1) reduces to

**Lemma 2.1.** For $n \geq 0$, there holds

$$\sum_{k=0}^{n} \frac{(b, q^{-1-2n}/b; q^2)_k}{(q^{2-2n}/b, bq^3; q^2)_k} q^{2k} = q^{-n}(1 - q^{n+1})(1 - bq)(1 - bq^{2n})/(1 - q)(1 - bq^n)(1 - bq^{n+1}).$$

*Proof. It is easy to verify that*

$$\frac{(b, q^{-1-2n}/b; q^2)_k}{(q^{2-2n}/b, bq^3; q^2)_k} q^{2k} + \frac{(b, q^{-1-2n}/b; q^2)_{n-k}}{(q^{2-2n}/b, bq^3; q^2)_{n-k}} q^{2n-2k} = \frac{(b, q^{-1-2n}/b; q^2)_k}{(q^{2-2n}/b, bq^3; q^2)_k} (q^{2k} + q^{4k-n}) = H_k - H_{k+1}, \quad (2.1)$$

*where*

$$H_k = \frac{q^{k-n}(1 - q^{n-2k+1})(1 - bq)(1 - bq^{2n})(b, bq^{2n-2k+3}; q^2)_k}{(1 - q)(1 - bq^n)(1 - bq^{n+1})(b, bq^{2n-2k+2}; q^2)_k}.$$

Summing (2.1) over $k$ from 0 to $n$, we get

$$2 \sum_{k=0}^{n} \frac{(b, q^{-1-2n}/b; q^2)_k}{(q^{2-2n}/b, bq^3; q^2)_k} q^{2k} = H_0 - H_{n+1} = 2H_0,$$

as desired. ☐

*Proof of Theorem 1.1.** Since the $\phi_3$ series in (1.1) is terminating, it suffices to prove it for $a = q^{2m}$, $m = 1, 2, \ldots$. The $a = q^2$ case is true by Lemma 2.1. Let

$$F_k(n, a, b, q) = \frac{(q^{-2n}, a, b, q^{1-2n}/ab; q^2)_k}{(q^2, q^{2-2n}/a, q^{2-2n}/b, abq; q^2)_k} q^{2k},$$

It is not difficult to verify that (or, see [5, (1.1)])

$$F_k(n, a, b, q) - F_k(n, a, b/q^2, q) = \alpha_n F_{k-1}(n-2, aq^2, b, q), \quad (2.2)$$

where

$$\alpha_n = \frac{(b/q^2 - q^{-2n}/ab)(1 - a)(1 - aq)(1 - q^{-2n})(1 - q^{-2n+2})q^2}{(1 - ab/q)(1 - abq)(1 - q^{2-2n}/a)(1 - q^{2-2n}/b)(1 - q^4-2n/b)}.$$  

Summing (2.2) over $k$ from 0 to $n$ gives

$$S(n-2, aq^2, b, q) = \alpha_n^{-1} \left( S(n, a, b, q) - S(n, a, b/q^2, q) \right), \quad (2.3)$$

where

$$S(n, a, b, q) = \sum_{k=0}^{n} F_k(n, a, b, q) = \left[ \frac{q^{-2n}, a, b, q^{3-2n}/ab}{q^{2-2n}/a, q^{4-2n}/b, abq; q^2, q^2} \right].$$
Suppose that (1.1) is true for \( a = q^{2n} \). Then by (2.3) we have

\[
S(n-2, a q^2, b, q) = \alpha_{n-1}^{-1} \left( \frac{q^{-n}(a, b, -q; q)_n(ab; q^2)_n}{(ab; q)_n(a, b; q^2)_n} - \frac{q^{-n}(a, b/q^2, -q; q)_n(ab/q^2; q^2)_n}{(ab/q^2; q)_n(a, b/q^2; q^2)_n} \right)
\]

\[
= \frac{q^{2-n}(aq^2, b, -q; q)_{n-2}(abq^2; q^2)_{n-2}}{(abq^2; q)_{n-2}(aq^2, b; q^2)_{n-2}}.
\]

Replacing \( n \) by \( n + 2 \), one sees that (1.1) is true for \( a q^2 = q^{2n+2} \). This completes the proof.

3 A proof of Conjecture 1.2

We first consider the \( a = q^2 \) case of (1.2).

