Proof of Andrews' conjecture on a ${}_4\phi_3$ summation

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Abstract. We give a new proof of a ${}_4\phi_3$ summation due to G.E. Andrews and confirm another ${}_4\phi_3$ summation conjectured by him recently. Some variations of these two ${}_4\phi_3$ summations are also given.

Keywords: basic hypergeometric series; Catalan numbers; ${}_4\phi_3$ summation; Andrews' conjecture

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1 Introduction

Recall that the basic hypergeometric series ${}_{r+1}\phi_r$ [3, p. 4] is defined as

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q,\, z\right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_k z^k}{(q, b_1, b_2, \dots, b_r; q)_k}, $$

where $(a_1, \dots, a_m; q)_n = \prod_{i=1}^{m}((1-a_i)(1-a_iq)\cdots(1-a_iq^{n-1}))$.

Recently, Andrews [1] gave a new ${}_4\phi_3$ summation formula as follows.

Theorem 1.1 (Andrews)**.** *For $n \geq 0$, there holds*

$$ {}_4\phi_3\left[\begin{matrix} q^{-2n},\, a,\, b,\, q^{1-2n}/ab \\ q^{2-2n}/a,\, q^{2-2n}/b,\, abq \end{matrix}; q^2,\, q^2\right] = \frac{q^{-n}(a, b, -q; q)_n (ab; q^2)_n}{(ab; q)_n (a, b; q^2)_n}. \tag{1.1} $$

Andrews' identity (1.1) is a deep extension of Shapilo's identity (see [6, p. 123, (5.12)] and [10, p. 31, Ex. 6.C.14])

$$ \sum_{k=0}^{n} C_{2k} C_{2n-2k} = 4^n C_n, $$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ are Catalan numbers.

At the end of his paper, Andrews [1] made the following conjecture:

Conjecture 1.2 (Andrews)**.** *For $n \geq 0$, there holds*

$$ \begin{aligned} &{}_4\phi_3\left[\begin{matrix} q^{-2n},\, a,\, b,\, q^{3-2n}/ab \\ q^{2-2n}/a,\, q^{4-2n}/b,\, abq \end{matrix}; q^2,\, q^2\right] \\ &\quad = \frac{(a, -q; q)_n (b; q)_{n-1} (ab; q^2)_{n-1} (abq^{2n-2}(b-q^2) + abq^{n-1}(q-1) + q - b)}{q^{n+1}(1 - abq^{2n-1})(ab; q)_{n-1} (a, b/q^2; q^2)_n}. \end{aligned} \tag{1.2} $$

Andrews proved (1.1) by using the q-binomial theorem and two special cases of the q-Pfaff-Saalschütz summation formula [3, p. 13, (1.7.2)]. In this paper, we first give a new proof of (1.1) along the lines of the proofs in [4,5]. Then we shall prove (1.2) similarly by using (1.1). Some variations of (1.1) and (1.2) are given in the last section.

2 A new proof of Theorem 1.1

For $a=q^2$, the identity (1.1) reduces to

Lemma 2.1. *For $n\geq 0$, there holds*

$$\sum_{k=0}^{n}\frac{(b,q^{-1-2n}/b;q^2)_k}{(q^{2-2n}/b,bq^3;q^2)_k}q^{2k}=\frac{q^{-n}(1-q^{n+1})(1-bq)(1-bq^{2n})}{(1-q)(1-bq^n)(1-bq^{n+1})}.$$

Proof. It is easy to verify that

$$\begin{aligned}\frac{(b,q^{-1-2n}/b;q^2)_k}{(q^{2-2n}/b,bq^3;q^2)_k}q^{2k}+\frac{(b,q^{-1-2n}/b;q^2)_{n-k}}{(q^{2-2n}/b,bq^3;q^2)_{n-k}}q^{2n-2k}&=\frac{(b,q^{-1-2n}/b;q^2)_k}{(q^{2-2n}/b,bq^3;q^2)_k}(q^{2k}+q^{4k-n})\\&=H_k-H_{k+1},\end{aligned}\tag{2.1}$$

where

$$H_k=\frac{q^{k-n}(1-q^{n-2k+1})(1-bq)(1-bq^{2n})(b,bq^{2n-2k+3};q^2)_k}{(1-q)(1-bq^n)(1-bq^{n+1})(bq,bq^{2n-2k+2};q^2)_k}.$$

