A \(q\)-ANALOGUE OF THE (I.2) SUPERCONGRUENCE OF VAN HAMME

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We give a \(q\)-analogue of the supercongruence (I.2) of Van Hamme. As a conclusion, we confirm a recent conjecture of Swisher. We also give a \(q\)-analogue of the corresponding \(\pi\) series (I.1) along with some similar results.

Keywords: Ramanujan; supercongruence; cyclotomic polynomials; \(q\)-Gamma function.

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1. Introduction

In 1914, Ramanujan [17] listed 17 infinite series representations of \(1/\pi\) (see also [3, pp. 352–354]). All the 17 formulas are similar to the following identity due to Bauer [4]:

\[
\sum_{k=0}^{\infty} (-1)^k (4k+1) \left(\frac{1}{2}\right)_k^3 = \frac{2}{\pi},
\]

(1.1)

where we use the Pochhammer symbol \((a)_k = a(a+1)\cdots(a+k-1)\).

In 1997, Van Hamme [20] conjectured that 13 Ramanujan-type series including (1.1) admit nice \(p\)-adic analogues, such as

\[
\sum_{k=0}^{p-1} (-1)^k (4k+1) \left(\frac{1}{2}\right)_k^3 \equiv p(-1)^{\frac{p-1}{2}} \pmod{p^3},
\]

(1.2)

where \(p\) is an odd prime. All of the 13 supercongruences are called Ramanujan-type supercongruences and have now been proved by using a variety of techniques (see [16] for a historic remark on this). Nevertheless, we still have many things to do on these supercongruences. Firstly, finding \(q\)-analogues of known supercongruences is worthwhile and usually challenging, especially for Van Hamme’s 13 supercongruences. Secondly, if there exists a \(q\)-analogue of a Ramanujan-type supercongruence,
it is natural to wonder whether there exists a corresponding $q$-analogue of the original Ramanujan-type series. Moreover, Swisher [18] has made many interesting conjectures on generalizations of Van Hamme’s supercongruences.

Many authors have contributed to $q$-analogues of congruences (see, for example, [2,5,7,9,10,11,12,13,19]). In particular, by establishing a complicated basic hypergeometric series identity, the author and Zeng [12] gave a $q$-analogue of Van Hamme’s supercongruence (H.2); Motivated by the work of Zudilin [21] and Ekhad and Zeilberger [6], the author [9,10] used the $q$-WZ method to obtain $q$-analogues of Van Hamme’s supercongruences (B.2), (E.2), and (F.2); The author and Wang [11] used a variation of the $q$-WZ method to prove a $q$-analogue of a theorem of Long [14, Theorem 1.1], which modulo $[p]^3$ reduces to a $q$-analogue of the supercongruence (C.2).

In this paper we shall give a $q$-analogue of Van Hamme’s supercongruence (I.2) (among the 13 supercongruences, Van Hamme [20] himself proved the cases (C.2), (H.2) and (I.2)).

**Entry (I.2) (Van Hamme [20]).** Let $p$ be an odd prime. Then

$$
\sum_{k=0}^{n-1} \frac{1}{k+1} \frac{(\frac{1}{2})^k}{k!} \equiv 2p^2 \pmod{p^3}. \quad (1.3)
$$

Following the notation in [8], the $q$-shifted factorial is defined by $(a;q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1})$ for $n \geq 1$ and $(a;q)_0 = 1$. The $q$-integer is defined as $[n] = [n]_q = 1 + q + \cdots + q^{n-1}$. We shall also use the $n$-th cyclotomic polynomial $\Phi_n(q)$, which may be defined by

$$
\Phi_n(q) = \prod_{1 \leq k \leq n, \gcd(k,n) = 1} (q - e^{2\pi i k/n}),
$$

where $i$ is the imaginary unit. It is well known that $\Phi_n(q)$ is an irreducible polynomial with integer coefficients and $\Phi_p(q) = [p]$ for any prime $p$. Throughout the paper, the polynomials are considered in the ring $\mathbb{Q}[q]$. Since $\Phi_p(q)$ is irreducible, the quotient ring $\mathbb{Q}[q]/\Phi_n(q)$ is a field, and so any polynomial which is not divisible by $\Phi_n(q)$ has an inverse in $\mathbb{Q}[q]/\Phi_n(q)$. Thus, rational functions in $\mathbb{Q}(q)$ can be interpreted modulo $\Phi_n(q)$ whenever they have an appropriate denominator.

