THREE DWORK-TYPE q-SUPERCONGRUENCES FROM THE q-DIXON SUM

VICTOR J. W. GUO

Abstract. Employing the method of creative microscoping introduced by the author and Zudilin and the q-Dixon sum, we establish three reduced Dwork-type q-supercongruences. As a conclusion, we get the following result: for any prime $p \equiv 2 \pmod 3$ and positive integer r,

$$\sum_{k=0}^{p^r-1} \frac{(-\frac{2}{3})_k^3}{k!^3} \equiv 0 \pmod{p^{3r}}.$$

Note that Hu and Wang have already proved that the above supercongruence also holds for primes $p \equiv 1 \pmod 3$.

1. Introduction

In 1997, Van Hamme [14] proposed the following conjecture: for any odd prime p,

$$\sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^3}{k!^3} \equiv \begin{cases} -\Gamma_p(\frac{1}{4})^4 \pmod{p^2} & \text{if } p \equiv 1 \pmod 4, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{1.1}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the *Pochhammer symbol*, and $\Gamma_p(x)$ is the *p-adic Gamma function* (see [1, §1.12]). In 2016, Long and Ramakrishna [9, Theorem 3] proved the following generalization of (1.1):

$$\sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^3}{k!^3} \equiv \begin{cases} -\Gamma_p(\frac{1}{4})^4 \pmod{p^3} & \text{if } p \equiv 1 \pmod 4, \\ -\frac{p^2}{16}\Gamma_p(\frac{1}{4})^4 \pmod{p^3} & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{1.2}$$

In 2020, Mao and Pan [10] established a similar result as follows: for any prime $p \equiv 1 \pmod 3$,

$$\sum_{k=0}^{p-1} \frac{(-\frac{2}{3})_k^3}{k!^3} \equiv 0 \pmod{p^2}. \tag{1.3}$$

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Later, Wei [16] proved that (1.3) is true modulo p^3, which may be written in the equivalent form: for primes $p \equiv 1 \pmod 3$,

$$\sum_{k=0}^{(p+2)/3} \frac{(-\frac{2}{3})_k^3}{k!^3} \equiv 0 \pmod{p^3}, \tag{1.4}$$

since the p-adic order of $(-\frac{2}{3})_k/k!$ is 1 for k in the range $(p+2)/3 < k \leqslant p-1$. In fact, much more is true. With the help of the creative microscoping method devised by the author and Zudilin along with the q-Dixon sum, Wei (see [16, Theorem 1.1]) gave a q-analogue of (1.4), which is equivalent to the following q-supercongruence: for any integer $n > 1$ satisfying $n \equiv 1 \pmod 6$,

$$\sum_{k=0}^{(n+2)/3} \frac{(1+q^{3k-1})(q^{-2};q^3)_k^3}{(1+q^{-1})(q^3;q^3)_k^3} q^{6k} \equiv 0 \pmod{\Phi_n(q)^3}. \tag{1.5}$$

Here and in what follows, we let $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ stand for the *q-shifted factorial* for $n \geqslant 0$, $(a;q)_\infty = \lim_{n\to\infty}(a;q)_n$, and we also adopt the abbreviated notation

$$(x_1, x_2, \dots, x_m; q)_n = (x_1;q)_n (x_2;q)_n \cdots (x_m;q)_n$$

for $n \geqslant 0$ or $n = \infty$. Moreover, $[n] = 1 + q + \cdots + q^{n-1}$ denotes the *q-integer*, and $\Phi_n(q)$ represents the n-th *cyclotomic polynomial* in q, which can be factorized as

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}} (q - \zeta^k),$$

where ζ is an n-th primitive root of unity. It is well known that $\Phi_n(q)$ is irreducible in the polynomial ring $\mathbb{Z}[q]$. Clearly, the supercongruence (1.4) follows from (1.5) by taking $n = p$ and $q \to 1$.