Summing (2.1) over k from 0 to n, we get

$$2\sum_{k=0}^{n}\frac{(b,q^{-1-2n}/b;q^2)_k}{(q^{2-2n}/b,bq^3;q^2)_k}q^{2k}=H_0-H_{n+1}=2H_0,$$

as desired. ∎

Proof of Theorem 1.1. Since the ${}_4\phi_3$ series in (1.1) is terminating, it suffices to prove it for $a=q^{2m}, m=1,2,\ldots$. The $a=q^2$ case is true by Lemma 2.1. Let

$$F_k(n,a,b,q)=\frac{(q^{-2n},a,b,q^{1-2n}/ab;q^2)_k}{(q^2,q^{2-2n}/a,q^{2-2n}/b,abq;q^2)_k}q^{2k},$$

It is not difficult to verify that (or, see [5, (1.1)])

$$F_k(n,a,b,q)-F_k(n,a,b/q^2,q)=\alpha_nF_{k-1}(n-2,aq^2,b,q),\tag{2.2}$$

where

$$\alpha_n=\frac{(b/q^2-q^{1-2n}/ab)(1-a)(1-aq)(1-q^{-2n})(1-q^{-2n+2})q^2}{(1-ab/q)(1-abq)(1-q^{2-2n}/a)(1-q^{2-2n}/b)(1-q^{4-2n}/b)}.$$

Summing (2.2) over k from 0 to n gives

$$S(n-2,aq^2,b,q)=\alpha_n^{-1}\left(S(n,a,b,q)-S(n,a,b/q^2,q)\right),\tag{2.3}$$

where

$$S(n,a,b,q)=\sum_{k=0}^{n}F_k(n,a,b,q)=\left[\begin{matrix}q^{-2n},\ a,\ b,\ q^{3-2n}/ab\\ q^{2-2n}/a,\ q^{4-2n}/b,\ abq\end{matrix};q^2,\ q^2\right].$$

Suppose that (1.1) is true for $a = q^{2m}$. Then by (2.3) we have

$$S(n-2, aq^2, b, q) = \alpha_n^{-1}\left(\frac{q^{-n}(a, b, -q; q)_n (ab; q^2)_n}{(ab; q)_n (a, b; q^2)_n} - \frac{q^{-n}(a, b/q^2, -q; q)_n (ab/q^2; q^2)_n}{(ab/q^2; q)_n (a, b/q^2; q^2)_n}\right)$$
$$= \frac{q^{2-n}(aq^2, b, -q; q)_{n-2}(abq^2; q^2)_{n-2}}{(abq^2; q)_{n-2}(aq^2, b; q^2)_{n-2}}.$$

Replacing n by $n+2$, one sees that (1.1) is true for $aq^2 = q^{2m+2}$. This completes the proof. ∎

3 A proof of Conjecture 1.2

We first consider the $a = q^2$ case of (1.2).

Lemma 3.1. *For $n \geq 0$, there holds*

$$\sum_{k=0}^{n} \frac{(b, q^{1-2n}/b; q^2)_k}{(q^{4-2n}/b, bq^3; q^2)_k} q^{2k}$$
$$= \frac{(1-q^{n+1})(1-bq)(1-bq^{2n-2})(bq^{2n}(b-q^2) + bq^{n+1}(q-1) + q - b)}{q^{n+1}(1-q)(1-b/q^2)(1-bq^{n-1})(1-bq^n)(1-bq^{2n+1})}. \tag{3.1}$$

