

AN EXTENSION OF A SUPERCONGRUENCE OF LONG AND RAMAKRISHNA

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ABSTRACT. We prove two supercongruences for specific truncated hypergeometric series. These include a uniparametric extension of a supercongruence that was recently established by Long and Ramakrishna. Our proofs involve special instances of various hypergeometric identities including Whipple's transformation and the Karlsson–Minton summation.

1. INTRODUCTION

Let $(a)_n = a(a+1)\cdots(a+n-1)$ denote the Pochhammer symbol. For complex numbers a_0, a_1, \dots, a_r and b_1, \dots, b_r , the (generalized) hypergeometric series ${}_{r+1}F_r$ is defined as

$${}_{r+1}F_r \left[\begin{matrix} a_0, & a_1, & \dots, & a_r \\ & b_1, & \dots, & b_r \end{matrix} ; z \right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_r)_k}{k! (b_1)_k \cdots (b_r)_k} z^k.$$

Summation and transformation formulas for generalized hypergeometric series play an important part in the investigation of supercongruences. See, for instance, [5, 10, 12, 13, 16–18, 20]. In particular, Long and Ramakrishna [16, Theorems 3 and 2] proved the following two supercongruences:

$$\sum_{k=0}^{p-1} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} \equiv \begin{cases} -\Gamma_p\left(\frac{1}{4}\right)^4 \pmod{p^3}, & \text{if } p \equiv 1 \pmod{4}, \\ -\frac{p^2}{16}\Gamma_p\left(\frac{1}{4}\right)^4 \pmod{p^3}, & \text{if } p \equiv 3 \pmod{4}, \end{cases} \quad (1.1)$$

and

$$\sum_{k=0}^{p-1} (6k+1) \frac{\left(\frac{1}{3}\right)_k^6}{k!^6} \equiv \begin{cases} -p\Gamma_p\left(\frac{1}{3}\right)^9 \pmod{p^6}, & \text{if } p \equiv 1 \pmod{6}, \\ -\frac{10}{27}p^4\Gamma_p\left(\frac{1}{3}\right)^9 \pmod{p^6}, & \text{if } p \equiv 5 \pmod{6}, \end{cases} \quad (1.2)$$

where p is an odd prime and $\Gamma_p(x)$ is the p -adic Gamma function. The restriction of the supercongruence (1.1) modulo p^2 was earlier established by Van Hamme [19, Equation (H.2)]. The supercongruence (1.2) is even stronger than a conjecture made by Van Hamme

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[19, Equation (D.2)] who asserted the corresponding supercongruence modulo p^4 for $p \equiv 1 \pmod{6}$. Long and Ramakrishna also mentioned that (1.2) does not hold modulo p^7 in general.

The first purpose of this paper is to prove the following supercongruence. Note that the $r = \pm 1$ cases partially confirm the $d = 5$ and $q \rightarrow 1$ case of [6, Conjectures 1 and 2].

Theorem 1. *Let $r \leq 1$ be an odd integer coprime with 5. Let p be a prime such that $p \equiv -\frac{r}{2} \pmod{5}$ and $p \geq \frac{5-r}{2}$. Then*

$$\sum_{k=0}^{p-1} (10k+r) \frac{\left(\frac{r}{5}\right)_k^5}{k!^5} \equiv 0 \pmod{p^4}, \quad (1.3)$$

Recently, the second author [14] established the following supercongruence related to (1.2):

$$\sum_{k=0}^{p-1} (6k-1) \frac{\left(-\frac{1}{3}\right)_k^6}{k!^6} \equiv \begin{cases} 140p^4 \Gamma_p\left(\frac{2}{3}\right)^9 \pmod{p^5}, & \text{if } p \equiv 1 \pmod{6}, \\ 378p \Gamma_p\left(\frac{2}{3}\right)^9 \pmod{p^5}, & \text{if } p \equiv 5 \pmod{6}, \end{cases} \quad (1.4)$$

where p is a prime.

The second purpose of this paper is to give the following common generalization of the second supercongruence in (1.2), restricted to modulo p^5 , and the first supercongruence in (1.4).