Our $q$-analogue of the entry (I.2) can be stated as follows:

**Theorem 1.1.** Let $n \geq 3$ be a positive odd integer. Then

$$
\sum_{k=0}^{n-1} \frac{(1-q^2)(q^2)^k q^{2k}}{(q^2; q^2)^k(q^2; q^2)_{k+1}} \equiv (1 + q)[n]_q^2 q^{\frac{n+1}{2}} \pmod{[n]^2\Phi_n(q)}. \quad (1.4)
$$

It is easy to see that when $n = p$ and $q \to 1$ the congruence (1.4) reduces to (1.3). Moreover, letting $n = p^r$ be an odd prime power and $q \to 1$ in (1.4), and noticing $\Phi_{p^r}(1) = p$, we get the following result, which confirms a recent conjecture of Swisher [18, (I.3)].
Corollary 1.2. Let \( p \) be an odd prime and \( r \) a positive integer. Then
\[
\sum_{k=0}^{p^r-1} \frac{1}{k+1} \left( \frac{1}{2} \right)_k^2 \equiv 2p^{2r} \pmod{p^{2r+1}}.
\]

We also have the following result, which is similar to Theorem 1.1.

Theorem 1.3. Let \( n \) be a positive odd integer. Then
\[
\sum_{k=0}^{n-1} \frac{(-1)^k}{k!^2} (q-1; q^2)_k^2 \equiv -(1+2q)[n]^2 \pmod{[n]^2 \Phi_n(q)}.
\] (1.5)

The \( n=p^r \) and \( q \to 1 \) case of (1.5) gives

Corollary 1.4. Let \( p \) be an odd prime and \( r \) a positive integer. Then
\[
\sum_{k=0}^{p^r-1} \frac{(-1)^k}{k!^2} \equiv -3p^{2r} \pmod{p^{2r+1}}.
\] (1.6)

Note that the corresponding \( \pi \) series (I.1) is
\[
\sum_{k=0}^{\infty} \frac{1}{k+1} \left( \frac{1}{2} \right)_k^2 = \frac{4}{\pi},
\]
while the infinite series related to (1.6) is
\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{k!^2} = \frac{4}{\pi}.
\] (1.7)

We have the following \( q \)-analogue of (I.1) and (1.7).

Theorem 1.5. For any complex number \( q \) with \( |q| < 1 \), we have
\[
\sum_{k=0}^{\infty} \frac{(1-q^2)(q^2; q^2)_k^2 q^{2k}}{(q^2; q^2)_k^2} = \frac{(1+q)(q; q^2)_{\infty}(q^3; q^2)_{\infty}}{(q^2; q^2)_{\infty}^2},
\] (1.8)
\[
\sum_{k=0}^{\infty} \frac{(q^{-1}; q^2)_k^2 q^{2k}}{(q^2; q^2)_k^2} = \frac{2(q; q^2)_{\infty}(q^3; q^2)_{\infty}}{(q^2; q^2)_{\infty}^2},
\] (1.9)
where \((a; q)_{\infty} = \lim_{n \to \infty} (a; q)_n\).

2. Proof of the theorems

Recall that the \( q \)-binomial coefficients \( \left[ \begin{array}{c} n \\ k \end{array} \right] \) are defined by
\[
\left[ \begin{array}{c} n \\ k \end{array} \right] = \begin{cases} \frac{(q^{n-k+1}; q)_k}{(q; q)_k} & \text{if } 0 \leq k \leq n, \\ 0 & \text{otherwise}. \end{cases}
\]
It is easy to see that
\[
\frac{(q; q^2)_k}{(q^2; q^2)_k} = \frac{1}{(-q; q)_k} \left[ \frac{2k}{k} \right] \quad \text{for } k \geq 0. \tag{2.1}
\]
We first give a preliminary result.