Pan, Tauraso, and Wang [12, Theorem 5.1] gave a uniform generalization of (1.2) and (1.4). In particular, they established the following supercongruence: for any odd prime $p \equiv 2 \pmod 3$,

$$\sum_{k=0}^{(2p+2)/3} \frac{(-\frac{2}{3})_k^3}{k!^3} \equiv 0 \pmod{p^3}. \tag{1.6}$$

Motivated by Wei's work [16], we shall give the following q-analogue of (1.6).

Theorem 1.1. *Let $n > 2$ be an integer with $n \equiv 2 \pmod 3$. Then*

$$\sum_{k=0}^{(2n+2)/3} \frac{(1+q^{3k-1})(q^{-2};q^3)_k^3}{(1+q^{-1})(q^3;q^3)_k^3} q^{6k} \equiv 0 \pmod{\Phi_n(q)^3}. \tag{1.7}$$

Wei [16, Conjecture 3.1] proposed the following conjecture: for any prime $p \equiv 1 \pmod 3$ and positive integer r,

$$\sum_{k=0}^{(p^r+2)/3} \frac{(-\frac{2}{3})_k^3}{k!^3} \equiv 0 \pmod{p^{3r}}. \tag{1.8}$$

The q-supercongruence (1.5) implies that the above supercongruence holds modulo p^3. Employing the creative microscoping method again, Ni [11] proved that (1.8) holds modulo p^{r+2}, and Hu and Wang [8] completely confirmed the supercongruence (1.8) and its companion: for any prime $p \equiv 1 \pmod 3$ and positive integer r,

$$\sum_{k=0}^{p^r-1} \frac{(-\frac{2}{3})_k^3}{k!^3} \equiv 0 \pmod{p^{3r}}. \tag{1.9}$$

The supercongruence (1.9) may be deemed a reduced Dwork-type supercongruence (see [2]). For more Dwork-type supercongruences, see [4, 7, 15].

We find that (1.9) is also true for any odd prime $p \equiv 2 \pmod 3$ and positive integer r. We shall prove this result by building the following q-supercongruence.

Theorem 1.2. *Let n be a positive integer with $n \equiv 5 \pmod 6$ and r a positive integer. Then*

$$\sum_{k=0}^{n^r-1} \frac{(1+q^{3k-1})(q^{-2};q^3)_k^3}{(1+q^{-1})(q^3;q^3)_k^3} q^{6k} \equiv 0 \pmod{[n^r]\prod_{j=1}^{r}\Phi_{n^j}(q)^2}. \tag{1.10}$$

It should be pointed out that Theorem 1.2 also holds for $n \equiv 1 \pmod 6$ and $n > 1$. This can be easily derived from combining [8, Theorem 1] and our Lemma 3.1.

Clearly, when $n = p$ is a prime and $q \to 1$, the q-supercongruence (1.10) reduces to the following supercongruence.

Corollary 1.3. *Let $p \equiv 2 \pmod 3$ be an odd prime and let r be a positive integer. Then*

$$\sum_{k=0}^{p^r-1} \frac{(-\frac{2}{3})_k^3}{k!^3} \equiv 0 \pmod{p^{3r}}.$$

Theorem 1.4. *Let n be a positive integer with $n \equiv 5 \pmod 6$ and r a positive integer. If r is odd, then*

$$\sum_{k=0}^{(2n^r+2)/3} \frac{(1+q^{3k-1})(q^{-2};q^3)_k^3}{(1+q^{-1})(q^3;q^3)_k^3} q^{6k} \equiv 0 \pmod{[n^r]\prod_{j=1}^{r}\Phi_{n^j}(q)^2}, \tag{1.11}$$

and if r is even, then

$$\sum_{k=0}^{(n^r+2)/3} \frac{(1+q^{3k-1})(q^{-2};q^3)_k^3}{(1+q^{-1})(q^3;q^3)_k^3} q^{6k} \equiv 0 \pmod{[n^r]\prod_{j=1}^{r}\Phi_{n^j}(q)^2}. \tag{1.12}$$

Similarly, we immediately get the following conclusion from the above theorem.

