A q-analogue of the (L.2) supercongruence of Van Hamme

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Abstract. The (L.2) supercongruence of Van Hamme was proved by Swisher recently. In this paper we provide a conjectural q-analogue of the (L.2) supercongruence of Van Hamme and prove a weaker form of it by using the q-WZ method. In the same way, we prove a complete q-analogue of the following congruence

$$\sum_{k=0}^{n}(6k+1)\binom{2k}{k}^3(-512)^{n-k}\equiv 0 \pmod{4(2n+1)\binom{2n}{n}},$$

which was conjectured by Z.-W. Sun and confirmed by B. He. We also provide a conjectural q-analogue of another congruence proved by Swisher.

Keywords: q-binomial coefficients; q-WZ method; cyclotomic polynomials; q-Gamma function; Jackson's ${}_6\phi_5$ summation.

2010 Mathematics Subject Classifications: 11B65 (Primary) 05A10, 05A30 (Secondary)

1. Introduction

In 1914, Ramanujan [18] discovered several infinite series for $1/\pi$ that enable us to compute π very accurately. The most impressive one might be

$$\sum_{k=0}^{\infty}\frac{(1/4)_k(1/2)_k(3/4)_k}{k!^3}(1103+26390k)(1/99)^{4k+2}=\frac{1}{2\sqrt{2}\pi}, \tag{1.1}$$

where $(a)_k=a(a+1)\cdots(a+k-1)$.

In 1997, Van Hamme [24] observed that 13 Ramanujan's or Ramanujan-like formulas for $1/\pi$, such as

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3}=\frac{2}{\pi}, \tag{1.2}$$

$$\sum_{k=0}^{\infty}(-1)^k(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3}=\frac{3\sqrt{3}}{2\pi}, \tag{1.3}$$

$$\sum_{k=0}^{\infty}(-1)^k(8k+1)\frac{(\frac{1}{4})_k^3}{k!^3}=\frac{2\sqrt{2}}{\pi}, \tag{1.4}$$

$$\sum_{k=0}^{\infty}(-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k}=\frac{2\sqrt{2}}{\pi}, \tag{1.5}$$

have very nice p-adic analogues:

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3}\equiv p\left(\frac{-1}{p}\right) \pmod{p^3}, \tag{1.6}$$

$$\sum_{k=0}^{\frac{p-1}{3}}(-1)^k(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3}\equiv p \pmod{p^3}, \quad \text{if} \quad p\equiv 1 \pmod{3}, \tag{1.7}$$

$$\sum_{k=0}^{\frac{p-1}{4}}(-1)^k(8k+1)\frac{(\frac{1}{4})_k^3}{k!^3}\equiv p\left(\frac{-2}{p}\right) \pmod{p^3}, \quad \text{if} \quad p\equiv 1 \pmod{4}, \tag{1.8}$$

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k}\equiv p\left(\frac{-2}{p}\right) \pmod{p^3}, \tag{1.9}$$

where p is an odd prime and $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol modulo p. Supercongruences of this type are called Ramanujan-type supercongruences. All of the 13 supercongruences have now been confirmed by different authors (see [16,22]). The supercongruence (1.6) was first proved by Mortenson [15] using a ${}_6F_5$ transformation and a technical evaluation of a quotient of Gamma functions, and later reproved by Zudilin [27] via the Wilf–Zeilberger method [25, 26] (the WZ pair was borrowed from [3]) and by Long [14] using hypergeometric identities. Swisher [22] used Long's method to prove 4 supercongruences of Van Hamme, including (1.7)–(1.9). Chen, Xie, and He [2] reproved (1.9) modulo p^2 via the WZ method again. He [11] has independently used Long's method to give a generalization of (1.7) and (1.8). Moreover, it is worth mentioning that the last supercongruence of Van Hamme was proved by Osburn and Zudilin [16] in 2016.

Motivated by Zudilin's work [27], the author [6, 7] uses the q-WZ method to obtain

q-analogues of (1.6)–(1.8): for any odd prime p,

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k q^{k^2}[4k+1]\frac{(q;q^2)_k^3}{(q^2;q^2)_k^3} \equiv [p]q^{\frac{(p-1)^2}{4}}(-1)^{\frac{p-1}{2}} \pmod{[p]^3}, \tag{1.10}$$

$$\sum_{k=0}^{\frac{p-1}{3}}(-1)^k q^{\frac{3k^2+k}{2}}[6k+1]\frac{(q;q^3)_k^3}{(q^3;q^3)_k^3} \equiv [p]q^{\frac{(p-1)(p-2)}{6}} \pmod{[p]^3}, \quad \text{if } p\equiv 1 \pmod 3, \tag{1.11}$$

$$\sum_{k=0}^{\frac{p-1}{4}}(-1)^k q^{2k^2+k}[8k+1]\frac{(q;q^4)_k^3}{(q^4;q^4)_k^3} \equiv [p]q^{\frac{(p-1)(p-3)}{8}}\left(\frac{-2}{p}\right) \pmod{[p]^3} \quad \text{if } p\equiv 1 \pmod 4, \tag{1.12}$$

where $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n\geqslant 1$ and $(a;q)_0=1$, and $[n]=[n]_q=1+q+\cdots+q^{n-1}$. Note that, for a polynomial $h(q)$ and two rational functions $f(q)$ and $g(q)$, we say that $f(q)$ is congruent to $g(q)$ modulo $h(q)$, denoted by $f(q)\equiv g(q) \pmod{h(q)}$, if the numerator of the reduced form of $f(q)-g(q)$ is divisible by $h(q)$. We point out that there are more general forms of (1.10)–(1.12) in [6, 7], and some other interesting q-congruences can be found in [13, 17, 19, 23].