Theorem 2. *Let $r \leq 1$ be an integer coprime with 3. Let p be a prime such that $p \equiv -r \pmod{3}$ and $p \geq 3-r$. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} (6k+r) \frac{\left(\frac{r}{3}\right)_k^6}{k!^6} &\equiv \frac{(-1)^{r+1} 80rp^4}{81} \cdot \frac{\Gamma_p\left(1+\frac{r}{3}\right)^2}{\Gamma_p\left(1+\frac{2r}{3}\right)^3 \Gamma_p\left(1-\frac{r}{3}\right)^4} \\ &\quad \times \sum_{k=0}^{1-r} \frac{(r-1)_k \left(\frac{r}{3}\right)_k^3}{(1)_k \left(\frac{2r}{3}\right)_k^3} \pmod{p^5}. \end{aligned} \quad (1.5)$$

Letting $r = 1$ and $r = -1$ in (1.5) and using (1.9) and (1.11), we arrive at the $p \equiv 5 \pmod{6}$ case of (1.2) modulo p^5 and the $p \equiv 1 \pmod{6}$ case of (1.4), respectively.

Our proof of Theorem 1 will require Whipple's well-poised ${}_7F_6$ transformation formula (see [2, p. 28]):

$$\begin{aligned} &{}_7F_6 \left[\begin{matrix} a, & 1 + \frac{1}{2}a, & b, & c, & d, & e, & -n \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+n \end{matrix} ; 1 \right] \\ &= \frac{(a+1)_n (a-d-e+1)_n}{(a-d+1)_n (a-e+1)_n} {}_4F_3 \left[\begin{matrix} 1+a-b-c, & d, & e, & -n \\ d+e-a-n, & 1+a-b, & 1+a-c \end{matrix} ; 1 \right], \end{aligned} \quad (1.6)$$

where n is a non-negative integer, and Karlsson–Minton’s summation formula (see, for example, [4, Equation (1.9.2)]):

$${}_{r+1}F_r \left[\begin{matrix} -n, & b_1 + m_1, & \dots, & b_r + m_r \\ & b_1, & \dots, & b_r \end{matrix} ; 1 \right] = 0, \quad (1.7)$$

where n, m_1, \dots, m_r are non-negative integers and $n > m_1 + \dots + m_r$. Our proof of Theorem 2 relies on a ${}_7F_6$ transformation formula slightly different from Whipple’s ${}_7F_6$ transformation formula (1.6), obtained as a result from combining (1.6) with a ${}_4F_3$ transformation formula. The transformation was already utilized by the second author to prove (1.4).

Furthermore, in order to prove Theorem 2, we require some properties of the p -adic Gamma function, collected in the following two lemmas.

Lemma 1. [3, Section 11.6] *Suppose p is an odd prime and $x \in \mathbb{Z}_p$. Then*

$$\Gamma_p(0) = 1, \quad \Gamma_p(1) = -1, \quad (1.8)$$

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{a_p(x)}, \quad (1.9)$$

$$\Gamma_p(x) \equiv \Gamma_p(y) \pmod{p} \quad \text{for } x \equiv y \pmod{p}, \quad (1.10)$$

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x & \text{if } v_p(x) = 0, \\ -1 & \text{if } v_p(x) > 0, \end{cases} \quad (1.11)$$

where $a_p(x) \in \{1, 2, \dots, p\}$ with $x \equiv a_p(x) \pmod{p}$ and $v_p(\cdot)$ denotes the p -order.

Lemma 2. [16, Lemma 17, (4)] *Let p be an odd prime. If $a \in \mathbb{Z}_p, n \in \mathbb{N}$ such that none of $a, a+1, \dots, a+n-1$ are in $p\mathbb{Z}_p$, then*

$$(a)_n = (-1)^n \frac{\Gamma_p(a+n)}{\Gamma_p(a)}. \quad (1.12)$$

In the following Sections 2 and 3, we give proofs of Theorems 1 and 2, respectively. The final Section 4 is devoted to a discussion and includes two conjectures.