Corollary 1.5. *Let* $p \equiv 2 \pmod 3$ *be an odd prime and let* r *be a positive integer. If* r *is odd, then*

$$\sum_{k=0}^{(2p^r+2)/3} \frac{(-\frac{2}{3})_k^3}{k!^3} \equiv 0 \pmod{p^{3r}},$$

and if r *is even, then*

$$\sum_{k=0}^{(p^r+2)/3} \frac{(-\frac{2}{3})_k^3}{k!^3} \equiv 0 \pmod{p^{3r}}.$$

Recall that the basic hypergeometric series ${}_{r+1}\phi_r$ is defined by

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_k}{(q, b_1, b_2, \dots, b_r; q)_k} z^k.$$

Then the q-Dixon sum [3, Appendix (II.13)] can be stated as follows:

$${}_4\phi_3 \left[\begin{matrix} a, -qa^{\frac12}, b, c \\ -a^{\frac12}, aq/b, aq/c \end{matrix} ; q, \frac{qa^{\frac12}}{bc} \right] = \frac{(aq, aq/bc, qa^{\frac12}/b, qa^{\frac12}/c; q)_\infty}{(aq/b, aq/c, qa^{\frac12}, qa^{\frac12}/bc; q)_\infty}, \tag{1.13}$$

where $|qa^{\frac12}/bc| < 1$. The sum (1.13) plays an important part in our proof of Theorems 1.1–1.4.

2. Proof of Theorem 1.1

In order to prove Theorem 1.1, we first establish the following q-congruence with an extra parameter a.

Theorem 2.1. *Let* $n > 2$ *be an integer with* $n \equiv 2 \pmod 3$, *and let* a *be an indeterminate. Then, modulo* $\Phi_n(q)(1-aq^{2n})(a-q^{2n})$,

$$\sum_{k=0}^{(2n+2)/3} \frac{(1+q^{3k-1})(aq^{-2}, q^{-2}/a, q^{-2}; q^3)_k}{(1+q^{-1})(aq^3, q^3/a, q^3; q^3)_k} q^{6k} \equiv 0. \tag{2.1}$$

Proof. Performing the parameter substitutions $q \mapsto q^3$, $a = q^{-2n-2}x^2$, $b = aq^{-2}$, $c = q^{-2}/a$ in (1.13), and then letting $x \to 1$, we obtain

$$\begin{aligned}
&\sum_{k=0}^{(2n+2)/3} \frac{(1+q^{3k-n-1})(aq^{-2}, q^{-2}/a, q^{-2n-2}; q^3)_k}{(1+q^{-1-n})(aq^{3-2n}, q^{3-2n}/a, q^3; q^3)_k} q^{6k-nk} \\
&\quad = \lim_{x\to 1} \sum_{k=0}^{\infty} \frac{(1+q^{3k-n-1}x)(aq^{-2}, q^{-2}/a, q^{-2n-2}x^2; q^3)_k}{(1+q^{-1-n}x)(aq^{3-2n}x^2, q^{3-2n}x^2/a, q^3; q^3)_k} q^{6k-nk} x^k \\
&\quad = \lim_{x\to 1} \frac{(q^{1-2n}x^2, q^{5-2n}x^2, q^{4-n}x/a, aq^{4-n}x; q^3)_\infty}{(q^{3-2n}x^2/a, aq^{3-2n}x^2, q^{2-n}x, q^{6-n}x; q^3)_\infty} \\
&\quad = \frac{2(q^{1-2n}, q^{5-2n}; q^3)_{(n+1)/3}}{(q^{3-2n}/a, aq^{3-2n}; q^3)_{(n+1)/3}}.
\end{aligned} \tag{2.2}$$

Since $q^n \equiv 1 \pmod{\Phi_n(q)}$ and $(q^{5-2n}; q^3)_{(n+1)/3}$ contains the factor $1 - q^{-n}$, from (2.2) we conclude that (2.1) holds modulo $\Phi_n(q)$.