Recall that the n-th *cyclotomic polynomial* $\Phi_n(q)$ is defined as

$$\Phi_n(q) := \prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}} (q-e^{2\pi i\frac{k}{n}}),$$

where i is the imaginary unit. It is clear that $\Phi_p(q)=[p]$ for any prime p. This paper was motivated by the following conjectural q-analogue of (1.9) (i.e., the (L.2) supercongruence of Van Hamme [24]).

Conjecture 1.1. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{\frac{n-1}{2}}(-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3} \equiv [n](-q)^{-\frac{(n-1)(n+5)}{8}} \pmod{[n]\Phi_n(q)^2}, \tag{1.13}$$

$$\sum_{k=0}^{n-1}(-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3} \equiv [n](-q)^{-\frac{(n-1)(n+5)}{8}} \pmod{[n]\Phi_n(q)^2}. \tag{1.14}$$

Note that, when $n=p$ is an odd prime, the congruences (1.13) and (1.14) are equivalent to each other, since $\frac{(q;q^2)_k}{(q^4;q^4)_k}\equiv 0 \pmod{[p]}$ for $\frac{p+1}{2}\leqslant k\leqslant p-1$. But they are not equivalent in general.

The first aim of this paper is to prove the following weaker form of Conjecture 1.1.

Theorem 1.2. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{\frac{n-1}{2}}(-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3} \equiv 0 \pmod{[n]}. \tag{1.15}$$

Moreover, if n is an odd prime power, then

$$\sum_{k=0}^{n-1}(-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3} \equiv [n](-q)^{-\frac{(n-1)(n+5)}{8}} \pmod{[n]\Phi_n(q)}. \tag{1.16}$$

Letting $q \to 1$ in (1.16), we obtain

Corollary 1.3. *Let p be an odd prime and r a positive integer. Then*

$$\sum_{k=0}^{p^r-1}(-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k} \equiv p\left(\frac{-2}{p}\right)^r \pmod{p^{r+1}}.$$

On the other hand, Z.-W. Sun [20, Conjecture 5.1(i)] made the following conjecture

$$\sum_{k=0}^{n}(6k+1)\binom{2k}{k}^3(-512)^{n-k} \equiv 0 \pmod{4(2n+1)\binom{2n}{n}}, \tag{1.17}$$

which was later proved by He [11] using the WZ method. The second aim of this paper is to prove the following q-analogue of (1.17).

Theorem 1.4. *Let n be a positive integer. Then*

$$\sum_{k=0}^{n}(-1)^k[6k+1]{\begin{bmatrix}2k\\k\end{bmatrix}}^3\frac{(-q;q)_n^6(-q^2;q^2)_n^3}{(-q;q)_k^6(-q^2;q^2)_k^3} \equiv 0 \pmod{(1+q^n)^2[2n+1]\begin{bmatrix}2n\\n\end{bmatrix}}, \tag{1.18}$$

where the q-binomial coefficients $\begin{bmatrix}m\\k\end{bmatrix}$ are defined by

$$\begin{bmatrix}m\\k\end{bmatrix} = \begin{bmatrix}m\\k\end{bmatrix}_q = \begin{cases}\dfrac{(q;q)_m}{(q;q)_k(q;q)_{m-k}} & \text{if } 0 \leqslant k \leqslant m,\\ 0 & \text{otherwise.}\end{cases}$$

We shall prove Theorem 1.2 in Section 2 using some properties of q-factorials and a q-WZ pair. In Section 3, we shall prove Theorem 1.4 using the same q-WZ pair and some properties of q-binomial coefficients. In section 4, we provide several related conjectures, including one on a q-analogue of Ramanujan's series (1.5) and another one on a q-analogue of the congruence $2^{\frac{p-1}{2}} \equiv (-1)^{\frac{p^2-1}{8}} \pmod{p}$ for any odd prime p.

2. Proof of Theorem 1.2

We first require three preliminary results.

Lemma 2.1. *If n is an odd prime power, then*

$$(-q^2;q^2)_{(n-1)/2} \equiv (-1)^{\frac{n^2-1}{8}} q^{\frac{n^2-1}{8}} \pmod{\Phi_n(q)}. \tag{2.1}$$

Proof. By the q-binomial theorem (see [1, p. 36, (3.3.6)]), for any odd positive integer n, we have

$$(-q^2;q^2)_{n-1}=\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\ k\end{bmatrix}_{q^2}q^{k^2+k}\equiv\sum_{k=0}^{n-1}(-1)^k=1 \pmod{\Phi_n(q)}, \tag{2.2}$$

since

$$\begin{bmatrix}n-1\\ k\end{bmatrix}_{q^2}=\prod_{j=1}^{k}\frac{1-q^{2n-2j}}{1-q^{2j}}\equiv\prod_{j=1}^{k}\frac{1-q^{-2j}}{1-q^{2j}}=(-1)^kq^{-k^2-k} \pmod{\Phi_n(q)}.$$

Note that

$$\begin{aligned}(-q^2;q^2)_{n-1}&=(-q^2;q^2)_{(n-1)/2}\prod_{k=1}^{\frac{n-1}{2}}(1+q^{2n-2k})\equiv(-q^2;q^2)_{(n-1)/2}\prod_{k=1}^{\frac{n-1}{2}}(1+q^{-2k})\\ &=(-q^2;q^2)_{(n-1)/2}^2q^{\frac{1-n^2}{4}} \pmod{\Phi_n(q)}.\end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3), we obtain $(-q^2;q^2)_{(n-1)/2}^2\equiv q^{\frac{n^2-1}{4}} \pmod{\Phi_n(q)}$. It follows that

$$(-q^2;q^2)_{(n-1)/2}\equiv\pm q^{\frac{n^2-1}{8}} \pmod{\Phi_n(q)}. \tag{2.4}$$

