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A FURTHER COMMON q-GENERALIZATION OF THE (B.2) AND (C.2) SUPERCONGRUENCES OF VAN HAMME

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In 2019, the author and Zudilin gave a parametric q-supercongruence, which is a common generalization of the (B.2) and (C.2) supercongruences of Van Hamme. In this paper, we further give a generalization of this q-supercongruence. As a corollary, we obtain the following supercongruence: for any prime $p > 2$, positive integer r, and nonnegative integer $s \leqslant \min\{10, (p^r-1)/4\}$,

$$\sum_{k=s}^{(p^r-1)/2+s} \frac{4k+1}{(-64)^k}\binom{2k-2s}{k-s}\binom{2k+2s}{k+s}\binom{2k}{k} \equiv p^r(-1)^{(p-1)r/2+s} \pmod{p^{r+2}}.$$

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1. Introduction

In 1914, Ramanujan [15] published 17 representations of $1/\pi$. A Ramanujan-type formula already in the literature is the following identity

$$\sum_{k=0}^{\infty} \frac{4k+1}{(-64)^k}\binom{2k}{k}^3 = \frac{2}{\pi}, \tag{1.1}$$

which was found by Bauer [1] in 1859. Van Hamme [18] observed that many Ramanujan-type formulas for $1/\pi$, like (1.1), have nice p-adic analogues (p is an odd prime), such as

$$\sum_{k=0}^{(p-1)/2} \frac{4k+1}{(-64)^k}\binom{2k}{k}^3 \equiv p(-1)^{(p-1)/2} \pmod{p^3}, \tag{1.2}$$

$$\sum_{k=0}^{(p-1)/2} \frac{4k+1}{256^k}\binom{2k}{k}^4 \equiv p \pmod{p^3} \tag{1.3}$$

(tagged (B.2) and (C.2) respectively in Van Hamme's list). The supercongruence (1.2) was first confirmed by Mortenson [14] employing a ${}_6F_5$ transformation formula, and later reproved by Zudilin [21] through the WZ pair in [3]. The supercongruence (1.3) was established by Van Hamme [18] himself, and Long [13, Theorem 1.1] further showed that it is true modulo p^4 for primes $p > 3$.

Using the q-WZ method, the author [6] gave a q-analogue of (1.2) as follows: for positive odd integers n,

$$\sum_{k=0}^{(n-1)/2} (-1)^k [4k+1] \frac{(q;q^2)_k^3}{(q^2;q^2)_k^3} q^{k^2} \equiv (-q)^{(n-1)^2/4}[n] \pmod{[n]\Phi_n(q)^2}. \tag{1.4}$$

Using the same method, the author and Wang [9] established a q-analogue of [13, Theorem 1.1 with $r = 1$]: for positive odd integers n,

$$\sum_{k=0}^{(n-1)/2} [4k+1] \frac{(q;q^2)_k^4}{(q^2;q^2)_k^4} \equiv q^{(1-n)/2}[n] + \frac{(n^2-1)(1-q)^2}{24} q^{(1-n)/2}[n]^3 \pmod{[n]\Phi_n(q)^3},$$

which modulo $[n]\Phi_n(q)^2$ reduces to the following q-analogue of (1.3):

$$\sum_{k=0}^{(n-1)/2} [4k+1] \frac{(q;q^2)_k^4}{(q^2;q^2)_k^4} \equiv q^{(1-n)/2}[n] \pmod{[n]\Phi_n(q)^2}. \tag{1.5}$$

Here and in what follows, $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ is the *q-shifted factorial*, and $[n] = 1+q+\cdots+q^{n-1}$ is the *q-integer*, and $\Phi_n(q)$ stands for the n-th *cyclotomic polynomial* in q, which can be written as

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}} (q-\zeta^k),$$

where ζ is an n-th primitive root of unity. For convenience, we will also adopt the abbreviated notation $(a_1,a_2,\dots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$.