2. PROOF OF THEOREM 1

Motivated by the work of McCarthy and Osburn [17] and Mortenson [18], we take the following choice of parameters in (1.6). Let $a = \frac{r}{5}, b = \frac{r+5}{10}, c = \frac{r+3p}{5}, d = \frac{r+3ip}{5}, e = \frac{r-3ip}{5}$, and $n = \frac{3p-r}{5}$, where $i^2 = -1$. Then we conclude that

$$\begin{aligned} & {}_6F_5 \left[\begin{matrix} \frac{r}{5}, & 1 + \frac{r}{10}, & \frac{r+3p}{5}, & \frac{r+3ip}{5}, & \frac{r-3ip}{5}, & \frac{r-3p}{5} \\ & \frac{r}{10}, & 1 - \frac{3p}{5}, & 1 - \frac{3ip}{5}, & 1 + \frac{3ip}{5}, & 1 + \frac{3p}{5} \end{matrix} ; 1 \right] \\ &= \frac{\left(1 + \frac{r}{5}\right)_{\frac{3p-r}{5}} \left(1 - \frac{r}{5}\right)_{\frac{3p-r}{5}}}{\left(1 - \frac{3ip}{5}\right)_{\frac{3p-r}{5}} \left(1 + \frac{3ip}{5}\right)_{\frac{3p-r}{5}}} {}_4F_3 \left[\begin{matrix} \frac{5-r-6p}{10}, & \frac{r+3ip}{5}, & \frac{r-3ip}{5}, & \frac{r-3p}{5} \\ \frac{2r-3p}{5}, & \frac{r+5}{10}, & \frac{5-3p}{5} & \end{matrix} ; 1 \right], \quad (2.1) \end{aligned}$$

It is easy to see that, for $k \geq 0$ and any p -adic integer b ,

$$(a + bp)_k (a - bp)_k (a + bip)_k (a - bip)_k \equiv (a)_k^4 \pmod{p^4}. \quad (2.2)$$

Hence, the left-hand side of (2.1) is congruent to

$$\begin{aligned} \sum_{k=0}^{\frac{3p-r}{5}} \frac{\left(1 + \frac{r}{10}\right)_k \left(\frac{r}{5}\right)_k^5}{\left(\frac{r}{10}\right)_k (1)_k^5} &= \frac{1}{r} \sum_{k=0}^{\frac{3p-r}{5}} (10k + r) \frac{\left(\frac{r}{5}\right)_k^5}{k!^5} \\ &\equiv \frac{1}{r} \sum_{k=0}^{p-1} (10k + r) \frac{\left(\frac{r}{5}\right)_k^5}{k!^5} \pmod{p^4}, \end{aligned}$$

where we have used the fact that $\frac{\left(\frac{r}{5}\right)_k}{k!} \equiv 0 \pmod{p}$ for $\frac{3p-r}{5} < k \leq p-1$ (the condition $p \geq \frac{5-r}{2}$ in the theorem is to guarantee $\frac{3p-r}{5} \leq p-1$). Since $\frac{3p-r}{5} \geq \frac{2p+r}{5}$, we have

$$\begin{aligned} \frac{\left(1 + \frac{r}{5}\right)_{\frac{3p-r}{5}} \left(1 - \frac{r}{5}\right)_{\frac{3p-r}{5}}}{\left(1 - \frac{3ip}{5}\right)_{\frac{3p-r}{5}} \left(1 + \frac{3ip}{5}\right)_{\frac{3p-r}{5}}} &= \frac{\left(1 + \frac{r}{5}\right)_{\frac{3p-r}{5}} \left(1 - \frac{r}{5}\right)_{\frac{3p-r}{5}}}{\prod_{j=1}^{\frac{3p-r}{5}} \left(j^2 + \frac{9p^2}{25}\right)} \\ &\equiv 0 \pmod{p^2}. \end{aligned}$$