On the other hand, making the parameter replacements $q \mapsto q^3$, $a = q^{-2}$, $b = q^{2n-2}$, $c = q^{-2n-2}$ in (1.13), we get

$$\sum_{k=0}^{(2n+2)/3} \frac{(1+q^{3k-1})(q^{2n-2}, q^{-2n-2}, q^{-2}; q^3)_k}{(1+q^{-1})(q^{3+2n}, q^{3-2n}, q^3; q^3)_k} q^{6k} = \frac{(q, q^5, q^{4-2n}, q^{4+2n}; q^3)_\infty}{(q^{3-2n}, q^{3+2n}, q^2, q^6; q^3)_\infty} = 0 \tag{2.3}$$

because $4 - 2n$ is negative and is divisible by 3. This means that, when $a - q^{-2n}$ or $a = q^{2n}$, the left-hand side of (2.1) is equal to 0. Namely, the q-congruence (2.1) holds modulo $1 - aq^{2n}$ and $a - q^{2n}$.

The proof of (2.1) then follows from the fact that $\Phi_n(q)$, $1 - aq^{2n}$, and $a - q^{2n}$ are pairwise coprime polynomials. □

Proof of Theorem 1.1. The denominator of (2.1) related to a is the factor $(aq^3, q^3/a; q^3)_{(2n+2)/3}$, which is coprime with $\Phi_n(q)$ when $a = 1$. On the other hand, $(1 - aq^{2n})(a - q^{2n})$ contains the factor $\Phi_n(q)^2$ when $a = 1$. Thus, letting $a = 1$ in (2.1), we conclude that (1.7) holds modulo $\Phi_n(q)^3$, as desired. □

3. Proof of Theorem 1.2

We first give the following lemma, which is somewhat similar to [5, Lemma 2.2].

Lemma 3.1. *Let $n > 2$ be an integer coprime with 6, and let a be an indeterminate. Then*

$$\sum_{k=0}^{m} \frac{(1+q^{3k-1})(aq^{-2}, q^{-2}/a, q^{-2}; q^3)_k}{(1+q^{-1})(aq^3, q^3/a, q^3; q^3)_k} q^{6k} \equiv 0 \pmod{[n]}, \tag{3.1}$$

$$\sum_{k=0}^{n-1} \frac{(1+q^{3k-1})(aq^{-2}, q^{-2}/a, q^{-2}; q^3)_k}{(1+q^{-1})(aq^3, q^3/a, q^3; q^3)_k} q^{6k} \equiv 0 \pmod{[n]}, \tag{3.2}$$

where $0 < m \leqslant n-1$ *and* $3m \equiv 2 \pmod{n}$*.*

Proof. For $n \equiv 5 \pmod{6}$, the q-congruence (2.1) implies that (3.1) holds modulo $\Phi_n(q)$. For $n \equiv 1 \pmod{6}$, like (2.2), we can prove that that

$$\sum_{k=0}^{(n+2)/3} \frac{(1+q^{3k-n/2-1})(aq^{-2}, q^{-2}/a, q^{-n-2}; q^3)_k}{(1+q^{-1-n/2})(aq^{3-n}, q^{3-n}/a, q^3; q^3)_k} q^{6k-nk/2} = 0. \tag{3.3}$$

Since $q^n \equiv 1 \pmod{\Phi_n(q)}$, letting $q \mapsto q^2$ and noticing that $\Phi_n(q)\Phi_n(-q) = \Phi_n(q^2)$ for odd n, from (3.3) we can easily deduce that (3.1) holds modulo $\Phi_n(q)$.