We now suppose that $n=p^r$ is an odd prime power. Then $2^{\frac{p^r-1}{2}}\equiv(-1)^{\frac{(p^2-1)r}{8}}=(-1)^{\frac{p^{2r}-1}{8}}$ $\pmod p$ since $2^{\frac{p-1}{2}}\equiv\left(\frac{2}{p}\right)=(-1)^{\frac{p^2-1}{8}} \pmod p$. Hence, letting $q=1$ in (2.4) and noticing that $\Phi_{p^r}(1)=p$, we are led to (2.1). ☐

Lemma 2.2. *Let n and k be positive integers with n odd. Then*

$$\frac{(q;q^2)_{(n-1)/2+k}(q;q^2)_{(n+1)/2-k}^2}{(q^2;q^2)_{(n-1)/2}^2(q^2;q^2)_{(n+1)/2-k}}\equiv 0 \pmod{1-q^n}, \tag{2.5}$$

and for $1\leqslant k\leqslant n$ with $k\neq\frac{n+1}{2}$ we have

$$\frac{(q;q^2)_{n+k-1}(q;q^2)_{n-k}^2}{(q^2;q^2)_{n-1}^2(q^2;q^2)_{n-k}}\equiv 0 \pmod{(1-q^n)\Phi_n(q)}. \tag{2.6}$$

Proof. It is well known that

$$q^m-1=\prod_{d\mid m}\Phi_d(q),$$

and so

$$(q^2;q^2)_m=(-1)^m\left(\prod_{d=1}^{m}\Phi_{2d}(q)^{\lfloor\frac{m}{d}\rfloor}\right)\left(\prod_{d=1}^{m}\Phi_{2d-1}(q)^{\lfloor\frac{m}{2d-1}\rfloor}\right), \tag{2.7}$$

$$(q;q^2)_m=\frac{(q;q)_{2m}}{(q^2;q^2)_m}=(-1)^m\prod_{d=1}^{m}\Phi_{2d-1}(q)^{\lfloor\frac{2m}{2d-1}\rfloor-\lfloor\frac{m}{2d-1}\rfloor}, \tag{2.8}$$

where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. Therefore,

$$\frac{(q;q^2)_{m+k}(q;q^2)_{m-k+1}^2}{(q^2;q^2)_m^2(q^2;q^2)_{m-k+1}}=-\prod_{d=1}^{m+k}\frac{\Phi_{2d-1}(q)^{\lfloor\frac{2m+2k}{2d-1}\rfloor+2\lfloor\frac{2m-2k+2}{2d-1}\rfloor-\lfloor\frac{m+k}{2d-1}\rfloor-3\lfloor\frac{m-k+1}{2d-1}\rfloor-2\lfloor\frac{m}{2d-1}\rfloor}}{\Phi_{2d}(q)^{2\lfloor\frac{m}{d}\rfloor+\lfloor\frac{m-k+1}{d}\rfloor}}. \tag{2.9}$$

Applying the following properties

$$\lfloor 2x\rfloor+\lfloor 2y\rfloor\geqslant\lfloor x\rfloor+\lfloor y\rfloor+\lfloor x+y\rfloor,\quad \lfloor 2y\rfloor\geqslant 2\lfloor y\rfloor, \tag{2.10}$$

we see that the exponent of $\Phi_{2d-1}(q)$ on the right-hand side of (2.9) is greater than or equal to

$$\left\lfloor\frac{2m+1}{2d-1}\right\rfloor-2\left\lfloor\frac{m}{2d-1}\right\rfloor,$$

which is clearly non-negative.

If $m=\frac{n-1}{2}$ or $m=n-1$, then for any d with $2d-1|n$, we have $\lfloor\frac{2m+1}{2d-1}\rfloor-2\lfloor\frac{m}{2d-1}\rfloor=1$, which means that the congruences (2.5) and (2.6) hold modulo $1-q^n$.

Furthermore, if $m=n-1$ and $1\leqslant k\leqslant n$, then the exponent of $\Phi_n(q)$ on the right-hand side of (2.9) is equal to

$$\left\lfloor\frac{2n+2k-2}{n}\right\rfloor+2\left\lfloor\frac{2n-2k}{n}\right\rfloor-\left\lfloor\frac{n+k-1}{n}\right\rfloor=\begin{cases}3, & \text{if } 1\leqslant k\leqslant\frac{n-1}{2},\\ 2 & \text{if } \frac{n+3}{2}\leqslant k\leqslant n.\end{cases}$$

This proves (2.6). □

Lemma 2.3. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{\frac{n-1}{2}}(-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3}=\sum_{k=1}^{\frac{n+1}{2}}\frac{(-1)^{\frac{n+1}{2}+k}(q;q^2)_{(n-1)/2+k}(q;q^2)_{(n+1)/2-k}^2}{(1-q)(q^4;q^4)_{(n-1)/2}^2(q^4;q^4)_{(n+1)/2-k}}, \tag{2.11}$$

$$\sum_{k=0}^{n-1}(-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3}=\sum_{k=1}^{n}\frac{(-1)^{n+k}(q;q^2)_{n+k-1}(q;q^2)_{n-k}^2}{(1-q)(q^4;q^4)_{n-1}^2(q^4;q^4)_{n-k}}. \tag{2.12}$$