Finally, by the congruences

$$(a + bip)_k (a - bip)_k \equiv (a + bp)_k (a - bp)_k \equiv (a)_k^2 \pmod{p^2} \quad (2.3)$$

for any p -adic integer b , we obtain

$$\begin{aligned} {}_4F_3 \left[\begin{matrix} \frac{5-r-6p}{10}, & \frac{r+3ip}{5}, & \frac{r-3ip}{5}, & \frac{r-3p}{5} \\ \frac{2r-3p}{5}, & \frac{r+5}{10}, & \frac{5-3p}{5} & \end{matrix} ; 1 \right] &\equiv {}_4F_3 \left[\begin{matrix} \frac{5-r-6p}{10}, & \frac{r}{5}, & \frac{r}{5}, & \frac{r-3p}{5} \\ \frac{2r-3p}{5}, & \frac{r+5}{10}, & \frac{5-3p}{5} & \end{matrix} ; 1 \right] \\ &\equiv {}_4F_3 \left[\begin{matrix} \frac{5-r-6p}{10}, & \frac{r+p}{5}, & \frac{r-p}{5}, & \frac{r-3p}{5} \\ \frac{2r-3p}{5}, & \frac{r+5}{10}, & \frac{5-3p}{5} & \end{matrix} ; 1 \right] \\ &= 0 \pmod{p^2}, \end{aligned}$$

where we have utilized Karlsson–Minton's summation (1.7) with $n = \frac{3p-r}{5}$, $b_1 = \frac{2r-3p}{5}$, $b_2 = \frac{r+5}{10}$, $b_3 = \frac{5-3p}{5}$, $m_1 = \frac{1-r}{2}$, $m_2 = \frac{2p+r-5}{10}$, and $m_3 = \frac{2p+r-5}{5}$ in the last step.

3. PROOF OF THEOREM 2

We can verify (1.5) for $r = 1$ and $p = 2$ by hand. In what follows, we assume that p is an odd prime. Recall the following transformation formula [14, Equation (4.2)]:

$$\begin{aligned}
& {}_7F_6 \left[\begin{matrix} t, & 1 + \frac{t}{2}, & -n, & t - a, & t - b, & t - c, & 1 - t - m + n + a + b + c \\ & \frac{t}{2}, & 1 + t + n, & 1 + a, & 1 + b, & 1 + c, & 2t + m - n - a - b - c \end{matrix} ; 1 \right] \\
&= \frac{(1+t)_n (a+b+2-m-t)_n (a+c+2-m-t)_n (b+c+2-m-t)_n}{(1+a)_n (1+b)_n (1+c)_n (a+b+c+1-m-2t)_n} \\
&\quad \times \frac{(a+b+1-m-t)(a+c+1-m-t)(b+c+1-m-t)}{(a+b+n+1-m-t)(a+c+n+1-m-t)(b+c+n+1-m-t)} \\
&\quad \times {}_4F_3 \left[\begin{matrix} -m, & -n, & a+b+c+1-m-2t, & a+b+c+1+n-m-t \\ a+b+1-m-t, & a+c+1-m-t, & b+c+1-m-t \end{matrix} ; 1 \right]. \quad (3.1)
\end{aligned}$$

Let ζ be a fifth primitive root of unity. Setting $m = 1 - r$, $t = \frac{r}{3}$, $n = \frac{2p-r}{3}$, $a = \frac{2p\zeta}{3}$, $b = \frac{2p\zeta^2}{3}$ and $c = \frac{2p\zeta^3}{3}$ in (3.1) and using $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$, the left-hand side of (3.1) becomes

$$\begin{aligned}
& {}_7F_6 \left[\begin{matrix} 1 + \frac{r}{6}, & \frac{r}{3}, & \frac{r-2p}{3}, & \frac{r-2p\zeta}{3}, & \frac{r-2p\zeta^2}{3}, & \frac{r-2p\zeta^3}{3}, & \frac{r-2p\zeta^4}{3} \\ & \frac{r}{6}, & 1 + \frac{2p}{3}, & 1 + \frac{2p\zeta}{3}, & 1 + \frac{2p\zeta^2}{3}, & 1 + \frac{2p\zeta^3}{3}, & 1 + \frac{2p\zeta^4}{3} \end{matrix} ; 1 \right] \\
&\equiv \frac{1}{r} \sum_{k=0}^{\frac{2p-r}{3}} (6k+r) \frac{\left(\frac{r}{3}\right)_k^6}{k!^6} \pmod{p^5},
\end{aligned}$$

where we have used the facts that none of the denominators in ${}_7F_6$ contain a multiple of p (the condition $p \geq 3 - r$ in the theorem is to guarantee $\frac{2p-r}{3} \leq p-1$) and

$$(u+vp)_k (u+vp\zeta)_k (u+vp\zeta^2)_k (u+vp\zeta^3)_k (u+vp\zeta^4)_k \equiv (u)_k^5 \pmod{p^5}.$$

