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FURTHER q-SUPERCONGRUENCES FROM ANDREWS' TERMINATING ${}_4\phi_3$ SUMMATION

Victor J. W. Guo

Abstract. Recently, Liu and Liu gave a q-supercongruence from Andrews' terminating q-analogue of Watson's formula. In this paper, employing the method of creative microscoping devised by the author and Zudilin in 2019, we deduce more q-supercongruences from Andrew's summation.

1. Introduction

In 2003, Rodriguez-Villegas [12] observed several interesting supercongruences between a truncated hypergeometric series associated to a Calabi–Yau manifold at a prime p and the number of its $\mathbb{F}_p$-points. In particular, he proposed four conjectures on supercongruences related to elliptic curves, one of which can be stated as follows: for any odd prime p,

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 16^{-k} \equiv (-1)^{(p-1)/2} \pmod{p^2}, \tag{1}$$

which was first confirmed by Mortenson [11].

In 2014, the author and Zeng [7] gave a q-analogue of (1) as follows:

$$\sum_{k=0}^{p-1} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} q^{2k} \equiv (-1)^{(p-1)/2} q^{(p^2-1)/4} \pmod{[p]^2}, \tag{2}$$

and a more general form of (2) was later given by the author, Pan, and Zhang [5]. Here and in what follows, assuming that $|q|<1$, the *q-shifted factorial* is defined as $(a;q)_0=1$ and $(a;q)_n=(1-a)(1-aq)\dots(1-aq^{n-1})$ for $n\geqslant 1$ or $n=\infty$. For simplicity, we will also use the shorthand notation $(a_1,\dots,a_m;q)_n=(a_1;q)_n\dots(a_m;q)_n$ for $n\geqslant 0$ or $n=\infty$. The n-th *cyclotomic polynomial* $\Phi_n(q)$ is defined by

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),$$

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where ζ denotes an n-th primitive root of unity. Let $[n]=[n]_q=(1-q^n)/(1-q)$ be the *q-integer*. In addition, for two rational functions $A(q)$ and $B(q)$, and a polynomial $P(q)\in\mathbb{Z}[q]$, the q-congruence $A(q)\equiv B(q)\pmod{P(q)}$ is meant that the numerator of $A(q)-B(q)$ is divisible by $P(q)$ in the polynomial ring $\mathbb{Z}[q]$.

In 2011, Z.-W. Sun [15, Conjecture 5.5] raised the following conjecture: for any odd prime p,

$$\sum_{k=0}^{p-1}\binom{2k}{k}^2 32^{-k}\equiv\begin{cases}2x-\dfrac{p}{2x}, & \text{if } p=x^2+y^2 \text{ with } x\equiv 1\pmod 4,\\ 0, & \text{if } p\equiv 3\pmod 4,\end{cases}\tag{3}$$

which was confirmed by Tauraso [16] and Z.-H. Sun [13, 14].

The author and Zeng [7, Corollary 2.7] gave a q-analogue of (3) for the second case as follows: for primes $p\equiv 3\pmod 4$,

$$\sum_{k=0}^{p-1}\frac{(q;q^2)_k^2 q^{2k}}{(q^2;q^2)_k(q^4;q^4)_k}\equiv 0\pmod{[p]^2}.\tag{4}$$

The author and Zudilin [8, eq. (54)] further generalized the above q-congruence to the modulus $\Phi_n(q)^2$ case where $n\equiv 3\pmod 4$ is a positive integer. For any rational number x and positive integer n satisfying the denominator of x is coprime with n, we let $\langle x\rangle_n$ stand for the *least non-negative residue* of x modulo n. Then a special case of [4, Theorem 1.3] implies the following result: for positive integers d, r, and n such that $\gcd(d,n)=1$ and $n\equiv\langle -r/d\rangle_n\equiv 1\pmod 2$,

$$\sum_{k=0}^{n-1}\frac{(q^r;q^d)_k(q^{d-r};q^d)_k q^{dk}}{(q^d;q^d)_k(q^{2d};q^{2d})_k}\equiv 0\pmod{\Phi_n(q)^2},\tag{5}$$

which is clearly a generalization of (4).