Employing the so-called 'creative microscoping' method, the author and Zudilin [9, Theorem 4.2 with $a = 1$] gave a generalization of (1.4) and (1.5) as follows: for odd n,

$$\begin{aligned}\sum_{k=0}^{(n-1)/2} & [4k+1] \frac{(q;q^2)_k^2 (q/c;q^2)_k (q;q^2)_k}{(q^2;q^2)_k^2 (cq^2;q^2)_k (q^2;q^2)_k} c^k \\ &\equiv \frac{(c/q)^{(n-1)/2}(q^2/c;q^2)_{(n-1)/2}}{(cq^2;q^2)_{(n-1)/2}}[n] \pmod{[n]\Phi_n(q)^2}.\end{aligned} \tag{1.6}$$

It is clear that the $c \to 0$ case of (1.6) reduces to (1.4), and the $c = 1$ case of (1.6) reduces to (1.5). Moreover, when $c = -1$, the q-supercongruence (1.6) gives

$$\sum_{k=0}^{(n-1)/2} (-1)^k [4k+1] \frac{(q;q^2)_k^2 (q^2;q^4)_k}{(q^2;q^2)_k^2 (q^4;q^4)_k} \equiv (-q)^{(1-n)/2}[n] \pmod{[n]\Phi_n(q)^2}, \tag{1.7}$$

which is another q-analogue of (1.2). For some other recent work on q-supercongruences, we refer the reader to [5,2,11,12,19,20].

In this paper, we shall give the following further generalization of (1.6).

Theorem 1.1. *Let $n>1$ be an odd integer and let $0\leqslant s\leqslant (n-1)/4$. Then*

$$\begin{aligned}\sum_{k=s}^{(n-1)/2+s}&[4k+1]\frac{(q;q^2)_{k-s}(q;q^2)_{k+s}(q/c;q^2)_k(q;q^2)_k}{(q^2;q^2)_{k-s}(q^2;q^2)_{k+s}(cq^2;q^2)_k(q^2;q^2)_k}c^k\\ &\equiv\frac{c^s(c/q)^{(n-1)/2}(q/c;q^2)_s(q^2/c;q^2)_{(n-1)/2}}{(cq;q^2)_s(cq^2;q^2)_{(n-1)/2}}[n]\pmod{\Phi_n(q)^3}.\end{aligned}\tag{1.8}$$

Furthermore, if $s\leqslant 10$, then (1.8) also holds modulo $[n]\Phi_n(q)^2$.

We now give some particular cases of (1.8). Letting $c\to 0$ in (1.8), we have the following corollary.

Corollary 1.2. *Let $n>1$ be an odd integer and let $0\leqslant s\leqslant (n-1)/4$. Then*

$$\sum_{k=s}^{(n-1)/2+s}(-1)^k[4k+1]\frac{(q;q^2)_{k-s}(q;q^2)_{k+s}(q;q^2)_k q^{k^2}}{(q^2;q^2)_{k-s}(q^2;q^2)_{k+s}(q^2;q^2)_k}\equiv(-q)^{(n-1)^2/4+s^2}[n]\pmod{\Phi_n(q)^3}.$$

Letting $c=1$ in (1.8), we get the following result due to Tang [17].

Corollary 1.3. *Let $n>1$ be an odd integer and let $0\leqslant s\leqslant (n-1)/4$. Then*

$$\sum_{k=s}^{(n-1)/2+s}[4k+1]\frac{(q;q^2)_{k-s}(q;q^2)_{k+s}(q;q^2)_k^2}{(q^2;q^2)_{k-s}(q^2;q^2)_{k+s}(q^2;q^2)_k^2}\equiv q^{(1-n)/2}[n]\pmod{\Phi_n(q)^3}.$$

Moreover, taking $c=-1$ in (1.8), we are led to the conclusion.

Corollary 1.4. *Let $n>1$ be an odd integer and let $0\leqslant s\leqslant (n-1)/4$. Then*

$$\sum_{k=s}^{(n-1)/2+s}(-1)^k[4k+1]\frac{(q;q^2)_{k-s}(q;q^2)_{k+s}(q^2;q^4)_k}{(q^2;q^2)_{k-s}(q^2;q^2)_{k+s}(q^4;q^4)_k}\equiv(-1)^s(-q)^{(1-n)/2}[n]\pmod{\Phi_n(q)^3}.$$

It is easy to see that the above three q-supercongruences are generalizations of (1.4), (1.5), and (1.7), respectively. For $n=p^r$, letting $q\to 1$ in these three q-supercongruences, we arrive at the following generalization of (1.2): for any odd prime p, positive integer r, and nonnegative integer $s\leqslant (p^r-1)/4$,

$$\sum_{k=s}^{(p^r-1)/2+s}\frac{4k+1}{(-64)^k}\binom{2k-2s}{k-s}\binom{2k+2s}{k+s}\binom{2k}{k}\equiv p^r(-1)^{(p-1)r/2+s}\pmod{p^3}.\tag{1.9}$$

Moreover, if $s\leqslant 10$, then (1.9) also holds modulo p^{r+2}.

2. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need two lemmas on q-congruences. The first one is a simple q-congruence modulo $\Phi_n(q)$.

Lemma 2.1. *Let $n>1$ be an odd integer and let $0\leqslant s\leqslant (n-1)/4$. Then*

$$\sum_{k=s}^{(n-1)/2+s}[4k+1]\frac{(aq;q^2)_{k-s}(q;q^2)_{k+s}(q/c;q^2)_k(q/a;q^2)_k}{(q^2;q^2)_{k-s}(q^2/a;q^2)_{k+s}(cq^2;q^2)_k(aq^2;q^2)_k}c^k\equiv 0 \pmod{\Phi_n(q)}. \tag{2.1}$$

Proof. The author and Schlosser [8, Lemma 3.1] gave the simple q-congruence: for $0\leqslant k\leqslant (n-1)/2$,

$$\frac{(aq;q^2)_{(n-1)/2-k}}{(q^2/a;q^2)_{(n-1)/2-k}}\equiv(-a)^{(n-1)/2-2k}\frac{(aq;q^2)_k}{(q^2/a;q^2)_k}q^{(n-1)^2/4+k} \pmod{\Phi_n(q)}. \tag{2.2}$$

It follows that, for $s\leqslant k\leqslant (n-1)/2-s$,

$$\begin{aligned}
&\frac{(aq;q^2)_{(n-1)/2-k-s}}{(q^2/a;q^2)_{(n-1)/2-k+s}}\\
&\quad=\frac{(aq;q^2)_{(n-1)/2-k-s}/(q^2/a;q^2)_{(n-1)/2-k-s}}{(1-q^{n+1-2k-2s}/a)(1-q^{n+3-2k-2s}/a)\cdots(1-q^{n+2s-1-2k}/a)}\\
&\quad\equiv\frac{(-a)^{(n-1)/2-2k-2s}(aq;q^2)_{k+s}q^{(n-1)^2/4+k+s}}{(q^2/a;q^2)_{k+s}(1-q^{1-2k-2s}/a)(1-q^{3-2k-2s}/a)\cdots(1-q^{2s-1-2k}/a)}\\
&\quad=(-a)^{(n-1)/2-2k}\frac{(aq;q^2)_{k-s}}{(q^2/a;q^2)_{k+s}}q^{(n-1)^2/4+4ks+k+s} \pmod{\Phi_n(q)},
\end{aligned} \tag{2.3}$$

where we have used the fact that $q^n\equiv 1 \pmod{\Phi_n(q)}$, and similarly, modulo $\Phi_n(q)$,

$$\frac{(aq;q^2)_{(n-1)/2-k+s}}{(q^2/a;q^2)_{(n-1)/2-k-s}}\equiv(-a)^{(n-1)/2-2k}\frac{(aq;q^2)_{k+s}}{(q^2/a;q^2)_{k-s}}q^{(n-1)^2/4-4ks+k-s}. \tag{2.4}$$

Using the q-congruences (2.2)–(2.4), we can easily verify that, for $N=(n-1)/2$ and $s\leqslant k\leqslant N-s$,

$$\begin{aligned}
&[4(N-k)+1]\frac{(aq;q^2)_{N-k-s}(q;q^2)_{N-k+s}(q/c;q^2)_{N-k}(q/a;q^2)_{N-k}}{(q^2;q^2)_{N-k-s}(q^2/a;q^2)_{N-k+s}(cq^2;q^2)_{N-k}(aq^2;q^2)_{N-k}}c^{N-k}\\
&\quad\equiv-[4k+1]\frac{(aq;q^2)_{k-s}(q;q^2)_{k+s}(q/c;q^2)_k(q/a;q^2)_k}{(q^2;q^2)_{k-s}(q^2/a;q^2)_{k+s}(cq^2;q^2)_k(aq^2;q^2)_k}c^k \pmod{\Phi_n(q)}.
\end{aligned}$$