**Lemma 2.1.** Let \( n \) be a positive odd integer. Then
\[
(-q; q)^2 n^{-1/2} \equiv q^{-n^2} \pmod{\Phi_n(q)}. \tag{2.2}
\]

**Proof.** By the \( q \)-binomial theorem (see [1, p. 36, (3.3.6)]), we have
\[
(-q; q)_{n-1} = \sum_{k=0}^{n-1} \left( \begin{array}{c} n-1 \\ k \end{array} \right) q^{(k+1)} \equiv n^{k} = 1 \pmod{\Phi_n(q)}, \tag{2.3}
\]

since
\[
\left[ \frac{n-1}{k} \right] = \prod_{j=1}^{k} \frac{1 - q^{n-j}}{1 - q^j} \equiv \prod_{j=1}^{k} \frac{1 - q^{-j}}{1 - q^j} = (-1)^k q^{-\left( \sum_{j=1}^{\infty} 1/(q^j) \right)} \"mod \Phi_n(q).\)
\tag{2.4}
\]

On the other hand, we have
\[
(-q; q)_{n-1} = (-q; q)_{(n-1)/2} \prod_{k=1}^{n-1} (1 + q^{-k}) \equiv (-q; q)_{(n-1)/2} \prod_{k=1}^{n-1} (1 + q^{-k})
\]
\[
= (-q; q)^2 \frac{1 - q^{-n}}{1 - q^{-n+1}} \pmod{\Phi_n(q)}. \tag{2.5}
\]

Combining (2.3) and (2.4), we obtain (2.2).

Now we can prove our main results.

**Proof of Theorem 1.1.** It is easy to see by induction that
\[
\sum_{k=0}^{m-1} \frac{(1 - q^2)(q; q^2)_k^2 q^{2k}}{(q^2; q^2)_k (q^2; q^2)_k+1} = (1 + q)[2m] \frac{(q; q^2)^2}{(q^2; q^2)^2}. \tag{2.6}
\]

In fact, for \( m = 1 \), both sides of (2.6) are equal to 1. Assume that (2.6) holds for \( m \). Then
\[
\sum_{k=0}^{m} (1 - q^2)(q; q^2)_k^2 q^{2k} \frac{(q; q^2)_k (q^2; q^2)_{k+1}}{(q^2; q^2)_k (q^2; q^2)_{k+1}} = (1 - q^2)(q; q^2)_m^2 q^{2m} + \sum_{k=0}^{m-1} \frac{(1 - q^2)(q; q^2)_k^2 q^{2k}}{(q^2; q^2)_k (q^2; q^2)_{k+1}}
\]
\[
= (1 + q)(q; q^2)_m^2 q^{2m} + (1 + q)[2m] \frac{(q; q^2)_m^2}{(q^2; q^2)^2}
\]
\[
= (1 + q)[2m + 1] \frac{(q; q^2)_{m+1}^2}{(q^2; q^2)_{m+1}^2}. \]
Namely, the identity (2.6) is also true for \( m + 1 \).

Putting \( m = \frac{n+1}{2} \) in (2.6), we get

\[
\sum_{k=0}^{\frac{n-1}{2}} (1-q^2)_{k}^2 q^{2k} = (1+q)[n+1] \frac{(q^2)^2}{(q^2)^2_{(n+1)/2}}
\]

\[
= \frac{(1+q)[n]}{[n+1][-q;j^2_{(n-1)/2}]} \left\lfloor \frac{n-1}{2} \right\rfloor (by \ (2.1)). \ (2.7)
\]

It is clear that \( \gcd([n], [n+1]) = 1 \) and \([n+1] \equiv 1 \pmod{\Phi_n(q)}\). It is also not difficult to see that \( \gcd([n], (-q; q)_{(n-1)/2}) = 1 \). Therefore, by (2.2) and (2.4), the right-hand side of (2.7) modulo \([n]q^n\Phi_n(q)\) reduces to

\[
\frac{(1+q)[n]^{2}}{q^{\frac{n-1}{2}}} q^{-2^{\left(\frac{n+1}{2}\right)}} = (1+q)[n]^{2} q^{-\frac{n-1}{2}} \equiv (1+q)[n]^{2} q^{\frac{n+1}{2}}.
\]

This completes the proof. \( \Box \)

**Proof of Theorem 1.3.** By induction on \( n \), we can prove that

\[
\sum_{k=0}^{\frac{n-1}{2}} (q^{-1}; q^2)_{k}^2 q^{2k} = \frac{(q; q^2)^2}{(q^2; q^2)^2_{n-1}}(2[2n-2] + q^{2n-2}). \ (2.8)
\]