Recently, Liu and Liu [10] gave a q-analogue of (3) for the first case as follows: for any positive odd integer $n\equiv 1\pmod 4$,

$$\sum_{k=0}^{n-1}\frac{(q;q^2)_k^2 q^{2k}}{(q^2;q^2)_k(q^4;q^4)_k}\equiv(-1)^{(n-1)/4}\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}q^{(n-1)(n+3)/8}\pmod{\Phi_n(q)^2}.\tag{6}$$

In this paper, we shall establish the following generalization of (6), which is also a compliment to (5).

Theorem 1.1. *Let d,r,n be positive integers such that $\gcd(d,n)=1$, n is odd, and $\langle -r/d\rangle_n\equiv 0\pmod 2$. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{n-1}\frac{(q^r;q^d)_k(q^{d-r};q^d)_k}{(q^d;q^d)_k(q^{2d};q^{2d})_k}q^{dk}$$

$$\equiv \frac{r_2(-1)^{\langle -r/d\rangle_n/2}(q^d;q^{2d})_{\langle -r/d\rangle_n/2}}{(r_1+r_2)(q^{2d};q^{2d})_{\langle -r/d\rangle_n/2}} q^{d\langle -r/d\rangle_n(\langle -r/d\rangle_n+2)/4}$$

$$+\frac{r_1(-1)^{\langle (r-d)/d\rangle_n/2}(q^d;q^{2d})_{\langle (r-d)/d\rangle_n/2}}{(r_1+r_2)(q^{2d};q^{2d})_{\langle (r-d)/d\rangle_n/2}} q^{d\langle (r-d)/d\rangle_n(\langle (r-d)/d\rangle_n+2)/4}, \tag{7}$$

where $r_1=(r+d\langle -r/d\rangle_n)/n$ *and* $r_2=(d-r+d\langle (r-d)/d\rangle_n)/n$.

It is easy to see that when $(d,r)=(2,1)$, the q-congruence (7) reduces to (6). Moreover, for $(d,r)=(2,3),(3,1),(4,1),(6,1)$, we deduce the following corollaries from Theorem 1.1.

Corollary 1.2. *Let* $n\equiv 3 \pmod 4$ *be a positive integer. Then, modulo* $\Phi_n(q)^2$,

$$\sum_{k=0}^{n-1}\frac{(q^3;q^2)_k(q^{-1};q^2)_k q^{2k}}{(q^2;q^2)_k(q^4;q^4)_k} \equiv \frac{(-1)^{(n-3)/4}}{2}\Bigg\{\frac{(q^2;q^4)_{(n-3)/4}}{(q^4;q^4)_{(n-3)/4}}q^{(n-3)(n+1)/8}$$

$$-\frac{(q^2;q^4)_{(n+1)/4}}{(q^4;q^4)_{(n+1)/4}}q^{(n+1)(n+5)/8}\Bigg\}. \tag{8}$$

Suppose that n is a prime. Letting $q\to 1$ in (8) and multiplying both sides by -1, we obtain the following supercongruence: for any prime $p\equiv 3 \pmod 4$,

$$\sum_{k=0}^{p-1}\frac{2k+1}{2k-1}\binom{2k}{k}^2 32^{-k}$$

$$\equiv \frac{(-1)^{(p-3)/4}}{2}\left\{\binom{(p+1)/2}{(p+1)/4}2^{-(p+1)/2}-\binom{(p-3)/2}{(p-3)/4}2^{-(p-3)/2}\right\} \pmod{p^2}.$$

Corollary 1.3. *Let* $n\equiv 1 \pmod 6$ *be a positive integer. Then, modulo* $\Phi_n(q)^2$,

$$\sum_{k=0}^{n-1}\frac{(q;q^3)_k(q^2;q^3)_k}{(q^3;q^3)_k(q^6;q^6)_k}q^{3k} \equiv (-1)^{(n-1)/6}\frac{2(q^3;q^6)_{(n-1)/6}}{3(q^6;q^6)_{(n-1)/6}}q^{(n-1)(n+5)/12}$$

$$+\frac{(q^3;q^6)_{(n-1)/3}}{3(q^6;q^6)_{(n-1)/3}}q^{(n-1)(n+2)/3}. \tag{9}$$

Letting n be a prime and taking $q\to 1$ in (9), we have the following supercongruence: for any prime $p\equiv 1 \pmod 6$,

$$\sum_{k=0}^{p-1}\frac{(\frac13)_k(\frac23)_k}{k!^2 2^k} \equiv (-1)^{(p-1)/6}\frac{2(\frac12)_{(p-1)/6}}{3(1)_{(p-1)/6}}+\frac{(\frac12)_{(p-1)/3}}{3(1)_{(p-1)/3}} \pmod{p^2},$$

where $(x)_a=x(x+1)\dots(x+a-1)$ is the Pochhammer symbol.