This indicates that the partial sum of the left-hand side of (2.1) truncated at $k=(n-1)/2-s$ is congruent to 0 modulo $\Phi_n(q)$. Furthermore, for k satisfying $(n-1)/2-s<k\leqslant (n-1)/2+s$, the q-shifted factorial $(q;q^2)_{k+s}$ has the factor $1-q^n$ and so each term indexed by k on the left-hand side of (2.1) is congruent to 0 modulo $\Phi_n(q)$. This proves the desired q-congruence (2.1). □

Following the monograph [4], the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined by (see [4])

$$
{}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q,\ z\right] = \sum_{k=0}^{\infty} \frac{(a_1;q)_k (a_2;q)_k \cdots (a_{r+1};q)_k}{(q;q)_k (b_1;q)_k \cdots (b_r;q)_k} z^k.
$$

Then a classical very-well-poised ${}_6\phi_5$ summation formula of Jackson (see [4, Appendix (II.21)]) can be written as follows:

$$
{}_6\phi_5\left[\begin{matrix} a,\ qa^{\frac12},\ -qa^{\frac12},\ b,\ c,\ q^{-n} \\ a^{\frac12},\ -a^{\frac12},\ aq/b,\ aq/c,\ aq^{n+1} \end{matrix}; q,\ \frac{aq^{n+1}}{bc}\right] = \frac{(aq;q)_n (aq/bc;q)_n}{(aq/b;q)_n (aq/c;q)_n}. \tag{2.5}
$$

We also require two q-congruences on the left-hand side of (2.1) modulo $1-aq^n$ and $a-q^n$, respectively.

Lemma 2.2. *Let $n>1$ be an odd integer and let $0 \leqslant s \leqslant (n-1)/2$. Then, modulo $(1-aq^n)(a-q^n)$,*

$$
\begin{aligned}
&\sum_{k=s}^{(n-1)/2+s} [4k+1] \frac{(aq;q^2)_{k-s} (q;q^2)_{k+s} (q/c;q^2)_k (q/a;q^2)_k}{(q^2;q^2)_{k-s} (q^2/a;q^2)_{k+s} (cq^2;q^2)_k (aq^2;q^2)_k} c^k \\
&\quad \equiv c^s (c/q)^{(n-1)/2} \frac{(q/a, q/c;q^2)_s (q^2/c;q^2)_{(n-1)/2}}{(cq/a, q;q^2)_s (cq^2;q^2)_{(n-1)/2}} [n].
\end{aligned} \tag{2.6}
$$

Proof. For $a=q^{-n}$, the left-hand side of (2.6) is equal to

$$
\begin{aligned}
&\sum_{k=s}^{(n-1)/2+s} [4k+1] \frac{(q^{1-n};q^2)_{k-s} (q;q^2)_{k+s} (q/c;q^2)_k (q^{1+n};q^2)_k}{(q^2;q^2)_{k-s} (q^{2+n};q^2)_{k+s} (cq^2;q^2)_k (q^{2-n};q^2)_k} c^k \\
&= \sum_{k=0}^{(n-1)/2} [4k+4s+1] \frac{(q^{1-n};q^2)_k (q;q^2)_{k+2s} (q/c;q^2)_{k+s} (q^{1+n};q^2)_{k+s}}{(q^2;q^2)_k (q^{2+n};q^2)_{k+2s} (cq^2;q^2)_{k+s} (q^{2-n};q^2)_{k+s}} c^{k+s} \\
&= [4s+1] \frac{(q/c;q^2)_s (q^{1+n};q^2)_s (q;q^2)_{2s}}{(cq^2;q^2)_s (q^{2-n};q^2)_s (q^{2+n};q^2)_{2s}} c^s \\
&\quad \times {}_6\phi_5\left[\begin{matrix} q^{4s+1}, q^{2s+\frac52}, -q^{2s+\frac52}, q^{2s+1}/c, q^{2s+1+n},\ q^{1-n} \\ q^{2s+\frac12}, -q^{2s+\frac12},\ cq^{2s+2},\ q^{2s+2-n}, q^{4s+2+n} \end{matrix}; q^2, c\right].
\end{aligned} \tag{2.7}
$$

