

A FURTHER q -SUPERCONGRUENCE FROM GASPER AND RAHMAN'S QUADRATIC SUMMATION

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ABSTRACT. Recently, employing Gasper and Rahman's quadratic summation and applying the method of creative microscoping in a novel way, Gu and Wang proved a q -supercongruence modulo the fourth power of a cyclotomic polynomial conjectured by Tang. In this paper, by making use of Gasper and Rahman's quadratic summation and the creative microscoping method again, we give a further generalization of Gu and Wang's result.

1. INTRODUCTION

Motivated by the work of Long [9] and Long and Ramakrishna [10], employing a ${}_7F_6$ summation of Gessel and Stanton [2], He [6] established the following supercongruence: for any odd prime p ,

$$\sum_{k=0}^{p-1} (6k+1) \frac{(\frac{1}{2})_k^3 (\frac{1}{4})_k}{k!^4 4^k} \equiv \begin{cases} (-1)^{(p+3)/4} p \Gamma_p(\frac{1}{2}) \Gamma_p(\frac{1}{4})^2 \pmod{p^2}, & \text{if } p \equiv 1 \pmod{4}, \\ 0 \pmod{p^2}, & \text{if } p \equiv 3 \pmod{4}, \end{cases} \quad (1.1)$$

where $(a)_b = \Gamma(a+b)/\Gamma(a)$ is the Pochhammer symbol also for b not being a non-negative integer, and $\Gamma_p(x)$ denotes Morita's p -adic Gamma function [11]. Note that the factor $(-1)^{(p+3)/4}$ was missing in He's original paper. Shortly afterwards, Liu [7] succeeded in proving that (1.1) is true modulo p^3 by applying another ${}_7F_6$ summation of Gessel and Stanton [2].

In 2021, using a quadratic transformation of Rahman [1, eq. (3.8.13)] and the creative microscoping method developed by the author and Zudilin [5], Liu and Wang [8] gave a q -analogue of Liu's generalization of (1.1) modulo p^3 as follows: for any positive odd integer n , modulo $[n]\Phi_n(q)^2$,

$$\begin{aligned} & \sum_{k=0}^{n-1} [6k+1] \frac{(q; q^2)_k^3 (q; q^4)_k}{(q^2; q^2)_k (q^4; q^4)_k^3} q^{k^2+k} \\ & \equiv \begin{cases} [n] \frac{(q^2; q^4)_{(n-1)/4}}{(q^4; q^4)_{(n-1)/4}} q^{(1-n)/4}, & \text{if } n \equiv 1 \pmod{4}, \\ 0, & \text{if } n \equiv 3 \pmod{4}. \end{cases} \end{aligned} \quad (1.2)$$

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In 2022, utilizing Gasper and Rahman's quadratic summation (see (2.1)) and the "creative microscoping" method again, Wei [13] further generalized (1.2) to the modulus $[n]\Phi_n(q)^3$ case. In particular, he proved that (1.2) holds modulo $[n]\Phi_n(q)^3$ for $n \equiv 3 \pmod{4}$, thus confirming an original conjecture of He [6] by letting n be an odd prime and $q \rightarrow 1$. Here and in what follows, the q -integer is defined as $[n] = \sum_{i=0}^{n-1} q^i$, the q -shifted factorial is defined as

$$(a; q)_n = \begin{cases} (1-a)(1-aq) \cdots (1-aq^{n-1}), & \text{if } n = 1, 2, \dots, \text{ or } \infty, \\ 1, & \text{if } n = 0, \\ \frac{1}{(1-aq^{-1})(1-aq^{-2}) \cdots (1-aq^{-n})}, & \text{if } n = -1, -2, \dots \end{cases}$$

For simplicity, we will often adopt the shorthand q -notation for a product of q -shifted factorials: $(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$ where $n \in \mathbb{Z}$ or $n = \infty$. Moreover, $\Phi_n(q)$ stands for the n -th cyclotomic polynomial in q , i.e.,

$$\Phi_n(q) = \prod_{\substack{1 \leq k \leq n \\ \gcd(k, n) = 1}} (q - \zeta^k),$$

where ζ is an n -th primitive root of unity.

Guo [4] and Tang [12] deduced more q -supercongruences from the aforementioned quadratic transformation of Rahman. For example, Tang [12] obtained a generalization of the second part of (1.2) modulo $\Phi_n(q)^3$: for all positive integers d , r , and n with $r < d \leq 2r$ and with $n \equiv -1 \pmod{2d}$,

$$\sum_{k=0}^{(rn+r-d)/d} [3dk + r] \frac{(q^r; q^{2d})_k (q^r, q^r, q^{d-r}; q^d)_k}{(q^d; q^d)_k (q^{2d}, q^{2d}, q^{d+2r}; q^{2d})_k} q^{d(k^2+k)/2} \equiv 0 \pmod{\Phi_n(q)^3}. \quad (1.3)$$

Recently, employing Gasper and Rahman's quadratic summation and applying the method of creative microscoping in a novel way, Gu and Wang [3] proved a refinement of a special case of (1.3): for all positive integers d , r , and n such that $d > 1$ is an odd, $r = (d+1)/2$, and $n \equiv -1 \pmod{2d}$, modulo $\Phi_n(q)^4$,

$$\sum_{k=0}^{(2n+2-d)(d-1)/(2d)} [3dk + r] \frac{(q^r; q^{2d})_k (q^r, q^r, q^{d-r}; q^d)_k}{(q^d; q^d)_k (q^{2d}, q^{2d}, q^{d+2r}; q^{2d})_k} q^{d(k^2+k)/2} \equiv 0, \quad (1.4)$$

which was originally conjectured by Tang [12, Conjecture 5.3].

