International Journal of Number Theory

A q-ANALOGUE OF THE (I.2) SUPERCONGRUENCE OF VAN HAMME

VICTOR J. W. GUO
School of Mathematical Sciences, Huaiyin Normal University
Huai'an 223300, Jiangsu, People's Republic of China
jwguo@hytc.edu.cn

Received (23 January 2018)
Accepted (2 July 2018)

We give a q-analogue of the supercongruence (I.2) of Van Hamme. As a conclusion, we confirm a recent conjecture of Swisher. We also give a q-analogue of the corresponding π series (I.1) along with some similar results.

Keywords: Ramanujan; supercongruence; cyclotomic polynomials; q-Gamma function.

Mathematics Subject Classification 2010: 11A07, 11B65, 05A10

1. Introduction

In 1914, Ramanujan [17] listed 17 infinite series representations of $1/\pi$ (see also [3, pp. 352–354]). All the 17 formulas are similar to the following identity due to Bauer [4]:

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3}=\frac{2}{\pi}, \tag{1.1}$$

where we use the Pochhammer symbol $(a)_k=a(a+1)\cdots(a+k-1)$.

In 1997, Van Hamme [20] conjectured that 13 Ramnujan-type series including (1.1) admit nice p-adic analogues, such as

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3}\equiv p(-1)^{\frac{p-1}{2}} \pmod{p^3}, \tag{1.2}$$

where p is an odd prime. All of the 13 supercongruences are called Ramanujan-type supercongruences and have now been proved by using a variety of techniques (see [16] for a historic remark on this). Nevertheless, we still have many things to do on these supercongruences. Firstly, finding q-analogues of known supercongruences is worthwhile and usually challenging, especially for Van Hamme's 13 supercongruences. Secondly, if there exists a q-analogue of a Ramanujan-type supercongruence, it is natural to wonder whether there exists a corresponding q-analogue of the original Ramnujan-type series. Moreover, Swisher [18] has made many interesting conjectures on generalizations of Van Hamme's supercongruences.

Many authors have contributed to q-analogues of congruences (see, for example, [2,5,7,9,10,11,12,13,19]). In particular, by establishing a complicated basic hypergeometric series identity, the author and Zeng [12] gave a q-analogue of Van Hamme's supercongruence (H.2); Motivated by the work of Zudilin [21] and Ekhad and Zeilberger [6], the author [9,10] used the q-WZ method to obtain q-analogues of Van Hamme's supercongruences (B.2), (E.2), and (F.2); The author and Wang [11] used a variation of the q-WZ method to prove a q-analogue of a theorem of Long [14, Theorem 1.1], which modulo $[p]^3$ reduces to a q-analogue of the supercongruence (C.2).

In this paper we shall give a q-analogue of Van Hamme's supercongruence (I.2) (among the 13 supercongruences, Van Hamme [20] himself proved the cases (C.2), (H.2) and (I.2)).

Entry (I.2) (Van Hamme [20]). Let p be an odd prime. Then

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{1}{k+1} \frac{(\frac{1}{2})_k^2}{k!^2} \equiv 2p^2 \pmod{p^3}. \tag{1.3}$$

Following the notation in [8], the *q-shifted factorial* is defined by $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geqslant 1$ and $(a;q)_0 = 1$. The *q-integer* is defined as $[n] = [n]_q = 1 + q + \cdots + q^{n-1}$. We shall also use the n-th *cyclotomic polynomial* $\Phi_n(q)$, which may be defined by

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}} (q - e^{2\pi i \frac{k}{n}}),$$

where i is the imaginary unit. It is well known that $\Phi_n(q)$ is an irreducible polynomial with integer coefficients and $\Phi_p(q) = [p]$ for any prime p. Throughout the paper, the polynomials are considered in the ring $\mathbb{Q}[q]$. Since $\Phi_n(q)$ is irreducible, the quotient ring $\mathbb{Q}[q]/\Phi_n(q)$ is a field, and so any polynomial which is not divisible by $\Phi_n(q)$ has an inverse in $\mathbb{Q}[q]/\Phi_n(q)$. Thus, rational functions in $\mathbb{Q}(q)$ can be interpreted modulo $\Phi_n(q)$ whenever they have an appropriate denominator.