Moreover, since $3m \equiv 2 \pmod{n}$, the q^3-shifted factorial $(q^{-2}; q^3)_k$ has the factor $1-q^n$ or $1-q^{2n}$ for k in the range $m < k \leqslant n-1$, and is clearly congruent to 0 modulo $\Phi_n(q)$. Meanwhile, the q^3-shifted factorial $(q^3; q^3)_k$ is coprime with $\Phi_n(q)$ for $m < k \leqslant n-1$. Hence, for each k satisfying $m < k \leqslant n-1$, the k-th term in (3.2) is always congruent to 0 modulo $\Phi_n(q)$. This along with (3.1) modulo $\Phi_n(q)$ establishes the truth of (3.2) modulo $\Phi_n(q)$.

We are now ready to prove (3.1) and (3.2) modulo $[n]$. Let $\zeta \neq 1$ denote an n-th root of unity, perhaps not primitive. In other words, ζ is a primitive root of unity of degree d such that $d \mid n$ and $d > 1$. Let $c_q(k)$ stand for the k-th term on the left-hand side of (3.2), i.e.,

$$c_q(k) = \frac{(1+q^{3k-1})(aq^{-2}, q^{-2}/a, q^{-2}; q^3)_k}{(1+q^{-1})(aq^3, q^3/a, q^3; q^3)_k} q^{6k}.$$

The q-congruences (3.1) and (3.2) modulo $\Phi_n(q)$ with $n \mapsto d$ give the following identity:

$$\sum_{k=0}^{m_1} c_\zeta(k) = \sum_{k=0}^{d-1} c_\zeta(k) = 0,$$

where $3m_1 \equiv 2 \pmod{d}$ and $0 < m_1 \leqslant d-1$. We have

$$\lim_{q \to \zeta} \frac{c_q(\ell d + k)}{c_q(\ell d)} = \frac{c_\zeta(\ell d + k)}{c_\zeta(\ell d)} = c_\zeta(k). \tag{3.4}$$

Therefore,

$$\sum_{k=0}^{n-1} c_\zeta(k) = \sum_{\ell=0}^{n/d-1} \sum_{k=0}^{d-1} c_\zeta(\ell d + k) = \sum_{\ell=0}^{n/d-1} c_\zeta(\ell d) \sum_{k=0}^{d-1} c_\zeta(k) = 0, \tag{3.5}$$

and

$$\sum_{k=0}^{m} c_\zeta(k) = \sum_{\ell=0}^{(m-m_1)/d-1} c_\zeta(\ell d) \sum_{k=0}^{d-1} c_\zeta(k) + c_\zeta(m-m_1) \sum_{k=0}^{m_1} c_\zeta(k) = 0.$$

This indicates that the two sums $\sum_{k=0}^{n-1} c_q(k)$ and $\sum_{k=0}^{m} c_q(k)$ are congruent to 0 modulo $\Phi_d(q)$. As this is true for any factor $d > 1$ of n, we conclude that they are congruent to 0 modulo

$$\prod_{d \mid n,\, d>1} \Phi_d(q) = [n],$$

thus building the q-congruences (3.1) and (3.2). □

We now give a parametric version of Theorem 1.2.

Theorem 3.2. *Let n be a positive integer with $n \equiv 5 \pmod 6$ and r a positive integer. Let a be an indeterminate. Then, modulo*

$$[n^r]\prod_{j=0}^{n^{r-1}-1}(1-aq^{(3j+2)n})(a-q^{(3j+2)n}),$$

we have

$$\sum_{k=0}^{n^r-1}\frac{(1+q^{3k-1})(aq^{-2},q^{-2}/a,q^{-2};q^3)_k}{(1+q^{-1})(aq^3,q^3/a,q^3;q^3)_k}q^{6k}\equiv 0. \tag{3.6}$$

Proof. Replacing n by n^r in (3.2), we see that (3.6) holds modulo $[n^r]$.