Making the parameter substitutions $q \mapsto q^2$, $a = q^{4s+1}$, $b \mapsto q^{2s+1}/c$, $c = q^{2s+1+n}$, and $n \mapsto (n-1)/2$ in Jackson's ${}_6\phi_5$ summation formula (2.5), we see that the right-hand side of (2.7) can be written as

$$
\begin{aligned}
&[4s+1]\frac{(q/c;q^2)_s(q^{1+n};q^2)_s(q;q^2)_{2s}c^s}{(cq^2;q^2)_s(q^{2-n};q^2)_s(q^{2+n};q^2)_{2s}}\frac{(q^{4s+3},cq^{1-n};q^2)_{(n-1)/2}}{(cq^{2s+2},q^{2s+2-n};q^2)_{(n-1)/2}}\\
&\quad=c^s[n+4s]\frac{(q/c,q^{1+n};q^2)_s(cq^{1-n};q^2)_{(n-1)/2}(q;q^2)_{(n-1)/2+2s}}{(q^{2+n};q^2)_{2s}(cq^2,q^{2-n};q^2)_{(n-1)/2+s}}\\
&\quad=c^s\frac{(q/c,q^{1+n};q^2)_s(cq^{1-n};q^2)_{(n-1)/2}(q;q^2)_{(n-1)/2}}{(cq^{1+n},q;q^2)_s(cq^2,q^{2-n};q^2)_{(n-1)/2}}[n]\\
&\quad=c^s(c/q)^{(n-1)/2}\frac{(q/c,q^{1+n};q^2)_s(q^2/c;q^2)_{(n-1)/2}}{(cq^{1+n},q;q^2)_s(cq^2;q^2)_{(n-1)/2}}[n],
\end{aligned}
$$

which is just the $a=q^{-n}$ case of the right-hand side of (2.6). Namely, the q-congruence (2.6) is true modulo $1-aq^n$.

For $a=q^n$, the left-hand side of (2.6) is equal to

$$
\begin{aligned}
&\sum_{k=s}^{(n-1)/2+s}[4k+1]\frac{(q^{1+n};q^2)_{k-s}(q;q^2)_{k+s}(q/c;q^2)_k(q^{1-n};q^2)_k}{(q^2;q^2)_{k-s}(q^{2-n};q^2)_{k+s}(cq^2;q^2)_k(q^{2+n};q^2)_k}c^k\\
&=\sum_{k=0}^{(n-1)/2}[4k+4s+1]\frac{(q^{1+n};q^2)_k(q;q^2)_{k+2s}(q/c;q^2)_{k+s}(q^{1-n};q^2)_{k+s}}{(q^2;q^2)_k(q^{2-n};q^2)_{k+2s}(cq^2;q^2)_{k+s}(q^{2+n};q^2)_{k+s}}c^{k+s}\\
&=[4s+1]\frac{(q/c;q^2)_s(q^{1-n};q^2)_s(q;q^2)_{2s}}{(cq^2;q^2)_s(q^{2+n};q^2)_s(q^{2-n};q^2)_{2s}}c^s\\
&\quad\times{}_6\phi_5\left[\begin{matrix}q^{4s+1},q^{2s+\frac52},-q^{2s+\frac52},q^{2s+1}/c,\ q^{1+n},\ q^{2s+1-n}\\ q^{2s+\frac12},-q^{2s+\frac12},cq^{2s+2},q^{4s+2-n},q^{2s+2+n}\end{matrix};q^2,c\right].
\end{aligned}\tag{2.8}
$$