Furthermore, for $\frac{2p-r}{3} < k \leq p-1$ we have $\left(\frac{r}{3}\right)_k \equiv 0 \pmod{p}$. Thus,

$$\begin{aligned}
& {}_7F_6 \left[\begin{matrix} 1 + \frac{r}{6}, & \frac{r}{3}, & \frac{r-2p}{3}, & \frac{r-2p\zeta}{3}, & \frac{r-2p\zeta^2}{3}, & \frac{r-2p\zeta^3}{3}, & \frac{r-2p\zeta^4}{3} \\ & \frac{r}{6}, & 1 + \frac{2p}{3}, & 1 + \frac{2p\zeta}{3}, & 1 + \frac{2p\zeta^2}{3}, & 1 + \frac{2p\zeta^3}{3}, & 1 + \frac{2p\zeta^4}{3} \end{matrix} ; 1 \right] \\
&\equiv \frac{1}{r} \sum_{k=0}^{p-1} (6k+r) \frac{\left(\frac{r}{3}\right)_k^6}{k!^6} \pmod{p^5}. \quad (3.2)
\end{aligned}$$

On the other hand, we determine the terminating hypergeometric series on the right-hand side of (3.1) modulo p :

$$\begin{aligned}
& \frac{(a+b+1-m-t)(a+c+1-m-t)(b+c+1-m-t)}{(a+b+n+1-m-t)(a+c+n+1-m-t)(b+c+n+1-m-t)} \\
& \times {}_4F_3 \left[\begin{matrix} -m, & -n, & a+b+c+1-m-2t, & a+b+c+1+n-m-t \\ a+b+1-m-t, & a+c+1-m-t, & b+c+1-m-t & ; 1 \end{matrix} \right] \\
& \equiv 8 \sum_{k=0}^{1-r} \frac{(r-1)_k \left(\frac{r}{3}\right)_k^3}{(1)_k \left(\frac{2r}{3}\right)_k^3} \pmod{p}. \tag{3.3}
\end{aligned}$$

Moreover,

$$\begin{aligned}
& \frac{(1+t)_n(a+b+2-m-t)_n(a+c+2-m-t)_n(b+c+2-m-t)_n}{(1+a)_n(1+b)_n(1+c)_n(a+b+c+1-m-2t)_n} \\
& = \frac{\left(1 + \frac{r}{3}\right)_{\frac{2p-r}{3}} \left(1 + \frac{2r+2p(\zeta+\zeta^2)}{3}\right)_{\frac{2p-r}{3}} \left(1 + \frac{2r+2p(\zeta+\zeta^3)}{3}\right)_{\frac{2p-r}{3}} \left(1 + \frac{2r+2p(\zeta^2+\zeta^3)}{3}\right)_{\frac{2p-r}{3}}}{(-1)^{\frac{2p-r}{3}} \left(1 + \frac{2p\zeta}{3}\right)_{\frac{2p-r}{3}} \left(1 + \frac{2p\zeta^2}{3}\right)_{\frac{2p-r}{3}} \left(1 + \frac{2p\zeta^3}{3}\right)_{\frac{2p-r}{3}} \left(1 + \frac{2p\zeta^4}{3}\right)_{\frac{2p-r}{3}}}. \tag{3.4}
\end{aligned}$$