For $a=q^{-(3j+2)n}$ or $a=q^{(3j+2)n}$ with j in the range $0\leqslant j\leqslant n^{r-1}-1$, the left-hand side of (3.6) is equal to

$$\sum_{k=0}^{((3j+2)n+2)/3}\frac{(1+q^{3k-1})(q^{-2-(3j+2)n},q^{-2+(3j+2)n},q^{-2};q^3)_k}{(1+q^{-1})(q^{3-(3j+2)n},q^{3+(3j+2)n},q^3;q^3)_k}q^{6k}, \tag{3.7}$$

where we have used the facts that $((3j+2)n+2)/3\leqslant n^r-1$ and $(q^{-2-(3j+2)n};q^3)_k=0$ for $k>((3j+2)n+2)/3$. Moreover, performing the parameter substitutions $q\mapsto q^3$, $a=q^{-2}$, $b=q^{(3j+2)n-2}$, $c=q^{-(3j+2)n-2}$ in (1.13), one sees that (3.7) vanishes, just like (2.3). This proves that (3.6) holds modulo $1-aq^{(3j+2)n}$ and $a-q^{(3j+2)n}$ for all $0\leqslant j\leqslant n^{r-1}-1$. Since all these moduli and $[n^r]$ are pairwise coprime polynomials in q, we accomplish the proof of (3.6). □

Proof of Theorem 1.2. It is easy to see that the $a=1$ case of

$$\prod_{j=0}^{n^{r-1}-1}(1-aq^{(3j+2)n})(a-q^{(3j+2)n})$$

contains the factor $\prod_{j=1}^{r}\Phi_{n^j}(q)^{2n^{r-j}}$. On the other hand, the factor involving a in the denominator of the left-hand side of (3.6) is $(aq^3,q^3/a;q^3)_{n^r-1}$. In view of $\gcd(n,3)=1$, when $a=1$ it merely contains the factor $\prod_{j=1}^{r}\Phi_{n^j}(q)^{2n^{r-j}-2}$ related to $\Phi_n(q),\Phi_{n^2}(q),\dots,\Phi_{n^r}(q)$. Therefore, taking $a=1$ in (3.6) we deduce that (1.10) is true modulo $\prod_{j=1}^{r}\Phi_{n^j}(q)^3$, where one product $\prod_{j=1}^{r}\Phi_{n^j}(q)$ arises from $[n^r]$.

Finally, the proof of (3.6) modulo $[n^r]$ is still valid for the $a=1$ case. Since the least common multiple of the polynomials $\prod_{j=1}^{r}\Phi_{n^j}(q)^3$ and $[n^r]$ is $[n^r]\prod_{j=1}^{r}\Phi_{n^j}(q)^2$, we compete the proof of (1.10). □

4. Proof of Theorem 1.4

Like the proof of Theorem 1.2, we first formulate a parametric version of Theorem 1.4.

Theorem 4.1. *Let n be a positive integer with $n \equiv 5 \pmod 6$ and r a positive integer. Let a be an indeterminate. If r is odd, then modulo*

$$[n^r] \prod_{j=0}^{(2n^{r-1}-2)/3} (1-aq^{(3j+2)n})(a-q^{(3j+2)n}),$$

we have

$$\sum_{k=0}^{(2n^r+2)/3} \frac{(1+q^{3k-1})(aq^{-2},q^{-2}/a,q^{-2};q^3)_k}{(1+q^{-1})(aq^3,q^3/a,q^3;q^3)_k} q^{6k} \equiv 0. \tag{4.1}$$

If r is even, then modulo

$$[n^r] \prod_{j=0}^{(n^{r-1}-2)/3} (1-aq^{(3j+2)n})(a-q^{(3j+2)n}),$$

we have

$$\sum_{k=0}^{(n^r+2)/3} \frac{(1+q^{3k-1})(aq^{-2},q^{-2}/a,q^{-2};q^3)_k}{(1+q^{-1})(aq^3,q^3/a,q^3;q^3)_k} q^{6k} \equiv 0. \tag{4.2}$$

Proof. We first consider the case where r is odd. Replacing n by n^r in (3.1), we see that (4.1) and (4.2) hold modulo $[n^r]$.