**Lemma 3.1.** For \( n \geq 0 \), there holds

\[
\sum_{k=0}^{n} \frac{(b, q^{1-2n}/b; q^2)_k q^{2k}}{(q^{1-2n}/b, bq^3; q^2)_k} = (1 - q^n)(1 - bq)(1 - bq^{2n-2})(bq^{2n}(b - q^2) + bq^{n+1}(q - 1) + q - b) q^n(1 - q)(1 - b/q^2)(1 - bq^{n-1})(1 - bq^{n})(1 - bq^{2n+1}).
\]

**(Proof.** Observe that

\[
\frac{(b, q^{1-2n}/b; q^2)_k q^{2k}}{(q^{1-2n}/b, bq^3; q^2)_k} = \frac{(b, q^{1-2n}/b, q^2)_{n-k}}{(q^{1-2n}/b, bq^3, q^2)_{n-k}} q^{2n-2k} = H_k - H_{k+1},
\]

where

\[
H_k = \frac{(1 - q^{n-2k+1})(1 - bq)(1 - bq^{2n-2})(bq^{2n-1} - bq^{2n-2k+1} + bq^{n}(q - 1) - bq^{2k-1} + 1)}{q^{n-k}(1 - q)(1 - b/q^2)(1 - bq^{n-1})(1 - bq^{n})(1 - bq^{2n+1})}
\]

\[
\times \frac{(b/q^2, bq^{2n-2k+3}; q^2)_k}{(b, bq^{2n-2k+3}; q^2)_k}.
\]

Then summing (3.2) over \( k \) from 0 to \( n \), we obtain (3.1). \)

Noticing that

\[
\frac{(q^{3-2n}/ab; q^2)_k}{(q^{4-2n}/b; q^2)_k} = \frac{(1 - 1/ab) (q^{1-2n}/ab; q^2)_k}{(1 - q^{1-2n}/ab) (q^{4-2n}/b; q^2)_k} + \frac{(1/ab - q^{1-2n}/ab) (q^{1-2n}/ab; q^2)_k}{(1 - q^{1-2n}/ab) (q^{2-2n}/b; q^2)_k},
\]

we have

\[
\phi_3 \left[ q^{-2n}, a, b, q^{3-2n}/ab \right] = \frac{bq^{2n-2}(1 - aq)}{(1 - abq^{2n-1})} \left[ q^{-2n}, a, b, q^{1-2n}/ab \right] + \frac{(1 - bq^{2n-2})}{(1 - abq^{2n-1})} \left[ q^{-2n}, a, b, q^{1-2n}/ab \right].
\]

(3.3)
Here we mention that the relation (3.3) is in fact a special case of Krattenthaler [7, (2.10)].
By (3.3) and (1.1), one sees that (1.2) is equivalent to the following result.

**Theorem 3.2.** For \( n \geq 0 \), there holds

\[
\begin{align*}
\phi_3 & \left[ \frac{q^{-2n}, a, b, q^{1-2n}/ab}{q^{2-2n}/a, q^{2-2n}/b, abq, q^2} \right] \\
& = \frac{(a, -q; q)_n(b; q)_{n-1}(abq^{2n-2}(b - q^2) + abq^{n-1}(q - 1) + q - b)}{bq^{3n-1}(1 - aq)(ab; q)_{n-1}(a, b/q^2, q^2)_n} \\
& \hspace{1cm} - \frac{(a, b, -q; q)_n(ab; q^2)_n}{bq^{3n-2}(1 - aq)(ab; q)_{n-1}(a, q^2)_n(b, q^2)_{n-1}}.
\end{align*}
\]

(3.4)

**Proof.** Let

\[
F_k(n, a, b, q) = \frac{(q^{-2n}, a, b, q^{1-2n}/ab; q^2)^k}{(q^2, q^{2-2n}/a, q^{1-2n}/b, abq; q^2)_k} q^{2k}.
\]

Similarly to (3.6), we have

\[
F_k(n, a, b, q) - F_k(n, a/q^2, b, q) = \beta_n F_{k-1}(n - 2, a, bq^2, q),
\]

(3.5)

where

\[
\beta_n = \frac{(a/q^2 - q^{1-2n}/ab)(1 - b)(1 - bq)(1 - q^{2n})(1 - q^{2n+2})q^2}{(1 - ab/q)(1 - abq)(1 - q^{2-2n}/a)(1 - q^{1-2n}/a)(1 - q^{4-2n}/b)}.
\]

Summing (3.5) over \( k \) from 0 to \( n \) yields that

\[
S(n, a, b, q) - S(n, a/q^2, b, q) = \beta_n S(n - 2, a, bq^2, q),
\]

(3.6)

where \( S(n, a, b, q) \) denotes the left-hand side of (3.4).

It suffices to prove (3.4) for \( a = q^{2n}, m = 1, 2, \ldots \). The \( a = q^2 \) case is true by (3.3), (1.1) and Lemma 3.1. We then can complete the proof of (3.4) by induction on \( n \) (firstly) and \( m \) (secondly) by checking that the right-hand side of (3.4) also satisfies the relation (3.6).

**Remark.** One may wonder, why not prove (1.2) directly by induction? The reason is that we cannot find a simple recurrence relation like (3.6) for the \( 4\phi_3 \) series in (1.2).