Proof. We define two rational functions in q:

$$F(n,k) = (-1)^{n+k}\frac{[6n-2k+1](q;q^2)_{n+k}(q;q^2)_{n-k}^2}{(q^4;q^4)_n^2(q^4;q^4)_{n-k}},$$

$$G(n,k) = \frac{(-1)^{n+k}(q;q^2)_{n+k-1}(q;q^2)_{n-k}^2}{(1-q)(q^4;q^4)_{n-1}^2(q^4;q^4)_{n-k}},$$

where we use the convention that $1/(q^4;q^4)_m = 0$ for $m = -1,-2,\dots$. The functions $F(n,k)$ and $G(n,k)$ satisfy the relation

$$F(n,k-1) - F(n,k) = G(n+1,k) - G(n,k). \tag{2.13}$$

Namely, they form a q-WZ pair. Indeed, we have the following expressions:

$$\frac{F(n,k-1)}{G(n,k)} = -\frac{(1-q^{6n-2k+3})(1-q^{2n-2k+1})^2}{(1-q^{4n-4k+4})(1-q^{4n})^2},$$

$$\frac{F(n,k)}{G(n,k)} = \frac{(1-q^{6n-2k+1})(1-q^{2n+2k-1})}{(1-q^{4n})^2},$$

$$\frac{G(n+1,k)}{G(n,k)} = -\frac{(1-q^{2n+2k-1})(1-q^{2n-2k+1})^2}{(1-q^{4n})^2(1-q^{4n-4k+4})}.$$

Then it is routine to verify the identity

$$\begin{aligned}
&-\frac{(1-q^{6n-2k+3})(1-q^{2n-2k+1})^2}{(1-q^{4n-4k+4})(1-q^{4n})^2} - \frac{(1-q^{6n-2k+1})(1-q^{2n+2k-1})}{(1-q^{4n})^2}\\
&= -\frac{(1-q^{2n+2k-1})(1-q^{2n-2k+1})^2}{(1-q^{4n})^2(1-q^{4n-4k+4})} - 1,
\end{aligned}$$

which is equivalent to (2.13) (dividing both sides by $G(n,k)$).

Let m be a positive odd integer. Summing (2.13) over $n = 0,1,\dots,\frac{m-1}{2}$, we obtain (via telescoping)

$$\sum_{n=0}^{\frac{m-1}{2}} F(n,k-1) - \sum_{n=0}^{\frac{m-1}{2}} F(n,k) = G\left(\frac{m+1}{2},k\right), \tag{2.14}$$

where we have used $G(0,k)=0$. Summing (2.14) over $k = 1,2,\dots,\frac{m+1}{2}$, we get

$$\sum_{n=0}^{\frac{m-1}{2}} F(n,0) = \sum_{n=0}^{\frac{m-1}{2}} F\left(n,\frac{m+1}{2}\right) + \sum_{k=1}^{\frac{m+1}{2}} G\left(\frac{m+1}{2},k\right) = \sum_{k=1}^{\frac{m+1}{2}} G\left(\frac{m+1}{2},k\right),$$

where we have used $F(n,k)=0$ for $n<k$ because $(q^4;q^4)_{n-k}$ is in the denominator. This proves that (2.11) holds for $n=m$.

Similarly, we have

$$\sum_{n=0}^{m-1} F(n,0) = \sum_{k=1}^{m} G(m,k).$$

That is, the identity (2.12) is true for $n=m$. □

Proof of Theorem 1.2. It is easy to see that

$$\begin{aligned}
&\frac{(q;q^2)_{(n-1)/2+k}(q;q^2)^2_{(n+1)/2-k}}{(1-q)(q^4;q^4)^2_{(n-1)/2}(q^4;q^4)_{(n+1)/2-k}} \\
&\quad = \frac{(q;q^2)_{(n-1)/2+k}(q;q^2)^2_{(n+1)/2-k}}{(1-q)(q^2;q^2)^2_{(n-1)/2}(q^2;q^2)_{(n+1)/2-k}} \frac{1}{(-q^2;q^2)^2_{(n-1)/2}(-q^2;q^2)_{(n+1)/2-k}}
\end{aligned}$$

By Lemma 2.2, we have

$$\frac{(q;q^2)_{(n-1)/2+k}(q;q^2)^2_{(n+1)/2-k}}{(1-q)(q^2;q^2)^2_{(n-1)/2}(q^2;q^2)_{(n+1)/2-k}} \equiv 0 \pmod{[n]}. \tag{2.15}$$

Moreover, we have $\gcd((-q^2;q^2)^2_{(n-1)/2}(-q^2;q^2)_{(n+1)/2-k},[n])=1$, since $(1-q^n,1+q^m)=1$ holds for all positive integers m and n with n odd. The proof of (1.15) then follows from (2.11) and (2.15).