Our q-analogue of the entry (I.2) can be stated as follows:

Theorem 1.1. *Let $n \geqslant 3$ be a positive odd integer. Then*

$$\sum_{k=0}^{\frac{n-1}{2}} \frac{(1-q^2)(q;q^2)_k^2 q^{2k}}{(q^2;q^2)_k (q^2;q^2)_{k+1}} \equiv (1+q)[n]^2 q^{\frac{n+1}{2}} \pmod{[n]^2\Phi_n(q)}. \tag{1.4}$$

It is easy to see that when $n = p$ and $q \to 1$ the congruence (1.4) reduces to (1.3). Moreover, letting $n = p^r$ be an odd prime power and $q \to 1$ in (1.4), and noticing $\Phi_{p^r}(1) = p$, we get the following result, which confirms a recent conjecture of Swisher [18, (I.3)].

Corollary 1.2. *Let p be an odd prime and r a positive integer. Then*

$$\sum_{k=0}^{\frac{p^r-1}{2}} \frac{1}{k+1}\frac{(\frac{1}{2})_k^2}{k!^2} \equiv 2p^{2r} \pmod{p^{2r+1}}.$$

We also have the following result, which is similar to Theorem 1.1.

Theorem 1.3. *Let n be positive odd integer. Then*

$$\sum_{k=0}^{n-1} \frac{(q^{-1};q^2)_k^2 q^{2k}}{(q^2;q^2)_k^2} \equiv -(1+2q)[n]^2 \pmod{[n]^2\Phi_n(q)}. \tag{1.5}$$

The $n=p^r$ and $q\to 1$ case of (1.5) gives

Corollary 1.4. *Let p be an odd prime and r a positive integer. Then*

$$\sum_{k=0}^{p^r-1} \frac{(-\frac{1}{2})_k^2}{k!^2} \equiv -3p^{2r} \pmod{p^{2r+1}}. \tag{1.6}$$

Note that the corresponding π series (I.1) is

$$\sum_{k=0}^{\infty} \frac{1}{k+1}\frac{(\frac{1}{2})_k^2}{k!^2} = \frac{4}{\pi},$$

while the infinite series related to (1.6) is

$$\sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k^2}{k!^2} = \frac{4}{\pi}. \tag{1.7}$$

We have the following q-analogue of (I.1) and (1.7).

Theorem 1.5. *For any complex number q with $|q|<1$, we have*

$$\sum_{k=0}^{\infty} \frac{(1-q^2)(q;q^2)_k^2 q^{2k}}{(q^2;q^2)_k(q^2;q^2)_{k+1}} = \frac{(1+q)(q;q^2)_\infty(q^3;q^2)_\infty}{(q^2;q^2)_\infty^2}, \tag{1.8}$$

$$\sum_{k=0}^{\infty} \frac{(q^{-1};q^2)_k^2 q^{2k}}{(q^2;q^2)_k^2} = \frac{2(q;q^2)_\infty(q^3;q^2)_\infty}{(q^2;q^2)_\infty^2}, \tag{1.9}$$

where $(a;q)_\infty = \lim_{n\to\infty}(a;q)_n$.

2. Proof of the theorems

Recall that the *q-binomial coefficients* $\begin{bmatrix} n \\ k \end{bmatrix}$ are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q^{n-k+1};q)_k}{(q;q)_k} & \text{if } 0\leqslant k\leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$\frac{(q;q^2)_k}{(q^2;q^2)_k}=\frac{1}{(-q;q)_k^2}\begin{bmatrix}2k\\k\end{bmatrix}\quad\text{for } k\geqslant 0. \tag{2.1}$$

We first give a preliminary result.

Lemma 2.1. *Let n be a positive odd integer. Then*

$$(-q;q)_{(n-1)/2}^2\equiv q^{\frac{n^2-1}{8}}\pmod{\Phi_n(q)}. \tag{2.2}$$

Proof. By the q-binomial theorem (see [1, p. 36, (3.3.6)]), we have

$$(-q;q)_{n-1}=\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}q^{\binom{k+1}{2}}\equiv\sum_{k=0}^{n-1}(-1)^k=1\pmod{\Phi_n(q)}, \tag{2.3}$$

since

$$\begin{bmatrix}n-1\\k\end{bmatrix}=\prod_{j=1}^{k}\frac{1-q^{n-j}}{1-q^j}\equiv\prod_{j=1}^{k}\frac{1-q^{-j}}{1-q^j}=(-1)^kq^{-\binom{k+1}{2}}\pmod{\Phi_n(q)}. \tag{2.4}$$