Making the parameter substitutions $q\mapsto q^2$, $a=q^{4s+1}$, $b\mapsto q^{2s+1}/c$, $c=q^{1+n}$, and $n\mapsto(n-1)/2-s$ in (2.5), we find that the right-hand side of (2.8) can be simplified as

$$
\begin{aligned}
&[4s+1]\frac{(q/c;q^2)_s(q^{1-n};q^2)_s(q;q^2)_{2s}c^s}{(cq^2;q^2)_s(q^{2+n};q^2)_s(q^{2-n};q^2)_{2s}}\frac{(q^{4s+3},cq^{2s+1-n};q^2)_{(n-1)/2-s}}{(cq^{2s+2},q^{4s+2-n};q^2)_{(n-1)/2-s}}\\
&\quad=c^s[n+2s]\frac{(q/c,q^{1-n};q^2)_s(cq^{2s+1-n};q^2)_{(n-1)/2-s}(q;q^2)_{(n-1)/2+s}}{(q^{2+n};q^2)_s(cq^2;q^2)_{(n-1)/2}(q^{2-n};q^2)_{(n-1)/2+s}}\\
&\quad=c^s\frac{(q/c,q^{1-n};q^2)_s(cq^{1-n};q^2)_{(n-1)/2}(q;q^2)_{(n-1)/2}}{(cq^{1-n};q^2)_s(cq^2,q^{2-n};q^2)_{(n-1)/2}(q;q^2)_s}[n]\\
&\quad=c^s(c/q)^{(n-1)/2}\frac{(q/c,q^{1-n};q^2)_s(q^2/c;q^2)_{(n-1)/2}}{(cq^{1-n};q^2)_s(cq^2;q^2)_{(n-1)/2}(q;q^2)_s}[n]
\end{aligned}
$$

which is the $a=q^{-n}$ case of the right-hand side of (2.6). Namely, the desired q-congruence (2.6) is true modulo $a-q^n$. Since $1-aq^n$ is coprime with $a-q^n$, we complete the proof of (2.6). □

Proof of Theorem 1.1. Note that $\Phi_n(q)$ and $(1-aq^n)(a-q^n)$ are coprime polynomials in q. Moreover, the right-hand sides of (2.6) is congruent to 0 modulo

$\Phi_n(q)$. Thus, we conclude that (2.6) holds modulo $\Phi_n(q)(1-aq^n)(a-q^n)$ for $0 \leqslant s \leqslant (n-1)/4$. Letting $a=1$ in this q-congruence yields that (1.8) is true modulo $\Phi_n(q)^3$.

In what follows, we shall prove that (1.8) is also true modulo $[n]$ for $s \leqslant 10$. Namely,

$$\sum_{k=s}^{(n-1)/2+s}[4k+1]\frac{(aq;q^2)_{k-s}(q;q^2)_{k+s}(q/c;q^2)_k(q/a;q^2)_k}{(q^2;q^2)_{k-s}(q^2/a;q^2)_{k+s}(cq^2;q^2)_k(aq^2;q^2)_k}c^k \equiv 0 \pmod{[n]},$$

or, equivalently,

$$\sum_{k=0}^{(n-1)/2}[4k+4s+1]\frac{(aq;q^2)_k(q;q^2)_{k+2s}(q/c;q^2)_{k+s}(q/a;q^2)_{k+s}}{(q^2;q^2)_k(q^2/a;q^2)_{k+2s}(cq^2;q^2)_{k+s}(aq^2;q^2)_{k+s}}c^{k+s} \equiv 0 \pmod{[n]}. \tag{2.9}$$

The proof is analogous to that of [8, Theorem 12.9] (or [10, Theorem 4.2]). For the reader's convenience, we give a detailed proof here.

Let $\zeta \neq 1$ denote an n-th root of unity, not necessarily primitive. In other words, ζ is a primitive root of unity of degree d for some $d \mid n$. Let $c_q(k)$ stand for the k-th summand on the left-hand side of (2.9). With the help of the mathematical software `Maple`, we can verify that (2.9) holds modulo $\Phi_n(q)$ for all non-negative integers $s \leqslant 10$ and positive odd integers $n \leqslant 4s-1$. This, together with (1.8), means that the q-congruence (1.8) is true modulo $\Phi_n(q)$ for all $0 \leqslant s \leqslant 10$ and odd $n>1$. The q-congruence is also true when the left-hand side is summing over k up to $n-1$, because each summand is congruent to 0 modulo $\Phi_n(q)$ for k satisfying $(n-1)/2 < k \leqslant n-1$. Taking $n=d$ leads to