Similarly, by (2.6), for $1 \leqslant k \leqslant n$ with $k \neq \frac{n+1}{2}$ we have

$$\frac{(q;q^2)_{n+k-1}(q;q^2)^2_{n-k}}{(1-q)(q^4;q^4)^2_{n-1}(q^4;q^4)_{n-k}} \equiv 0 \pmod{[n]\Phi_n(q)}.$$

Therefore, modulo $[n]\Phi_n(q)$, the identity (2.12) reduces to

$$\begin{aligned}
&\sum_{k=0}^{n-1}(-1)^k[6k+1]\frac{(q;q^2)^3_k}{(q^4;q^4)^3_k} \\
&\equiv \frac{(-1)^{n+\frac{n+1}{2}}(q;q^2)_{(3n-1)/2}(q;q^2)^2_{(n-1)/2}}{(1-q)(q^4;q^4)^2_{n-1}(q^4;q^4)_{(n-1)/2}} \\
&= \frac{(-1)^{\frac{n-1}{2}}(q;q^2)_{(n-1)/2}[n](q^{n+2};q^2)_{n-1}(q;q^2)^2_{(n-1)/2}}{(q^4;q^4)^2_{n-1}(q^4;q^4)_{(n-1)/2}} \\
&\equiv \frac{(-1)^{\frac{n-1}{2}}(q;q^2)_{(n-1)/2}[n](q^2;q^2)_{n-1}(q;q^2)^2_{(n-1)/2}}{(q^4;q^4)^2_{n-1}(q^4;q^4)_{(n-1)/2}} \\
&= \frac{(-1)^{\frac{n-1}{2}}[n]}{(-q^2;q^2)^2_{n-1}(-q^2;q^2)_{(n-1)/2}(-q;q)^3_{n-1}} {\begin{bmatrix} n-1 \\ \frac{n-1}{2} \end{bmatrix}}^2_{q^2} \pmod{[n]\Phi_n(q)},
\end{aligned} \tag{2.16}$$

where we have used the fact $\frac{A_1(q)[n]}{B_1(q)} \equiv \frac{A_2(q)[n]}{B_2(q)} \pmod{[n]\Phi_n(q)}$ if $\frac{A_1(q)}{B_1(q)} \equiv \frac{A_2(q)}{B_2(q)} \pmod{\Phi_n(q)}$ and the denominators of the reduced forms of $\frac{A_1(q)}{B_1(q)}$ and $\frac{A_2(q)}{B_2(q)}$ are both relatively prime to $[n]$. By the proof of (2.1), we have $(-q;q)_{n-1} \equiv (-q^2;q^2)_{n-1} \equiv 1 \pmod{\Phi_n(q)}$ and $\begin{bmatrix} n-1 \\ \frac{n-1}{2} \end{bmatrix}_{q^2} \equiv (-1)^{\frac{n-1}{2}} q^{\frac{1-n^2}{4}} \pmod{\Phi_n(q)}$. Thus, from (2.16) we obtain

$$\sum_{k=0}^{n-1} (-1)^k [6k+1] \frac{(q;q^2)_k^3}{(q^4;q^4)_k^3} \equiv \frac{(-1)^{\frac{n-1}{2}} [n] q^{\frac{1-n^2}{2}}}{(-q^2;q^2)_{(n-1)/2}} \pmod{[n]\Phi_n(q)}, \tag{2.17}$$

which means that the congruence (1.14) modulo $[n]$ is true. If n is an odd prime power, then by Lemma 2.1 and noticing that $q^{\frac{3(1-n^2)}{8}} \equiv q^{-\frac{(n-1)(n+5)}{8}} \pmod{\Phi_n(q)}$, the congruence (2.17) is equivalent to (1.16). □

3. Proof of Theorem 1.4

We need two divisibility results on q-binomial coefficients.

Lemma 3.1. [7, Lemma 4.1] *Let n be a positive integer. Then*

$$(-q;q)_n^3 \begin{bmatrix} 4n+1 \\ 2n \end{bmatrix} \equiv 0 \pmod{(1+q^n)^2(-q;q)_{2n}}.$$

Lemma 3.2. *Let n and k be positive integers with $k \leqslant n+1$. Then*

$$\frac{(q;q^2)_{n+k}(q;q^2)_{n-k+1}^2(-q;q)_n^6}{(1-q)(q^2;q^2)_n^2(q^2;q^2)_{n-k+1}} \equiv 0 \pmod{(1+q^n)^2[2n+1]\begin{bmatrix} 2n \\ n \end{bmatrix}}. \tag{3.1}$$

Proof. Since

$$(1+q^n)^2[2n+1]\begin{bmatrix} 2n \\ n \end{bmatrix} = \frac{(1+q^n)^2(q;q)_{2n+1}}{(1-q)(q;q)_n^2},$$

to prove (3.1), it is equivalent to prove that

$$\frac{(q;q^2)_{n+k}(q;q^2)_{n-k+1}^2(-q;q)_n^2(-q;q)_{n-1}^2}{(q^2;q^2)_{n-k+1}(q;q)_{2n+1}} \tag{3.2}$$

is a polynomial in q with integer coefficients. Noticing (2.7), (2.8), and

$$(-q;q)_n = \frac{(q^2;q^2)_n}{(q;q)_n} = \prod_{d=1}^{n} \Phi_{2d}(q)^{\lfloor \frac{n}{d} \rfloor - \lfloor \frac{n}{2d} \rfloor},$$

the expression (3.2) can be factorized into

$$\left(\prod_{d=1}^{n} \Phi_{2d}(q)^{2\lfloor \frac{n}{d} \rfloor + 2\lfloor \frac{n-1}{d} \rfloor - 2\lfloor \frac{n}{2d} \rfloor - 2\lfloor \frac{n-1}{2d} \rfloor - \lfloor \frac{n-k+1}{d} \rfloor - \lfloor \frac{2n+1}{2d} \rfloor} \right)$$
$$\times \left(\prod_{d=2}^{n+k} \Phi_{2d-1}(q)^{\lfloor \frac{2n+2k}{2d-1} \rfloor + 2\lfloor \frac{2n-2k+2}{2d-1} \rfloor - \lfloor \frac{n+k}{2d-1} \rfloor - 3\lfloor \frac{n-k+1}{2d-1} \rfloor - \lfloor \frac{2n+1}{2d-1} \rfloor} \right).$$