Corollary 1.4. *Let* $n\equiv 1 \pmod 8$ *be a positive integer. Then, modulo* $\Phi_n(q)^2$,

$$\sum_{k=0}^{n-1}\frac{(q;q^4)_k(q^3;q^4)_k}{(q^4;q^4)_k(q^8;q^8)_k}q^{4k} \equiv (-1)^{(n-1)/8}\Bigg\{\frac{3(q^4;q^8)_{(n-1)/8}}{4(q^8;q^8)_{(n-1)/8}}q^{(n-1)(n+7)/16}$$

$$+\frac{(q^4;q^8)_{4(n-1)/8}}{4(q^6;q^6)_{3(n-1)/8}}q^{(3n-3)(3n+5)/16}\Bigg\}. \tag{10}$$

Likewise, when n is a prime and $q\to 1$ in (10), we get the following result: for any prime $p\equiv 1 \pmod 8$,

$$\sum_{k=0}^{p-1}\frac{(\frac14)_k(\frac34)_k}{k!^2 2^k}\equiv(-1)^{(p-1)/8}\left(\frac{3(\frac12)_{(p-1)/8}}{4(1)_{(p-1)/8}}+\frac{(\frac12)_{3(p-1)/8}}{4(1)_{3(p-1)/8}}\right) \pmod{p^2}.$$

Corollary 1.5. *Let $n\equiv 1 \pmod{12}$ be a positive integer. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{n-1}\frac{(q;q^6)_k(q^5;q^6)_k}{(q^6;q^6)_k(q^{12};q^{12})_k}q^{6k}\equiv(-1)^{(n-1)/12}\Bigg\{\frac{5(q^6;q^{12})_{(n-1)/12}q^{(n-1)(n+11)/24}}{6(q^{12};q^{12})_{(n-1)/12}}$$
$$+\frac{(q^6;q^{12})_{5(n-1)/12}q^{(5n-5)(5n+7)/24}}{6(q^{12};q^{12})_{5(n-1)/12}}\Bigg\}. \tag{11}$$

Similarly as before, letting n be a prime and taking the limits as $q\to 1$ in (11), we are led to the following supercongruence: for any prime $p\equiv 1 \pmod{12}$,

$$\sum_{k=0}^{p-1}\frac{(\frac16)_k(\frac56)_k}{k!^2 2^k}\equiv(-1)^{(p-1)/12}\left(\frac{5(\frac12)_{(p-1)/12}}{6(1)_{(p-1)/12}}+\frac{(\frac12)_{5(p-1)/12}}{6(1)_{5(p-1)/12}}\right) \pmod{p^2}.$$

We shall prove Theorem 1.1 by using the method of "creative microscoping" introduced by the author and Zudilin [8] in 2019. For more recent work related to this method, we refer the reader to [3, 6, 9, 17, 18].

2. Proof of Theorem 1.1

Following Gasper and Rahman [2], the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined by

$${}_{r+1}\phi_r\left[\begin{matrix}a_1,a_2,\dots,a_{r+1}\\ b_1,b_2,\dots,b_r\end{matrix};q,\,z\right]=\sum_{k=0}^{\infty}\frac{(a_1,a_2,\dots,a_{r+1};q)_k z^k}{(q,b_1,\dots,b_r;q)_k}.$$

We shall use Andrews' terminating q-analogue of Watson's formula (see [1] or [2, (II.17)]):

$${}_4\phi_3\left[\begin{matrix}q^{-n},\,a^2q^{n+1},\,b,\,-b\\ aq,\,-aq,\,b^2\end{matrix};q,q\right]=\begin{cases}0, & \text{if } n \text{ is odd},\\ \dfrac{b^n(q,a^2q^2/b^2;q^2)_{n/2}}{(a^2q^2,\,b^2q;q^2)_{n/2}}, & \text{if } n \text{ is even}.\end{cases} \tag{12}$$