$$\sum_{k=0}^{(d-1)/2} c_\zeta(k) = \sum_{k=0}^{d-1} c_\zeta(k) = 0.$$

Observing that

$$\frac{c_\zeta(\ell d+k)}{c_\zeta(\ell d)} = \lim_{q\to\zeta}\frac{c_q(\ell d+k)}{c_q(\ell d)} = \frac{c_\zeta(k)}{c_\zeta(0)},$$

we obtain

$$\begin{aligned}\sum_{k=0}^{(n-1)/2} c_\zeta(k) &= \sum_{\ell=0}^{(n/d-3)/2}\sum_{k=0}^{d-1} c_\zeta(\ell d+k) + \sum_{k=0}^{(d-1)/2} c_\zeta((n-d)/2+k)\\ &= \frac{1}{c_\zeta(0)}\sum_{\ell=0}^{(n/d-3)/2} c_\zeta(\ell d)\sum_{k=0}^{d-1} c_\zeta(k) + \sum_{k=0}^{(d-1)/2} c_\zeta((n-d)/2+k)\\ &= 0.\end{aligned}$$

This proves that $\sum_{k=0}^{(n-1)/2} c_q(k)$ is congruent to 0 modulo $\Phi_d(q)$. Since every cyclotomic polynomial $\Phi_d(q)$ is irreducible in the ring $\mathbb{Z}[q]$, we deduce that the left-hand side of (2.9) is congruent to 0 modulo

$$\prod_{d|n,\, d>1} \Phi_d(q) = [n].$$

Therefore, the q-congruence (1.8) holds modulo $[n]$. Noticing that the least common multiple of $\Phi_n(q)^3$ and $[n]$ is $[n]\Phi_n(q)^2$, we accomplish the proof. □

3. Concluding remarks

It is natural to suspect that the condition $s \leqslant 10$ for (1.8) holding modulo $[n]\Phi_n(q)^2$ is not necessary. Namely, we believe that the following stronger version of Theorem 1.1 should be true.

Conjecture 3.1. *The q-supercongruence* (1.8) *holds modulo* $[n]\Phi_n(q)^2$. *In particular, the supercongruence* (1.9) *holds modulo* p^{r+2}.

In light of the proof the second part of Theorem 1.1, to prove Conjecture 3.1, it suffices to show the following q-congruence: for any non-negative integer s and odd integer $n > 1$,

$$\sum_{k=s}^{(n-1)/2+s} [4k+1] \frac{(q;q^2)_{k-s}(q;q^2)_{k+s}(q/c;q^2)_k(q;q^2)_k}{(q^2;q^2)_{k-s}(q^2;q^2)_{k+s}(cq^2;q^2)_k(q^2;q^2)_k} c^k \equiv 0 \pmod{\Phi_n(q)}.$$

In 2012, using the WZ method, Sun [16] obtained the following refinement of (1.2): for any prime $p > 3$,

$$\sum_{k=0}^{(p-1)/2} (-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3} \equiv p(-1)^{(p-1)/2} + p^3 E_{p-3} \pmod{p^4}, \tag{3.1}$$

where E_{p-3} is the $(p-3)$th Euler number, which may be defined by

$$\frac{2}{e^x + e^{-x}} = \sum_{k=0}^{\infty} E_k \frac{x^k}{k!}.$$

Recently, a q-analogue of (3.1) was given by the author [7]. We do not know wether the supercongruence (1.9) for $r = 1$ can be generalized to the modulus p^4 case for general s. However, we find that the following refinement of (1.9) for $s = (p^r - 1)/6$ seems to be true.

Conjecture 3.2. *Let p be an odd prime and $r \geqslant 1$ with $p^r \equiv 1 \pmod 6$, and let $s = (p^r - 1)/6$. Then*

$$\sum_{k=s}^{(p^r-1)/2+s} \frac{4k+1}{(-64)^k}\binom{2k-2s}{k-s}\binom{2k+2s}{k+s}\binom{2k}{k} \equiv p^r(-1)^{(p-1)r/2+s} \pmod{p^{r+3}}.$$

It is worth mentioning that we cannot expect that the previous q-analogue of (1.9) hold modulo $[n]\Phi_n(q)^3$ for $s = (n-1)/6$. We hope that an interested reader can make progress on Conjecture 3.2 at least for the $r = 1$ case.

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