Proof. Observe that

$$\frac{(b, q^{1-2n}/b; q^2)_k}{(q^{4-2n}/b, bq^3; q^2)_k} q^{2k} + \frac{(b, q^{1-2n}/b; q^2)_{n-k}}{(q^{4-2n}/b, bq^3; q^2)_{n-k}} q^{2n-2k} = H_k - H_{k+1}, \tag{3.2}$$

where

$$H_k = \frac{(1-q^{n-2k+1})(1-bq)(1-bq^{2n-2})(b^2q^{2n-1} - bq^{2n-2k+1} + bq^n(q-1) - bq^{2k-1} + 1)}{q^{n-k}(1-q)(1-b/q^2)(1-bq^{n-1})(1-bq^n)(1-bq^{2n+1})}$$
$$\times \frac{(b/q^2, bq^{2n-2k+3}; q^2)_k}{(bq, bq^{2n-2k}; q^2)_k}.$$

Then summing (3.2) over k from 0 to n, we obtain (3.1). ∎

Noticing that

$$\frac{(q^{3-2n}/ab; q^2)_k}{(q^{4-2n}/b; q^2)_k} = \frac{(1-1/aq)}{(1-q^{1-2n}/ab)} \frac{(q^{1-2n}/ab; q^2)_k}{(q^{4-2n}/b; q^2)_k} + \frac{(1/aq - q^{1-2n}/ab)}{(1-q^{1-2n}/ab)} \frac{(q^{1-2n}/ab; q^2)_k}{(q^{2-2n}/b; q^2)_k},$$

we have

$${}_4\phi_3\left[\begin{matrix} q^{-2n},\ a,\ b,\ q^{3-2n}/ab \\ q^{2-2n}/a,\ q^{4-2n}/b,\ abq \end{matrix}; q^2,\ q^2\right]$$
$$= \frac{bq^{2n-2}(1-aq)}{(1-abq^{2n-1})}\left[\begin{matrix} q^{-2n},\ a,\ b,\ q^{1-2n}/ab \\ q^{2-2n}/a,\ q^{4-2n}/b,\ abq \end{matrix}; q^2,\ q^2\right]$$
$$+ \frac{(1-bq^{2n-2})}{(1-abq^{2n-1})}\left[\begin{matrix} q^{-2n},\ a,\ b,\ q^{1-2n}/ab \\ q^{2-2n}/a,\ q^{2-2n}/b,\ abq \end{matrix}; q^2,\ q^2\right]. \tag{3.3}$$

Here we mention that the relation (3.3) is in fact a special case of Krattenthaler [7, (2.10)]. By (3.3) and (1.1), one sees that (1.2) is equivalent to the following result.

Theorem 3.2. *For $n \geq 0$, there holds*

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-2n},\, a,\, b,\, q^{1-2n}/ab \\ q^{2-2n}/a,\, q^{4-2n}/b,\, abq \end{matrix}; q^2,\, q^2\right] \\
&= \frac{(a,-q;q)_n(b;q)_{n-1}(ab;q^2)_{n-1}(abq^{2n-2}(b-q^2)+abq^{n-1}(q-1)+q-b)}{bq^{3n-1}(1-aq)(ab;q)_{n-1}(a,b/q^2;q^2)_n} \\
&\quad - \frac{(a,b,-q;q)_n(ab;q^2)_n}{bq^{3n-2}(1-aq)(ab;q)_n(a,q^2)_n(b;q^2)_{n-1}}. \qquad (3.4)
\end{aligned}
$$

Proof. Let

$$
F_k(n,a,b,q) = \frac{(q^{-2n},a,b,q^{1-2n}/ab;q^2)_k}{(q^2,q^{2-2n}/a,q^{4-2n}/b,abq;q^2)_k} q^{2k},
$$

Similarly to (3.6), we have

$$
F_k(n,a,b,q) - F_k(n,a/q^2,b,q) = \beta_n F_{k-1}(n-2,a,bq^2,q), \qquad (3.5)
$$

where

$$
\beta_n = \frac{(a/q^2 - q^{1-2n}/ab)(1-b)(1-bq)(1-q^{-2n})(1-q^{-2n+2})q^2}{(1-ab/q)(1-abq)(1-q^{2-2n}/a)(1-q^{4-2n}/a)(1-q^{4-2n}/b)}.
$$

Summing (3.5) over k from 0 to n yields that

$$
S(n,a,b,q) - S(n,a/q^2,b,q) = \beta_n S(n-2,a,bq^2,q), \qquad (3.6)
$$

where $S(n,a,b,q)$ denotes the left-hand side of (3.4).