The $b\to 0$ case of (12) reduces to

(13)
$$
{}_4\phi_3\left[\begin{matrix} q^{-n},\, a^2q^{n+1},\, 0,\, 0\\ aq,\, -aq,\, 0\end{matrix};q,q\right]=\begin{cases}0, & \text{if } n \text{ is odd},\\ \dfrac{(-1)^{n/2}(q;q^2)_{n/2}}{(a^2q^2;q^2)_{n/2}}a^nq^{n(n+2)/4}, & \text{if } n \text{ is even}.\end{cases}
$$

Theorem 2.1. *Let d,r,n be positive integers such that $\gcd(d,n)=1$, n is odd, and $\langle -r/d\rangle_n\equiv 0 \pmod 2$. Let a be an indeterminate. Then, modulo $(1-aq^{r_1n})(a-q^{r_2n})$,*

$$
\begin{aligned}
&\sum_{k=0}^{n-1}\frac{(aq^r;q^d)_k(q^{d-r}/a;q^d)_k}{(q^d;q^d)_k(q^{2d};q^{2d})_k}q^{dk}\\
&\quad\equiv\frac{(-1)^{\langle -r/d\rangle_n/2}(q^d;q^{2d})_{\langle -r/d\rangle_n/2}(a^{r_1+r_2}-a^{r_2}q^{r_1r_2n})}{(q^{2d};q^{2d})_{\langle -r/d\rangle_n/2}(a^{r_1+r_2}-1)}q^{d\langle -r/d\rangle_n(\langle -r/d\rangle_n+2)/4}\\
&\qquad+\frac{(-1)^{\langle (r-d)/d\rangle_n/2}(q^d;q^{2d})_{\langle (r-d)/d\rangle_n/2}(1-a^{r_2}q^{r_1r_2n})}{(q^{2d};q^{2d})_{\langle (r-d)/d\rangle_n/2}(1-a^{r_1+r_2})}\\
&\qquad\times q^{d\langle (r-d)/d\rangle_n(\langle (r-d)/d\rangle_n+2)/4},
\end{aligned}\tag{14}
$$

where $r_1=(r+d\langle -r/d\rangle_n)/n$ and $r_2=(d-r+d\langle (r-d)/d\rangle_n)/n$.

Proof. For $a=q^{-r_1n}$, the left-hand side of (14) can be written as

$$
\begin{aligned}
&\sum_{k=0}^{\langle -r/d\rangle_n}\frac{(q^{-d\langle -r/d\rangle_n};q^d)_k(q^{d+d\langle -r/d\rangle_n}/a;q^d)_k}{(q^d;q^d)_k(q^{2d};q^{2d})_k}q^{dk}\\
&\quad={}_4\phi_3\left[\begin{matrix} q^{-d\langle -r/d\rangle_n},\, q^{d+d\langle -r/d\rangle_n},\, 0,\, 0\\ q^d,\, -q^d,\, 0\end{matrix};q^d,\,q^d\right].
\end{aligned}\tag{15}
$$

Performing the parameter substitutions $n\mapsto\langle -r/d\rangle_n$, $q\mapsto q^d$, and $a=1$ in (13), since $\langle -r/d\rangle_n\equiv 0\pmod 2$, we see that the right-hand side of (15) is equal to

$$
\frac{(-1)^{\langle -r/d\rangle_n/2}(q^d;q^{2d})_{\langle -r/d\rangle_n/2}}{(q^{2d};q^{2d})_{\langle -r/d\rangle_n/2}}q^{d\langle -r/d\rangle_n(\langle -r/d\rangle_n+2)/4},
$$

which is exactly the right-hand side of (14) with $a=q^{-r_1n}$. Namely, the q-congruence (14) is true modulo $1-aq^{r_1n}$.

Similarly, for $a=q^{r_2n}$, the left-hand side of (14) is equal to

$$
\frac{(-1)^{\langle (r-d)/d\rangle_n/2}(q^d;q^{2d})_{\langle (r-d)/d\rangle_n/2}}{(q^{2d};q^{2d})_{\langle (r-d)/d\rangle_n/2}}q^{d\langle (r-d)/d\rangle_n(\langle (r-d)/d\rangle_n+2)/4},
$$

which is the value of the right-hand side of (14) at $a=q^{r_2n}$. This means that the q-congruence (14) is true modulo $a-q^{r_2n}$.