Note that

$$\left(1 + \frac{r}{3}\right)_{\frac{2p-r}{3}} = \frac{2p}{3} \left(1 + \frac{r}{3}\right)_{\frac{2p-r-3}{3}}, \tag{3.5}$$

and

$$\begin{aligned}
& \left(1 + \frac{2r+2p(\zeta+\zeta^2)}{3}\right)_{\frac{2p-r}{3}} \left(1 + \frac{2r+2p(\zeta+\zeta^3)}{3}\right)_{\frac{2p-r}{3}} \left(1 + \frac{2r+2p(\zeta^2+\zeta^3)}{3}\right)_{\frac{2p-r}{3}} \\
& = \frac{5p^3}{27} \left(1 + \frac{2r+2p(\zeta+\zeta^2)}{3}\right)_{\frac{p-2r-3}{3}} \left(1 + \frac{2r+2p(\zeta+\zeta^3)}{3}\right)_{\frac{p-2r-3}{3}} \\
& \quad \times \left(1 + \frac{2r+2p(\zeta^2+\zeta^3)}{3}\right)_{\frac{p-2r-3}{3}} \left(\frac{3+p(2\zeta+2\zeta^2+1)}{3}\right)_{\frac{p+r}{3}} \\
& \quad \times \left(\frac{3+p(2\zeta+2\zeta^3+1)}{3}\right)_{\frac{p+r}{3}} \left(\frac{3+p(2\zeta^2+2\zeta^3+1)}{3}\right)_{\frac{p+r}{3}}. \tag{3.6}
\end{aligned}$$

Combining (3.5) and (3.6), we arrive at

$$\begin{aligned} & \frac{\left(1 + \frac{r}{3}\right)_{\frac{2p-r}{3}} \left(1 + \frac{2r+2p(\zeta+\zeta^2)}{3}\right)_{\frac{2p-r}{3}} \left(1 + \frac{2r+2p(\zeta+\zeta^3)}{3}\right)_{\frac{2p-r}{3}} \left(1 + \frac{2r+2p(\zeta^2+\zeta^3)}{3}\right)_{\frac{2p-r}{3}}}{(-1)_{\frac{2p-r}{3}} \left(1 + \frac{2p\zeta}{3}\right)_{\frac{2p-r}{3}} \left(1 + \frac{2p\zeta^2}{3}\right)_{\frac{2p-r}{3}} \left(1 + \frac{2p\zeta^3}{3}\right)_{\frac{2p-r}{3}} \left(1 + \frac{2p\zeta^4}{3}\right)_{\frac{2p-r}{3}}} \\ & \equiv \frac{(-1)_{\frac{2p-r}{3}} 10p^4}{81} \cdot \frac{\left(1 + \frac{r}{3}\right)_{\frac{2p-r-3}{3}} \left(1 + \frac{2r}{3}\right)_{\frac{p-2r-3}{3}} (1)_{\frac{p+r}{3}}^3}{(1)_{\frac{2p-r}{3}}^4} \pmod{p^5}. \end{aligned} \quad (3.7)$$

It follows from (3.2)–(3.4) and (3.7) that

$$\begin{aligned} \sum_{k=0}^{p-1} (6k+r) \frac{\left(\frac{r}{3}\right)_k^6}{k!^6} & \equiv \frac{(-1)_{\frac{2p-r}{3}} 80rp^4}{81} \cdot \frac{\left(1 + \frac{r}{3}\right)_{\frac{2p-r-3}{3}} \left(1 + \frac{2r}{3}\right)_{\frac{p-2r-3}{3}} (1)_{\frac{p+r}{3}}^3}{(1)_{\frac{2p-r}{3}}^4} \\ & \quad \times \sum_{k=0}^{1-r} \frac{(r-1)_k \left(\frac{r}{3}\right)_k^3}{(1)_k \left(\frac{2r}{3}\right)_k^3} \pmod{p^5}. \end{aligned} \quad (3.8)$$