In fact, both sides of (2.8) are equal to 1 for \( n = 1 \). Suppose that (2.8) holds for \( n \). Then

\[
\sum_{k=0}^{n} (q^{-1}; q^2)_{k}^2 q^{2k} = \frac{(q^{-1}; q^2)_{n}q^{2n}}{(q^2; q^2)^2_{n-1}} + \sum_{k=0}^{n-1} (q^{-1}; q^2)_{k}^2 q^{2k}
\]

\[
= \frac{(q^{-1}; q^2)_{n}q^{2n}}{(q^2; q^2)^2_{n}} + \frac{(q; q^2)^2}{(q^2; q^2)^2_{n-1}}(2[2n-2] + q^{2n-2})
\]

\[
= \frac{(q; q^2)^2}{(q^2; q^2)^2_{n}} (2[2n] + q^{2n}).
\]

Namely, the identity (2.8) also holds for \( n + 1 \).

By (2.1), the right-hand side of (2.8) can be written as

\[
\left[ \frac{2n-2}{n-1} \right]^{2} [2[2n-2] + q^{2n-2}] = \frac{2n-2}{n-2} \frac{2[2n-2] + q^{2n-2}}{(-q; q)_{n-1}^4 [n-1]^2}.
\]

(2.9)

It is easy see that \([2n-2] \equiv -q^{-1} - q^{-2} \pmod{\Phi_n(q)}, [n-1] \equiv -q^{-1} \pmod{\Phi_n(q)}, \) and \(q^{n-1} \equiv q^{-1} \pmod{\Phi_n(q)}\). It is also not difficult to see that \( \gcd([n], [n-1]) = 1 \), and \( \gcd([n], (-q; q)_{n-1}) = 1 \) for odd \( n \). We can substitute these congruences and (2.3) into the right-hand side of (2.9) to obtain the desired congruence (1.5). \( \Box \)

**Proof of Theorem 1.5.** Letting \( m \to \infty \) in (2.6) and noticing that

\[
\lim_{m \to \infty} \left[ m \right] = \frac{1}{1-q},
\]

we obtain (1.8). Similarly, letting \( n \to \infty \) in (2.8), we are led to (1.9). \( \Box \)
3. Concluding remarks

It is not difficult to see that the left-hand side of (1.8) converges uniformly on the interval [0, 1), and so

$$\lim_{q \to 1^-} \sum_{k=0}^{\infty} \frac{(1 - q^2)(q^2; q^2)_k q^{2k}}{(q^2; q^2)_k (q^2; q^2)_{k+1}} = \sum_{k=0}^{\infty} \frac{1}{k+1} \frac{1}{k!^2}.$$ 

Recall that the $q$-Gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}, \quad 0 < q < 1.$$ 

It is easy to see that $\lim_{q \to 1^-} \Gamma_q(x) = \Gamma(x)$ (see [8, p. 20]), and so

$$\lim_{q \to 1^-} \frac{(1 + q)(q; q^2)\infty (q^3; q^2)\infty}{(q^2; q^2)_\infty^2} = \lim_{q \to 1^-} \frac{2}{\Gamma_q(x)} \Gamma_q(x) = \frac{2}{\Gamma(\frac{1}{2}) \Gamma(\frac{3}{2})} = \frac{4}{\pi}.$$ 

This means that (1.8) is indeed a $q$-analogue of (I.1). Similarly, the identity (1.9) is a $q$-analogue of (1.7).

We point out that $q$-analogues of the series (B.1), (E.1), and (F.1) in Van Hamme’s paper [20] can be easily deduced from Jackson’s $6\phi_5$ summation (see [8, Appendix (II.21)]).

However, for the series (H.1) in [20]:

$$\sum_{k=0}^{\infty} \frac{(\frac{3}{4})_k^3}{k!^3} = \frac{\pi}{\Gamma(\frac{3}{4})^4}, \quad (3.1)$$

we have not found such a $q$-analogue, though the author and Zeng [12, Corollary 1.2] proved that

$$\sum_{k=0}^{p-1} \frac{(q^3; q^3)_k^3 (q^2; q^2)_k (q^4; q^4)_k q^{2k}}{(q^2; q^2)_k (q^4; q^4)_k} \equiv 0 \pmod{[p]^2} \quad \text{for any prime } p \equiv 3 \pmod{4},$$

which provides us with a plausible choice for the $q$-analogue of the left-hand side of (3.1).

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References

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