Since $1-aq^{r_1n}$ and $a-q^{r_2n}$ are relatively prime polynomials in q, we finish the proof of the theorem. □

Proof of Theorem 1.1. Observe that $1-a^{r_2}q^{r_1r_2n}$ is divisible by $1-aq^{r_1n}$, and $a^{r_1}-q^{r_1r_2n}$ is divisible by $a-q^{r_2n}$. In light of the identities

$$1=\frac{(a^{r_2}-1)+(1-a^{r_2}q^{r_1r_2n})}{a^{r_2}(1-q^{r_1r_2n})}=\frac{(1-a^{r_1})+(a^{r_1}-q^{r_1r_2n})}{1-q^{r_1r_2n}},$$

we can rewrite (14) as follows: modulo $(1-aq^{r_1n})(a-q^{r_2n})$,

$$\begin{aligned}
&\sum_{k=0}^{n-1}\frac{(aq^r;q^d)_k(q^{d-r}/a;q^d)_k}{(q^d;q^d)_k(q^{2d};q^{2d})_k}q^{dk}\\
&\equiv\frac{(-1)^{\langle -r/d\rangle_n/2}(q^d;q^{2d})_{\langle -r/d\rangle_n/2}(a^{r_1}-q^{r_1r_2n})(a^{r_2}-1)}{(q^{2d};q^{2d})_{\langle -r/d\rangle_n/2}(a^{r_1+r_2}-1)(1-q^{r_1r_2n})}q^{d\langle -r/d\rangle_n(\langle -r/d\rangle_n+2)/4}\\
&\quad+\frac{(-1)^{\langle (r-d)/d\rangle_n/2}(q^d;q^{2d})_{\langle (r-d)/d\rangle_n/2}(1-a^{r_2}q^{r_1r_2n})(1-a^{r_1})}{(q^{2d};q^{2d})_{\langle (r-d)/d\rangle_n/2}(1-a^{r_1+r_2})(1-q^{r_1r_2n})}\\
&\quad\times q^{d\langle (r-d)/d\rangle_n(\langle (r-d)/d\rangle_n+2)/4}.
\end{aligned}\tag{16}$$

In view of $q^n\equiv 1 \pmod{\Phi_n(q)}$, from the proof of Theorem 2.1 we conclude that, modulo $\Phi_n(q)$,

$$\begin{aligned}
&\frac{(-1)^{\langle -r/d\rangle_n/2}(q^d;q^{2d})_{\langle -r/d\rangle_n/2}}{(q^{2d};q^{2d})_{\langle -r/d\rangle_n/2}}q^{d\langle -r/d\rangle_n(\langle -r/d\rangle_n+2)/4}\\
&\quad\equiv\frac{(-1)^{\langle (r-d)/d\rangle_n/2}(q^d;q^{2d})_{\langle (r-d)/d\rangle_n/2}}{(q^{2d};q^{2d})_{\langle (r-d)/d\rangle_n/2}}q^{d\langle (r-d)/d\rangle_n(\langle (r-d)/d\rangle_n+2)/4}.
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{(-1)^{\langle -r/d\rangle_n/2}(q^d;q^{2d})_{\langle -r/d\rangle_n/2}(a^{r_1}-q^{r_1r_2n})(a^{r_2}-1)}{(q^{2d};q^{2d})_{\langle -r/d\rangle_n/2}(a^{r_1+r_2}-1)}q^{d\langle -r/d\rangle_n(\langle -r/d\rangle_n+2)/4}\\
&+\frac{(-1)^{\langle (r-d)/d\rangle_n/2}(q^d;q^{2d})_{\langle (r-d)/d\rangle_n/2}(1-a^{r_2}q^{r_1r_2n})(1-a^{r_1})}{(q^{2d};q^{2d})_{\langle (r-d)/d\rangle_n/2}(1-a^{r_1+r_2})}\\
&\quad\times q^{d\langle (r-d)/d\rangle_n(\langle (r-d)/d\rangle_n+2)/4}\\
&\quad\equiv 0 \pmod{\Phi_n(q)}.
\end{aligned}$$

This implies that the denominator of (the reduced form of) the right-hand side of (16) is coprime with $\Phi_n(q)$.

The limit of $(1-aq^{r_1n})(a-q^{r_2n})$ as $a\to 1$ contains the factor $\Phi_n(q)^2$. Therefore, taking $a\to 1$ in (16) and applying L'Hôpital's rule, we arrive at (7). □

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Victor J. W. Guo
School of Mathematics
Hangzhou Normal University
Hangzhou 311121, People's Republic of China
E-mail address: jwguo@math.ecnu.edu.cn