For $a=q^{-(3j+2)n}$ or $a=q^{(3j+2)n}$ with j in the range $0\leqslant j\leqslant (2n^{r-1}-2)/3$, the left-hand side of (3.6) can be written as (3.7), where we have used $((3j+2)n+2)/3\leqslant (2n^r+2)/3$ and $(q^{-2-(3j+2)n};q^3)_k=0$ for $k>((3j+2)n+2)/3$. We have mentioned in the previous section that the sum (3.7) is equal to 0. This proves the truth of (3.6) modulo $1-aq^{(3j+2)n}$ and $a-q^{(3j+2)n}$ for all $0\leqslant j\leqslant (2n^{r-1}-2)/3$. Since all these moduli and $[n^r]$ are pairwise coprime polynomials in q, we complete the proof of (4.1) .

Similarly, we can prove (4.2). □

Proof of Theorem 1.4. We first consider the case where r is odd. It is not hard to see that the $a=1$ case of

$$\prod_{j=0}^{(2n^{r-1}-2)/3} (1-aq^{(3j+2)n})(a-q^{(3j+2)n})$$

has the factor $\prod_{j=1}^r \Phi_{n^j}(q)^{2\lfloor (2n^{r-j}+2)/3\rfloor}$, where $\lfloor x\rfloor$ denotes the integer part of a real number x. On the other hand, the factor involving a in the denominator of the left-hand side of (4.1) is $(aq^3,q^3/a;q^3)_{(2n^r+2)/3}$. Since $\gcd(n,3)=1$, when $a=1$ it only has the factor $\prod_{j=1}^r \Phi_{n^j}(q)^{2\lfloor (2n^{r-j}-1)/3\rfloor}$ related to $\Phi_n(q),\Phi_{n^2}(q),\dots,\Phi_{n^r}(q)$. Therefore, taking $a=1$ in (4.1) we deduce that (1.10) is true modulo $\prod_{j=1}^r \Phi_{n^j}(q)^3$, one product $\prod_{j=1}^r \Phi_{n^j}(q)$ arising from $[n^r]$.

Moreover, the q-congruence (4.1) modulo $[n^r]$ is also valid for the $a=1$ case. The proof of (1.11) then follows from the fact that the least common multiple of the polynomials $\prod_{j=1}^{r}\Phi_{n^j}(q)^3$ and $[n^r]$ is $[n^r]\prod_{j=1}^{r}\Phi_{n^j}(q)^2$.

We now consider the r even case. When $a=1$, the polynomial

$$\prod_{j=0}^{(n^{r-1}-2)/3}(1-aq^{(3j+2)n})(a-q^{(3j+2)n})$$

has the factor $\prod_{j=1}^{r}\Phi_{n^j}(q)^{2\lfloor (n^{r-j}+2)/3\rfloor}$. On the other hand, the polynomial $(aq^3,q^3/a;q^3)_{(n^r+2)/3}$ in the denominator of the left-hand side of (4.2) only contains the factor $\prod_{j=1}^{r}\Phi_{n^j}(q)^{2\lfloor (n^{r-j}-1)/3\rfloor}$ related to $\Phi_n(q),\Phi_{n^2}(q),\dots,\Phi_{n^r}(q)$. Note that (4.2) also holds modulo $[n^r]$ for $a=1$. The proof of (1.12) then follows from (4.2) by putting $a=1$. $\square$

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School of Mathematics, Hangzhou Normal University, Hangzhou 311121, People's Republic of China
E-mail address: jwguo@math.ecnu.edu.cn