It suffices to prove (3.4) for $a = q^{2m}, m = 1,2,\ldots$. The $a = q^2$ case is true by (3.3), (1.1) and Lemma 3.1. We then can complete the proof of (3.4) by induction on n (firstly) and m (secondly) by checking that the right-hand side of (3.4) also satisfies the relation (3.6). ∎

Remark. One may wonder, why not prove (1.2) directly by induction? The reason is that we cannot find a simple recurrence relation like (3.6) for the ${}_4\phi_3$ series in (1.2).

4 Concluding remarks

Letting $(a,b,q) \to (a^{-1},b^{-1},q^{-1})$ in (1.1), we obtain the following variation

$$
{}_4\phi_3\left[\begin{matrix} q^{-2n},\, a,\, b,\, q^{1-2n}/ab \\ q^{2-2n}/a,\, q^{2-2n}/b,\, abq \end{matrix}; q^2,\, q^4\right] = \frac{(a,b,-q;q)_n(ab;q^2)_n}{(ab;q)_n(a,b;q^2)_n}. \qquad (4.1)
$$

Since

$$
(q^{3-2n}/ab;q^2)_k = \frac{1}{1-q^{1-2n}/ab}(q^{1-2n}/ab;q^2)_k - \frac{q^{1-2n+2k}/ab}{1-q^{1-2n}/ab}(q^{1-2n}/ab;q^2)_k,
$$

combining (1.1) and (4.1) leads to

$$ {}_4\phi_3\left[\begin{matrix} q^{-2n},\, a,\, b,\, q^{3-2n}/ab \\ q^{2-2n}/a,\, q^{2-2n}/b,\, abq \end{matrix};q^2,\, q^2\right] = \frac{(a,b,-q;q)_n(ab;q^2)_n}{(1-abq^{2n-1})(ab;q)_{n-1}(a,b;q^2)_n}. $$

Moreover, replacing b by bq^2 in (1.2), we have

$$ \begin{aligned} &{}_4\phi_3\left[\begin{matrix} q^{-2n},\, a,\, bq^2,\, q^{1-2n}/ab \\ q^{2-2n}/a,\, q^{2-2n}/b,\, abq^3 \end{matrix};q^2,\, q^2\right] \\ &= \frac{(a,-q;q)_n(bq^2;q)_{n-1}(abq^2;q^2)_{n-1}(abq^{2n+1}(b-1)+abq^n(q-1)+1-bq)}{q^n(1-abq^{2n+1})(abq^2;q)_{n-1}(a,b;q^2)_n}. \end{aligned} \tag{4.2} $$

Substituting $(a,b,q)\to(a^{-1},b^{-1},q^{-1})$ in (4.2), we get

$$ \begin{aligned} &{}_4\phi_3\left[\begin{matrix} q^{-2n},\, a,\, bq^2,\, q^{1-2n}/ab \\ q^{2-2n}/a,\, q^{2-2n}/b,\, abq^3 \end{matrix};q^2,\, q^4\right] \\ &= \frac{(a,-q;q)_n(bq^2;q)_{n-1}(abq^2;q^2)_{n-1}(abq^{2n}(bq-1)+bq^n(1-q)+1-b)}{(1-abq^{2n+1})(abq^2;q)_{n-1}(a,b;q^2)_n}. \end{aligned} \tag{4.3} $$

Since

$$ (aq^2;q^2)_k = \frac{(a;q^2)_k}{1-a} - \frac{a(a;q^2)_k q^{2k}}{1-a}, $$

combining (4.2) and (4.3) immediately yields that

$$ {}_4\phi_3\left[\begin{matrix} q^{-2n},\, aq^2,\, bq^2,\, q^{1-2n}/ab \\ q^{2-2n}/a,\, q^{2-2n}/b,\, abq^3 \end{matrix};q^2,\, q^2\right] = \frac{q^{-n}(aq,bq,-q;q)_n(abq^2;q^2)_n}{(1-abq^{2n+1})(abq^2;q)_{n-1}(a,b;q^2)_n}. \tag{4.4} $$