On the other hand, we have

$$\begin{aligned}(-q;q)_{n-1}&=(-q;q)_{(n-1)/2}\prod_{k=1}^{\frac{n-1}{2}}(1+q^{n-k})\equiv(-q;q)_{(n-1)/2}\prod_{k=1}^{\frac{n-1}{2}}(1+q^{-k})\\&=(-q;q)_{(n-1)/2}^2q^{\frac{1-n^2}{8}}\pmod{\Phi_n(q)}.\end{aligned} \tag{2.5}$$

Combining (2.3) and (2.4), we obtain (2.2). □

Now we can prove our main results.

Proof of Theorem 1.1. It is easy to see by induction that

$$\sum_{k=0}^{m-1}\frac{(1-q^2)(q;q^2)_k^2q^{2k}}{(q^2;q^2)_k(q^2;q^2)_{k+1}}=(1+q)[2m]\frac{(q;q^2)_m^2}{(q^2;q^2)_m^2}. \tag{2.6}$$

In fact, for $m=1$, both sides of (2.6) are equal to 1. Assume that (2.6) holds for m. Then

$$\begin{aligned}\sum_{k=0}^{m}\frac{(1-q^2)(q;q^2)_k^2q^{2k}}{(q^2;q^2)_k(q^2;q^2)_{k+1}}&=\frac{(1-q^2)(q;q^2)_m^2q^{2m}}{(q^2;q^2)_m(q^2;q^2)_{m+1}}+\sum_{k=0}^{m-1}\frac{(1-q^2)(q;q^2)_k^2q^{2k}}{(q^2;q^2)_k(q^2;q^2)_{k+1}}\\&=\frac{(1-q^2)(q;q^2)_m^2q^{2m}}{(q^2;q^2)_m(q^2;q^2)_{m+1}}+(1+q)[2m]\frac{(q;q^2)_m^2}{(q^2;q^2)_m^2}\\&=\frac{(1+q)(q;q^2)_{m+1}^2}{(1-q)(q^2;q^2)_m(q^2;q^2)_{m+1}}\\&=(1+q)[2m+2]\frac{(q;q^2)_{m+1}^2}{(q^2;q^2)_{m+1}^2}.\end{aligned}$$

Namely, the identity (2.6) is also true for $m+1$.

Putting $m=\frac{n+1}{2}$ in (2.6), we get

$$\sum_{k=0}^{\frac{n-1}{2}} \frac{(1-q^2)(q;q^2)_k^2 q^{2k}}{(q^2;q^2)_k(q^2;q^2)_{k+1}} = (1+q)[n+1]\frac{(q;q^2)_{(n+1)/2}^2}{(q^2;q^2)_{(n+1)/2}^2}$$

$$= \frac{(1+q)[n]^2}{[n+1](-q;q)_{(n-1)/2}^4}\begin{bmatrix} n-1 \\ \frac{n-1}{2}\end{bmatrix}^2 \quad \text{(by (2.1))}. \tag{2.7}$$

It is clear that $\gcd([n],[n+1])=1$ and $[n+1]\equiv 1 \pmod{\Phi_n(q)}$. It is also not difficult to see that $\gcd([n],(-q;q)_{(n-1)/2})=1$. Therefore, by (2.2) and (2.4), the right-hand side of (2.7) modulo $[n]^2\Phi_n(q)$ reduces to

$$\frac{(1+q)[n]^2}{q^{\frac{n^2-1}{4}}}q^{-2\binom{(n+1)/2}{2}} = (1+q)[n]^2 q^{\frac{1-n^2}{2}} \equiv (1+q)[n]^2 q^{\frac{n+1}{2}}.$$

This completes the proof. □

Proof of Theorem 1.3. By induction on n, we can prove that

$$\sum_{k=0}^{n-1}\frac{(q^{-1};q^2)_k^2 q^{2k}}{(q^2;q^2)_k^2} = \frac{(q;q^2)_{n-1}^2}{(q^2;q^2)_{n-1}^2}(2[2n-2]+q^{2n-2}). \tag{2.8}$$