It is clear that $\lfloor \frac{n}{d} \rfloor - \lfloor \frac{n-k+1}{d} \rfloor \geqslant 0$ (since $k \geqslant 1$), $\lfloor \frac{2n+1}{2d} \rfloor = \lfloor \frac{n}{d} \rfloor$, and

$$\left\lfloor \frac{n-1}{d} \right\rfloor - \left\lfloor \frac{n}{2d} \right\rfloor - \left\lfloor \frac{n-1}{2d} \right\rfloor \geqslant 0.$$

So, the exponent of $\Phi_{2d}(q)$ is non-negative. Moreover, by (2.10), we have

$$\left\lfloor \frac{2n+2k}{2d-1} \right\rfloor + \left\lfloor \frac{2n-2k+2}{2d-1} \right\rfloor \geqslant \left\lfloor \frac{n+k}{2d-1} \right\rfloor + \left\lfloor \frac{n-k+1}{2d-1} \right\rfloor + \left\lfloor \frac{2n+1}{2d-1} \right\rfloor,$$

$$\left\lfloor \frac{2n-2k+2}{2d-1} \right\rfloor \geqslant 2 \left\lfloor \frac{n-k+1}{2d-1} \right\rfloor,$$

This implies that the exponent of $\Phi_{2d-1}(q)$ is also non-negative and therefore (3.2) is a product of cyclotomic polynomials. □

Similarly as before, summing (2.13) over n from 0 to N, we obtain

$$\sum_{n=0}^{N} F(n,k-1) - \sum_{n=0}^{N} F(n,k) = G\left(N+1,k\right). \tag{3.3}$$

Furthermore, summing (3.3) over k from 1 to N, we get

$$\sum_{n=0}^{N} F(n,0) - \sum_{n=0}^{N} F(n,N) = \sum_{k=1}^{N} G\left(N+1,k\right). \tag{3.4}$$

Since

$$\begin{aligned}\sum_{n=0}^{N} F(n,N) = F(N,N) &= [4N+1]\frac{(q;q^2)_{2N}}{(q^4;q^4)_N^2}\\ &= \frac{[4N+1]}{(-q^2;q^2)_N^2(-q;q)_{2N}(-q;q)_N^2}\begin{bmatrix}4N\\2N\end{bmatrix}\begin{bmatrix}2N\\N\end{bmatrix},\end{aligned}$$

by Lemma 3.1 we have

$$\begin{aligned}(-q;q)_N^6(-q^2;q^2)_N^3\sum_{n=0}^{N} F(n,N) &= (-q;q)_N^4(-q^2;q^2)_N\frac{[2N+1]}{(-q;q)_{2N}}\begin{bmatrix}4N+1\\2N\end{bmatrix}\begin{bmatrix}2N\\N\end{bmatrix}\\ &\equiv 0 \pmod{(1+q^N)^2[2N+1]\begin{bmatrix}2N\\N\end{bmatrix}}.\end{aligned}$$

Additionally, by Lemma 3.2, for $1 \leqslant k \leqslant N$, we have

$$\begin{aligned}(-q;q)_N^6(-q^2;q^2)_N^3 G(N+1,k) &= \frac{(q;q^2)_{N+k}(q;q^2)_{N-k+1}^2(-q;q)_N^6}{(1-q)(q^2;q^2)_N^2(q^2;q^2)_{N-k+1}}\frac{(-q^2;q^2)_N}{(-q^2;q^2)_{N-k+1}}\\ &\equiv 0 \pmod{(1+q^N)^2[2N+1]\begin{bmatrix}2N\\N\end{bmatrix}}.\end{aligned}$$

Therefore, from (3.4) we deduce that

$$(-q;q)_N^6(-q^2;q^2)_N^3\sum_{n=0}^{N} F(n,0)\equiv 0 \pmod{(1+q^N)^2[2N+1]{2N\brack N}}.$$

Namely, the congruence (1.18) holds for $n=N$ by noticing that

$$\frac{(q;q^2)_k}{(q^4;q^4)_k}={2k\brack k}\frac{1}{(-q;q)_k^2(-q^2;q^2)_k}.$$

□

4. Concluding remarks and open problems

It seems that the condition "n is an odd prime power" in Lemma 2.1 is not necessary. Namely, we have the following conjecture.

Conjecture 4.1. *The congruence* (2.1) *holds for all positive odd integers n.*

Pan [17, (1.4)] has given a q-analogue of Fermat's little theorem: $(q^m;q^m)_{p-1}/(q;q)_{p-1}\equiv 1 \pmod{[p]}$ for any prime p and positive integer m with $\gcd(p,m)=1$. More general, for all positive integers m and n with $\gcd(m,n)=1$, we have

$$\frac{(q^m;q^m)_{n-1}}{(q;q)_{n-1}}=\prod_{j=1}^{n-1}\frac{1-q^{mj}}{1-q^j}\equiv 1 \pmod{\Phi_n(q)}.$$

We now suppose that n is a positive odd integer. Similarly to the proof of (2.1), we can show that

$$(q^m;q^m)_{(n-1)/2}^2/(q;q)_{(n-1)/2}^2\equiv q^{\frac{(m-1)(n^2-1)}{8}} \pmod{\Phi_n(q)}. \tag{4.1}$$

We have a generalization of Conjecture 4.1 as follows.