By Lemmas 1 and 2, we have

$$\begin{aligned} & \frac{\left(1 + \frac{r}{3}\right)_{\frac{2p-r-3}{3}} \left(1 + \frac{2r}{3}\right)_{\frac{p-2r-3}{3}} (1)_{\frac{p+r}{3}}^3}{(1)_{\frac{2p-r}{3}}^4} \\ & \stackrel{(1.12)}{=} \frac{(-1)_{\frac{2p-r}{3}+r} \Gamma_p\left(\frac{2p}{3}\right) \Gamma_p\left(\frac{p}{3}\right)^3 \Gamma_p\left(1 + \frac{p+r}{3}\right)^3 \Gamma_p(1)}{\Gamma_p\left(1 + \frac{r}{3}\right) \Gamma_p\left(1 + \frac{2r}{3}\right)^3 \Gamma_p\left(1 + \frac{2p-r}{3}\right)^4} \\ & \stackrel{(1.10)}{=} \frac{(-1)_{\frac{2p-r}{3}+r} \Gamma_p(0)^4 \Gamma_p\left(1 + \frac{r}{3}\right)^2 \Gamma_p(1)}{\Gamma_p\left(1 + \frac{2r}{3}\right)^3 \Gamma_p\left(1 - \frac{r}{3}\right)^4} \pmod{p} \\ & \stackrel{(1.8)}{=} \frac{(-1)_{\frac{2p-r}{3}+r+1} \Gamma_p\left(1 + \frac{r}{3}\right)^2}{\Gamma_p\left(1 + \frac{2r}{3}\right)^3 \Gamma_p\left(1 - \frac{r}{3}\right)^4}. \end{aligned} \quad (3.9)$$

The proof of (1.5) then follows from (3.8) and (3.9).

4. DISCUSSION

We know that many supercongruences have nice q -analogues (see [1, 6–9, 11, 15]). For example, we have the following conjectural q -analogue of (1.3): for the same p and r as in Theorem 1,

$$\sum_{k=0}^{p-1} [10k+r] \frac{(q^r; q^5)_k^5}{(q^5; q^5)_k^5} q^{\frac{5(3-r)k}{2}} \equiv 0 \pmod{[p]^4}, \quad (4.1)$$

where $[n] = 1 + q + \cdots + q^{n-1}$ is the q -integer and $(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$ denotes the q -shifted factorial.

Although there are q -analogues of Whipple's well-poised ${}_7F_6$ transformation and of Karlsson–Minton's summation (see [4, Appendix (II.27) and (III.18)]), we are unable to give a proof of (4.1). This is because we only know a q -analogue of (2.3) (see [7, Lemma 1]) but do not know any q -analogues of (2.2). Besides, we do not know how to prove (4.1) by using the method of 'creative microscoping' devised in [9] either.

While in Theorem 2 we were able to provide a common generalization of the second supercongruence in (1.2) (restricted to modulo p^5) and the first supercongruence in (1.4), it appears to be rather difficult to extend Theorem 1 to a higher supercongruence involving the p -adic Gamma function in the spirit of Theorem 2, even in the special cases $r = 1$ or $r = -1$.

We end our paper with two further conjectures for future research. Conjecture 1 concerns a stronger version of Theorem 2 and includes the second supercongruence in (1.2) as a special case. Conjecture 2 concerns a common generalization of the first supercongruence in (1.2) and the second supercongruence in (1.4).

Conjecture 1. *The supercongruence (1.5) holds modulo p^6 for any prime $p > 3$.*

Conjecture 2. *Let $r \leq 1$ be an integer coprime with 3. Let $p \geq 7$ be a prime such that $p \equiv r \pmod{3}$ and $p \geq 3 - 2r$. Then*

$$\sum_{k=0}^{p-1} (6k+r) \frac{\left(\frac{r}{3}\right)_k^6}{k!^6} \equiv \frac{(-1)^r 8rp}{3} \cdot \frac{\Gamma_p\left(1 + \frac{r}{3}\right)^2}{\Gamma_p\left(1 + \frac{2r}{3}\right)^3 \Gamma_p\left(1 - \frac{r}{3}\right)^4} \times \sum_{k=0}^{1-r} \frac{(r-1)_k \left(\frac{r}{3}\right)_k^3}{(1)_k \left(\frac{2r}{3}\right)_k} \pmod{p^6}. \quad (4.2)$$

The $r = 1$ case of Conjecture 1 was proved by Long and Ramakrishna [16].

We now explain the difficulties we encountered trying to prove Conjecture 1 for arbitrary integer r satisfying the conditions stated in Theorem 2. It is reasonable to follow the method successfully used by Long and Ramakrishna [16] to establish the desired modulo p^6 supercongruence in the $r = 1$ case.