In fact, both sides of (2.8) are equal to 1 for $n=1$. Suppose that (2.8) holds for n. Then

$$\begin{aligned}\sum_{k=0}^{n}\frac{(q^{-1};q^2)_k^2 q^{2k}}{(q^2;q^2)_k^2} &= \frac{(q^{-1};q^2)_n^2 q^{2n}}{(q^2;q^2)_n^2} + \sum_{k=0}^{n-1}\frac{(q^{-1};q^2)_k^2 q^{2k}}{(q^2;q^2)_k^2} \\ &= \frac{(q^{-1};q^2)_n^2 q^{2n}}{(q^2;q^2)_n^2} + \frac{(q;q^2)_{n-1}^2}{(q^2;q^2)_{n-1}^2}(2[2n-2]+q^{2n-2}) \\ &= \frac{(q;q^2)_n^2}{(q^2;q^2)_n^2}(2[2n]+q^{2n}).\end{aligned}$$

Namely, the identity (2.8) also holds for $n+1$.

By (2.1), the right-hand side of (2.8) can be written as

$$\begin{bmatrix} 2n-2 \\ n-1\end{bmatrix}^2 \frac{(2[2n-2]+q^{2n-2})}{(-q;q)_{n-1}^4} = \begin{bmatrix} 2n-2 \\ n-2\end{bmatrix}^2 \frac{(2[2n-2]+q^{2n-2})[n]^2}{(-q;q)_{n-1}^4[n-1]^2}. \tag{2.9}$$

It is easy see that $[2n-2]\equiv -q^{-1}-q^{-2} \pmod{\Phi_n(q)}$, $[n-1]\equiv -q^{-1} \pmod{\Phi_n(q)}$, and $q^{n-1}\equiv q^{-1} \pmod{\Phi_n(q)}$. It is also not difficult to see that $\gcd([n],[n-1])=1$, and $\gcd([n],(-q;q)_{n-1})=1$ for odd n. We can substitute these congruences and (2.3) into the right-hand side of (2.9) to obtain the desired congruence (1.5). □

Proof of Theorem 1.5. Letting $m\to\infty$ in (2.6) and noticing that

$$\lim_{m\to\infty}[m] = \frac{1}{1-q},$$

we obtain (1.8). Similarly, letting $n\to\infty$ in (2.8), we are led to (1.9). □

3. Concluding remarks

It is not difficult to see that the left-hand side of (1.8) converges uniformly on the interval $[0,1)$, and so

$$\lim_{q\to 1^-}\sum_{k=0}^{\infty}\frac{(1-q^2)(q;q^2)_k^2 q^{2k}}{(q^2;q^2)_k(q^2;q^2)_{k+1}}=\sum_{k=0}^{\infty}\frac{1}{k+1}\frac{(\frac{1}{2})_k^2}{k!^2}.$$

Recall that the q-Gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x)=\frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},\quad 0<q<1.$$

It is easy to see that $\lim_{q\to 1^-}\Gamma_q(x)=\Gamma(x)$ (see [8, p. 20]), and so

$$\lim_{q\to 1^-}\frac{(1+q)(q;q^2)_\infty(q^3;q^2)_\infty}{(q^2;q^2)_\infty^2}=\lim_{q\to 1^-}\frac{2}{\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\frac{3}{2})}=\frac{2}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}=\frac{4}{\pi}.$$

This means that (1.8) is indeed a q-analogue of (I.1). Similarly, the identity (1.9) is a q-analogue of (1.7).

We point out that q-analogues of the series (B.1), (E.1), and (F.1) in Van Hamme's paper [20] can be easily deduced from Jackson's ${}_6\phi_5$ summation (see [8, Appendix (II.21)]).

However, for the series (H.1) in [20]:

$$\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k^3}{k!^3}=\frac{\pi}{\Gamma(\frac{3}{4})^4},\tag{3.1}$$

we have not found such a q-analogue, though the author and Zeng [12, Corollary 1.2] proved that

$$\sum_{k=0}^{p-1}\frac{(q;q^2)_k^2(q^2;q^4)_k q^{2k}}{(q^2;q^2)_k^2(q^4;q^4)_k}\equiv 0 \pmod{[p]^2}\quad\text{for any prime } p\equiv 3 \pmod 4,$$

which provides us with a plausible choice for the q-analogue of the left-hand side of (3.1).

Acknowledgments

The author would like to thank the anonymous referee for a careful reading of a previous version of this paper. This work was partially supported by the National Natural Science Foundation of China (grant 11771175), the Natural Science Foundation of Jiangsu Province (grant BK20161304), and the Qing Lan Project of Education Committee of Jiangsu Province.

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