Conjecture 4.2. *Let $m,n>1$ be positive integers with n odd and* $\gcd(m,n)=1$. *Then*

$$\frac{(q^m;q^m)_{(n-1)/2}}{(q;q)_{(n-1)/2}}\equiv\begin{cases}\left(\frac{m}{n}\right)q^{\frac{(m-1)(n^2-1)}{16}} \pmod{\Phi_n(q)}, & \text{if } 16\mid(m-1)(n^2-1),\\ \left(\frac{m}{n}\right)q^{\frac{(m-1)(n^2-1)+8n}{16}} \pmod{\Phi_n(q)}, & \text{if } 16\nmid(m-1)(n^2-1),\end{cases} \tag{4.2}$$

where $\left(\frac{m}{n}\right)$ is the Jacobi symbol.

Similarly as Lemma 2.1, we can prove the following result.

Theorem 4.3. *Conjecture* 4.2 *is true for all odd prime powers n.*

Proof. It is clear that (4.1) is equivalent to

$$(q^m;q^m)^2_{(n-1)/2}/(q;q)^2_{(n-1)/2} \equiv q^{\frac{(m-1)(n^2-1)}{8}+n} \pmod{\Phi_n(q)}. \tag{4.3}$$

Moreover, if $(m-1)(n^2-1)/8$ is odd, then $(m-1)(n^2-1)/8+n$ is even. By (4.1) and (4.3), we know that

$$\frac{(q^m;q^m)_{(n-1)/2}}{(q;q)_{(n-1)/2}} \equiv \begin{cases} \pm q^{\frac{(m-1)(n^2-1)}{16}} \pmod{\Phi_n(q)}, & \text{if } 16 \mid (m-1)(n^2-1), \\ \pm q^{\frac{(m-1)(n^2-1)+8n}{16}} \pmod{\Phi_n(q)}, & \text{if } 16 \nmid (m-1)(n^2-1). \end{cases} \tag{4.4}$$

It remains to determine the sign of the right-hand side of (4.4). We now assume that $n=p^r$ is an odd prime power. Then $m^{\frac{p-1}{2}} \equiv \left(\frac{m}{p}\right) \pmod{p}$ and, by the binomial theorem, $(p^r-1)/2 = (((p-1)+1)^r-1)/2 \equiv (p-1)r/2 \pmod{p-1}$. Since $m^{p-1}\equiv 1 \pmod{p}$, we conclude that $m^{\frac{p^r-1}{2}} \equiv m^{\frac{(p-1)r}{2}} = \left(\frac{m}{p}\right)^r = \left(\frac{m}{p^r}\right) = \left(\frac{m}{n}\right) \pmod{p}$. Therefore, taking $q=1$ in (4.4) and noticing that $\Phi_{p^r}(1)=p$, we deduce that the sign $\pm$ in (4.4) must be $\left(\frac{m}{n}\right)$. □

For any positive odd integer n, it is easy to see that $\Phi_n(q^2)=\Phi_n(q)\Phi_n(-q)$. Replacing q by q^2 in (4.2) and noticing that $q^n\equiv 1 \pmod{\Phi_n(q)}$, we obtain the following conjectural congruence:

$$\frac{(q^{2m};q^{2m})_{(n-1)/2}}{(q^2;q^2)_{(n-1)/2}} \equiv \left(\frac{m}{n}\right) q^{\frac{(m-1)(n^2-1)}{8}} \pmod{\Phi_n(q)},$$

which reduces to (2.1) when $m=2$.

Let us turn back to Swisher's work [22, Corollary 1.4]. She proves the following interesting congruence:

$$\sum_{k=0}^{\frac{p-1}{2}} (-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k}\sum_{j=1}^{k}\left(\frac{1}{(2j-1)^2}-\frac{1}{16j^2}\right) \equiv 0 \pmod{p}.$$

We provide a q-analogue of this congruence as follows.

Conjecture 4.4. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{\frac{n-1}{2}} (-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3}\sum_{j=1}^{k}\left(\frac{q^{2j-1}}{[2j-1]^2}-\frac{q^{4j}}{[4j]^2}\right) \equiv 0 \pmod{\Phi_n(q)}.$$

Swisher [22] has made many interesting conjectural supercongruences on generalizations of Van Hamme's 13 Ramanujan-type supercongruences. For instance, She [22, (L.3)] conjectured that, for any odd prime p and positive integer r,

$$\sum_{k=0}^{\frac{p^r-1}{2}} (-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k} \equiv (-1)^{\frac{(p-1)(p+5)}{8}} p \sum_{k=0}^{\frac{p^{r-1}-1}{2}} (-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k} \pmod{p^{3r}}. \tag{4.5}$$

If the supercongruence (4.5) is true, then we can easily conclude that

$$\sum_{k=0}^{\frac{p^r-1}{2}}(-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k}\equiv(-1)^{\frac{(p-1)(p+5)r}{8}}p^r \pmod{p^{r+2}},$$

which is the $n=p^r$ and $q=1$ case of our conjectural congruence (1.13) by noticing that $(-1)^{\frac{(p-1)(p+5)r}{8}}=(-1)^{\frac{(p^r-1)(p^r+5)}{8}}$. That is, the congruence (1.13) coincides with Swisher's Conjecture (L.3).

If the conjectural congruence (1.14) is true, then

$$\sum_{k=0}^{p^r-1}(-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k}\equiv(-1)^{\frac{(p-1)(p+5)r}{8}}p^r \pmod{p^{r+2}}. \tag{4.6}$$

Motivated by Swisher's Conjecture (L.3) and the conjectures of Z.-W. Sun [21], we would like to raise the following conjecture, which is a refinement of (4.6).