### 4 Concluding remarks

Letting \((a, b, q) \to (a^{-1}, b^{-1}, q^{-1})\) in (1.1), we obtain the following variation

\[
\phi_3 \left[ \frac{q^{-2n}, a, b, q^{1-2n}/ab}{q^{2-2n}/a, q^{2-2n}/b, abq, q^2} ; q^2, q^4 \right] = \frac{(a, b, -q; q)_n(ab; q^2)_n}{(ab; q)_n(a, b; q^2)_n}.
\]

(4.1)

Since

\[
(q^{3-2n}/ab; q^2)_k = \frac{1}{1 - q^{1-2n}/ab} (q^{1-2n}/ab; q^2)_k - \frac{q^{1-2n+2k}/ab}{1 - q^{1-2n}/ab} (q^{1-2n}/ab; q^2)_k,
\]

4
combining (1.1) and (4.1) leads to

\[
4\phi_3 \left[ \frac{q^{2n}, a, b, q^{-2n}/ab}{q^{-2n}/a, q^{2n}/b, abq^3 ; q^2, q^2} \right] = \frac{(a, b, -q; q)_n(ab; q^2)_n}{(1 - abq^{2n-1})(ab; q)_{n-1}(a, b; q^2)_n}.
\]

Moreover, replacing \(b\) by \(bq^2\) in (1.2), we have

\[
4\phi_3 \left[ \frac{q^{2n}, a, bq^2, q^{1-2n}/ab}{q^{2n}/a, q^{2n}/b, abq^3 ; q^2, q^2} \right] = \frac{(a, -q; q)_n(bq^2; q)_{n-1}(abq^2; q^2)_{n-1}(abq^{2n+1}(b - 1) + abq^n(q - 1) + 1 - bq)}{(1 - abq^{2n+1})(abq^2; q)_{n-1}(a, b; q^2)_n}.
\]

Substituting \((a, b, q) \rightarrow (a^{-1}, b^{-1}, q^{-1})\) in (4.2), we get

\[
4\phi_3 \left[ \frac{q^{2n}, aq^2, bq^2, q^{1-2n}/ab}{q^{2n}/a, q^{2n}/b, abq^3 ; q^2, q^2} \right] = \frac{q^{-n}(aq, bq, -q; q)_n(abq^2; q^2)_n}{(1 - abq^{2n+1})(abq^2; q)_{n-1}(a, b; q^2)_n}.
\]

Since

\[
(qa^2; q^2)_k = \frac{(a; q^2)_k}{1 - a} - \frac{a(a; q^2)_kq^{2k}}{1 - a},
\]

combining (4.2) and (4.3) immediately yields that

\[
4\phi_3 \left[ \frac{q^{2n}, aq^2, bq^2, q^{1-2n}/ab}{q^{2n}/a, q^{2n}/b, abq^3 ; q^2, q^2} \right] = \frac{q^{-n}(aq, bq, -q; q)_n(abq^2; q^2)_n}{(1 - abq^{2n+1})(abq^2; q)_{n-1}(a, b; q^2)_n}.
\]

Noticing that

\[
\frac{(q^{2n}; q^2)_k}{(q^{2n}; q^2)_k} = \frac{(q^{-2n-2}; q^2)_k}{(q^{2}; q^2)_k} + q^{-2n-2}\frac{(q^{2n}; q^2)_{k-1}}{(q^{2}; q^2)_{k-1}}
\]

and \((x; q)_k = (1 - x)(xq; q)_{k-1}\), we have

\[
4\phi_3 \left[ \frac{q^{2n}, a, b, q^{-1-2n}/ab}{q^{-2n}/a, q^{-2n}/b, abq^3 ; q^2, q^2} \right] = 4\phi_3 \left[ \frac{q^{-2n}, a, b, q^{-1-2n}/ab}{q^{-2n}/a, q^{-2n}/b, abq^3 ; q^2, q^2} \right] + \frac{q^{-2n}(1 - a)(1 - b)(1 - q^{-1-2n}/ab)}{(1 - q^{-2n}/a)(1 - q^{-2n}/b)(1 - abq)} 4\phi_3 \left[ \frac{q^{2n}, aq^2, bq^2, q^{1-2n}/ab}{q^{2n}/a, q^{2n}/b, abq^3 ; q^2, q^2} \right].
\]

Plugging the formulas (1.1) \((n \rightarrow n + 1)\) and (4.4) into (4.5), and making some simplification, we obtain the following new neat \(4\phi_3\) summation formula:

\[
4\phi_3 \left[ \frac{q^{2n}, a, b, q^{-1-2n}/ab}{q^{-2n}/a, q^{-2n}/b, abq^3 ; q^2, q^2} \right] = \frac{(aq, bq, -q; q)_n(abq^2; q^2)_n}{(abq; q)_n(aq^2, bq^2; q^2)_n}.
\]
Note that a computer proof of (1.2) and (4.6) has been given by Mu [8] based on the \( q \)-Zeilberger algorithm [2,9,11] after reading a previous version of this paper.

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**References**