Let ζ be a fifth primitive root of unity. Numerical calculation suggests

$${}_7F_6 \left[\begin{matrix} 1 + \frac{r}{6}, & \frac{r}{3}, & \frac{r}{3} - x, & \frac{r}{3} - \zeta x, & \frac{r}{3} - \zeta^2 x, & \frac{r}{3} - \zeta^3 x, & \frac{r}{3} - \zeta^4 x \\ & \frac{r}{6}, & 1 + x, & 1 + \zeta x, & 1 + \zeta^2 x, & 1 + \zeta^3 x, & 1 + \zeta^4 x \end{matrix}; 1 \right]_{\frac{2p-r}{3}} \in p\mathbb{Z}_p[[x^5]] \quad (4.3)$$

(the $\frac{2p-r}{3}$ as a subindex of the ${}_7F_6$ series means that the respective series is truncated and contains only its first $\frac{2p-r}{3} + 1$ terms, just as on the right-hand side of (4.4)). The case $r = 1$ of this assertion was proved by Long and Ramakrishna in [16] using Bailey's ${}_9F_8$ transformation. Letting $x = \frac{2p}{3}$ in (4.3) (in which case the series gets naturally truncated from the top, due to the appearance of $\frac{r-2p}{3}$, a negative integer, as an upper parameter),

we obtain

$$\begin{aligned} & {}_7F_6 \left[\begin{matrix} 1 + \frac{r}{6}, & \frac{r}{3}, & \frac{r-2p}{3}, & \frac{r-2p\zeta}{3}, & \frac{r-2p\zeta^2}{3}, & \frac{r-2p\zeta^3}{3}, & \frac{r-2p\zeta^4}{3} \\ & \frac{r}{6}, & 1 + \frac{2p}{3}, & 1 + \frac{2p\zeta}{3}, & 1 + \frac{2p\zeta^2}{3}, & 1 + \frac{2p\zeta^3}{3}, & 1 + \frac{2p\zeta^4}{3} \end{matrix}; 1 \right] \\ & \equiv \frac{1}{r} \sum_{k=0}^{\frac{2p-r}{3}} (6k+r) \frac{\left(\frac{r}{3}\right)_k^6}{k!^6} \pmod{p^6}. \end{aligned} \quad (4.4)$$

For $r = 1$, the left-hand side of (4.4) is summable and reduces to a closed form (see [16, (4.6)]) which was shown by Long and Ramakrishna to be congruent to a product of p -adic Gamma functions modulo p^6 .

For an arbitrary integer r satisfying the conditions in Theorem 2, the left-hand side of (4.4) is a multiple of a ${}_4F_3$ series (see Equation (3.3) and (3.4)) instead of a product. Since, in our proof of Theorem 2, we show the right-hand side of (3.4) is a multiple of p^4 , in order to determine the left-hand side of (4.4) modulo p^6 , it would suffice to evaluate the left-hand side of (3.3) modulo p^2 and show that it agrees with the right-hand side of (3.3) (modulo p^2). However, putting $m = 1 - r$, $t = \frac{r}{3}$, $n = \frac{2p-r}{3}$, $a = \frac{2p\zeta}{3}$, $b = \frac{2p\zeta^2}{3}$ and $c = \frac{2p\zeta^3}{3}$ in (3.3), numerical calculation suggests that (3.3) (valid as a congruence modulo p) is in general invalid as a congruence modulo p^2 . (A counterexample is, for instance, $r = -1$ and $p = 13$.) We can only deduce that the left-hand side of (3.3) is congruent to the following form modulo p^2 :

$$8 \sum_{k=0}^{1-r} \frac{(r-1)_k \left(\frac{r}{3}\right)_k^3}{(1)_k \left(\frac{2r}{3}\right)_k^3} + pf(r),$$

where $f(r)$ is not always divisible by p .

The proof of (4.3) in the case $r = 1$ given by Long and Ramakrishna in [16] is rather involved. While it is feasible that one can establish its r -extension (4.3) using Bailey's ${}_9F_8$ transformation as well, we find it hard to determine $f(r)$, quite in contrast to the case $r = 1$. To conclude, we do not see how Long and Ramakrishna's method would extend to prove Conjecture 1.

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