Conjecture 4.5. *Let p be an odd prime and r a positive integer. Then*

$$\sum_{k=0}^{p^r-1}(-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k}\equiv(-1)^{\frac{(p-1)(p+5)}{8}}p\sum_{k=0}^{p^{r-1}-1}(-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k} \pmod{p^{3r}}.$$

Moreover, since the supercongruences (1.6)–(1.9) have very nice q-analogues, it is natural to ask whether their original π series (1.2)–(1.5) have similar q-analogues or not. This is true for (1.2)–(1.4). In fact, letting $n\to\infty$, $a=b=c=q$, and $q\to q^2,q^3,q^4$ in Jackson's ${}_6\phi_5$ summation (see [4, Appendix (II.20)]):

$${}_6\phi_5\left[\begin{matrix} a,\ qa^{\frac{1}{2}},\ -qa^{\frac{1}{2}},\ b,\ c,\ d \\ a^{\frac{1}{2}},\ -a^{\frac{1}{2}},\ aq/b,\ aq/c,\ aq/d\end{matrix};q,\ \frac{aq}{bcd}\right]=\frac{(aq;q)_\infty(aq/bc;q)_\infty(aq/bd;q)_\infty(aq/cd;q)_\infty}{(aq/b;q)_\infty(aq/c;q)_\infty(aq/d;q)_\infty(aq/bcd;q)_\infty},$$

where $(a;q)_\infty=\lim_{n\to\infty}(a;q)_n$ and the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r\left[\begin{matrix} a_1,a_2,\dots,a_{r+1} \\ b_1,b_2,\dots,b_r\end{matrix};q,\ z\right]=\sum_{k=0}^{\infty}\frac{(a_1;q)_k(a_2;q)_k\cdots(a_{r+1};q)_k z^k}{(q;q)_k(b_1;q)_k(b_2;q)_k\cdots(b_r;q)_k},$$

we are led to the following q-series identities:

$$\begin{aligned}
\sum_{k=0}^{\infty}(-1)^k q^{k^2}[4k+1]\frac{(q;q^2)_k^3}{(q^2;q^2)_k^3}&=\frac{(q;q^2)_\infty(q^3;q^2)_\infty}{(q^2;q^2)_\infty^2},\\
\sum_{k=0}^{\infty}(-1)^k q^{\frac{3k^2+k}{2}}[6k+1]\frac{(q;q^3)_k^3}{(q^3;q^3)_k^3}&=\frac{(q^2;q^3)_\infty(q^4;q^3)_\infty}{(q^3;q^3)_\infty^2},\\
\sum_{k=0}^{\infty}(-1)^k q^{2k^2+k}[8k+1]\frac{(q;q^4)_k^3}{(q^4;q^4)_k^3}&=\frac{(q^3;q^4)_\infty(q^5;q^4)_\infty}{(q^4;q^4)_\infty^2},
\end{aligned} \tag{4.7}$$

which are q-analogues of (1.2)–(1.4), respectively.

We have the following conjectural q-analogue of (1.5).

Conjecture 4.6. *For any complex number q with $|q| < 1$, there holds*

$$\sum_{k=0}^{\infty}(-1)^k q^{3k^2}[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3} = \frac{(q^3;q^4)_\infty (q^5;q^4)_\infty}{(q^4;q^4)_\infty^2}. \tag{4.8}$$

Note that the right-sides of (4.7) and (4.8) are the same. It is easy to see that the left-hand side of (4.8) converges uniformly on the interval $[0,1)$, and so

$$\lim_{q\to 1^-}\sum_{k=0}^{\infty}(-1)^k q^{3k^2}[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3} = \sum_{k=0}^{\infty}(-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k}.$$

On the other hand, the q-Gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0 < q < 1$$

(see [4, p. 20]), and we have

$$\lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x).$$

It follows that

$$\lim_{q\to 1^-}\frac{(q^3;q^4)_\infty (q^5;q^4)_\infty}{(q^4;q^4)_\infty^2} = \lim_{q\to 1^-}\frac{1}{\Gamma_{q^4}(\frac{3}{4})\Gamma_{q^4}(\frac{5}{4})} = \frac{1}{\Gamma(\frac{3}{4})\Gamma(\frac{5}{4})} = \frac{2\sqrt{2}}{\pi}.$$

This means that (4.8) is indeed a q-analogue of (1.5).

Remark. Conjecture 1.1 has recently been confirmed by Guo and Zudilin [10, Theorem 4.4], and Conjecture 4.6 has been proved by Guo and Liu [8], Hou, Krattenthaler, and Sun [12], and Guo and Zudilin [9]. It was pointed out by the editor that Conjecture 4.6 can also be deduced from the following terminating quadratic summation of Gessel and Stanton [5, (6.8)]:

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k(a;q^{\frac{1}{2}})_k(aq/c;q^{\frac{1}{2}})_k(c/aq^{\frac{1}{2}};q^{\frac{1}{2}})_k(1-aq^{\frac{3k}{2}})}{(aq^{n+\frac{1}{2}};q^{\frac{1}{2}})_k(q;q)_k(c;q)_k(a^2q^{\frac{3}{2}}/c;q)_k(1-a)}q^{nk+\frac{k^2+k}{4}}a^k = \frac{(aq^{\frac{1}{2}};q^{\frac{1}{2}})_{2n}}{(c;q)_n(a^2q^{\frac{3}{2}}/c;q)_n}$$

by letting $n\to\infty$, $q\to q^4$, $c\to q^4$, and $a\to q$.

Acknowledgments. The author thanks the anonymous referee for many valuable comments and suggestions. This work was partially supported by the National Natural Science Foundation of China (grant 11771175).

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