Noticing that

$$ \frac{(q^{-2n};q^2)_k}{(q^2;q^2)_k} = \frac{(q^{-2n-2};q^2)_k}{(q^2;q^2)_k} + q^{-2n-2}\frac{(q^{-2n};q^2)_{k-1}}{(q^2;q^2)_{k-1}} $$

and $(x;q)_k=(1-x)(xq;q)_{k-1}$, we have

$$ \begin{aligned} &{}_4\phi_3\left[\begin{matrix} q^{-2n},\, a,\, b,\, q^{-1-2n}/ab \\ q^{-2n}/a,\, q^{-2n}/b,\, abq \end{matrix};q^2,\, q^2\right] \\ &= {}_4\phi_3\left[\begin{matrix} q^{-2n-2},\, a,\, b,\, q^{-1-2n}/ab \\ q^{-2n}/a,\, q^{-2n}/b,\, abq \end{matrix};q^2,\, q^2\right] \\ &\quad + \frac{q^{-2n}(1-a)(1-b)(1-q^{-1-2n}/ab)}{(1-q^{-2n}/a)(1-q^{-2n}/b)(1-abq)}\,{}_4\phi_3\left[\begin{matrix} q^{-2n},\, aq^2,\, bq^2,\, q^{1-2n}/ab \\ q^{2-2n}/a,\, q^{2-2n}/b,\, abq^3 \end{matrix};q^2,\, q^2\right]. \end{aligned} \tag{4.5} $$

Plugging the formulas (1.1) ($n\to n+1$) and (4.4) into (4.5), and making some simplification, we obtain the following new neat ${}_4\phi_3$ summation formula:

$$ {}_4\phi_3\left[\begin{matrix} q^{-2n},\, a,\, b,\, q^{-1-2n}/ab \\ q^{-2n}/a,\, q^{-2n}/b,\, abq \end{matrix};q^2,\, q^2\right] = \frac{(aq,bq,-q;q)_n(abq^2;q^2)_n}{(abq;q)_n(aq^2,bq^2;q^2)_n}. \tag{4.6} $$

Note that a computer proof of (1.2) and (4.6) has been given by Mu [8] based on the q-Zeilberger algorithm [2, 9, 11] after reading a previous version of this paper.

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References

[1] G.E. Andrews, On Shapiro's Catalan convolution, Adv. Appl. Math. 46 (2011), 15–24.

[2] H. Böing and W. Koepf, Algorithms for q-hypergeometric summation in computer algebra, J. Symbolic Comput. 28 (1999), 777–799.

[3] G. Gasper and M. Rahman, Basic hypergeometric series, Second Edition, Encyclopedia of Mathematics and Its Applications, Vol. 96, Cambridge University Press, Cambridge, 2004.

[4] V.J.W. Guo, Elementary proofs of some q-identities of Jackson and Andrews-Jain, Discrete Math. 295 (2005), 63–74.

[5] V.J.W. Guo and J. Zeng, Short proofs of summation and transformation formulas for basic hypergeometric series, J. Math. Anal. Appl. 327 (2007), 310–325.

[6] T. Koshy, Catalan Numbers with Applications, Oxford University Press, New York, 2009.

[7] C. Krattenthaler, A systematic list of two- and three-term contiguous relations for basic hypergeometric series, unpublished manuscript, 1993, available at `http://www.mat.univie.ac.at/~kratt/papers.html`.

[8] Y.-P. Mu, Andrews' conjecture on a ${}_4\phi_3$ summation and its extensions, Electron. J. Combin. 18 (2) (2012), #P36.

[9] P. Paule and A. Riese, A Mathematica q-analogue of Zeilberger's algorithm based on an algebraically motivated approach to q-hypergeometric telescoping, In: Special Functions, q-Series and Related Topics, Fields Inst. Commun. 14 (1997), 179–210, Amer. Math. Soc., Providence, Rhode Island.

[10] R.P. Stanley, Catalan addendum, `http://www-math.mit.edu/~rstan/ec/catadd.pdf`, 6 October 2008 version.

[11] H. Wilf and D. Zeilberger, An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities, Invent. Math. 108 (1992), 575–633.