For any positive odd integer $d > 1$ and positive integer n with $n \equiv -1 \pmod{2d}$, we define the function $f(d, n)$ as follows:

$$f(d, n) = \begin{cases} (n+1)/6, & \text{if } d = 3, \\ (n-4)/5, & \text{if } d = 5, \\ \lfloor (d+1)/4 \rfloor, & \text{if } d \geq 7 \text{ and } n = 2d-1, \\ (n+1)/d + (d-1)/2, & \text{if } d \geq 7 \text{ and } n \geq 4d-1, \end{cases} \quad (1.5)$$

where $\lfloor x \rfloor$ denotes the largest integer not exceeding x . In this paper, inspired by Gu and Wang's work [3], we shall establish the following further generalization of (1.4).

Theorem 1.1. *Let $d > 1$ be an odd integer, and $r = (d + 1)/2$. Let n be a positive integer satisfying $n \equiv -1 \pmod{2d}$, and let $0 \leq s \leq f(d, n)$. Then, modulo $\Phi_n(q)^4$,*

$$\sum_{k=s}^{(2n+2-d)(d-1)/(2d)} [3dk + r] \frac{(q^r; q^d)_{k-2s} (q^r; q^d)_{k+2s} (q^{d-r}; q^d)_k (q^r; q^{2d})_k}{(q^d; q^d)_k (q^{2d}; q^{2d})_{k-s} (q^{2d}; q^{2d})_{k+s} (q^{d+2r}; q^{2d})_k} q^{d(k^2+k)/2} \equiv 0. \quad (1.6)$$

It is clear that the $s = 0$ case reduces to (1.4). Taking $d = 3, 5$, we deduce two conclusions from Theorem 1.1.

Corollary 1.2. *Let n be a positive integer satisfying $n \equiv 5 \pmod{6}$, and let $0 \leq s \leq (n + 1)/6$. Then, modulo $\Phi_n(q)^4$,*

$$\sum_{k=s}^{(2n-1)/3} [9k + 2] \frac{(q^2; q^3)_{k-2s} (q^2; q^3)_{k+2s} (q; q^3)_k (q^2; q^6)_k}{(q^3; q^3)_k (q^6; q^6)_{k-s} (q^6; q^6)_{k+s} (q^7; q^6)_k} q^{3(k^2+k)/2} \equiv 0.$$

In particular, for any prime $p \equiv 5 \pmod{6}$ and $0 \leq s \leq (p + 1)/6$,

$$\sum_{k=s}^{(2p-1)/3} \frac{(9k + 2) \left(\frac{2}{3}\right)_{k-2s} \left(\frac{2}{3}\right)_{k+2s} \left(\frac{1}{3}\right)_k^2}{k!(k-s)!(k+s)! \left(\frac{7}{6}\right)_k 4^k} \equiv 0 \pmod{p^4}.$$

Corollary 1.3. *Let n be a positive integer satisfying $n \equiv 9 \pmod{10}$, and let $0 \leq s \leq (n - 4)/5$. Then, modulo $\Phi_n(q)^4$,*

$$\sum_{k=s}^{(4n-6)/5} [15k + 3] \frac{(q^3; q^5)_{k-2s} (q^3; q^5)_{k+2s} (q^2; q^5)_k (q^3; q^{10})_k}{(q^5; q^5)_k (q^{10}; q^{10})_{k-s} (q^{10}; q^{10})_{k+s} (q^{11}; q^{10})_k} q^{5(k^2+k)/2} \equiv 0.$$

In particular, for any prime $p \equiv 9 \pmod{10}$ and $0 \leq s \leq (p - 4)/5$,

$$\sum_{k=s}^{(4p-6)/5} \frac{(5k + 1) \left(\frac{3}{5}\right)_{k-2s} \left(\frac{3}{5}\right)_{k+2s} \left(\frac{2}{5}\right)_k \left(\frac{3}{10}\right)_k}{k!(k-s)!(k+s)! \left(\frac{11}{10}\right)_k 4^k} \equiv 0 \pmod{p^4}.$$

We shall prove Theorem 1.1 by making use of the creative microscoping method [5] and Gasper and Rahman's quadratic summation (2.1) again.

2. PROOF OF THEOREM 1.1

Recall that the *basic hypergeometric series* ${}_{r+1}\phi_r$ (see [1]) is defined as

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_k z^k}{(q, b_1, \dots, b_r; q)_k}.$$

Then Gasper and Rahman's quadratic summation (see [1, (3.8.12)]) can be written as follows:

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{1 - aq^{3k}}{1 - a} \frac{(a, b, q/b; q)_k (d, f, a^2q/df; q^2)_k}{(aq/d, aq/f, df/a; q)_k (q^2, aq^2/b, abq; q^2)_k} q^k \\ & + \frac{(aq, f/a, b, q/b; q)_{\infty} (d, aq^2/df, fq^2/d, df^2q/a^2; q^2)_{\infty}}{(a/f, fq/a, aq/d, df/a; q)_{\infty} (aq^2/b, abq, fq/ab, bf/a; q^2)_{\infty}} \end{aligned}$$

$$\begin{aligned}
& \times {}_3\phi_2 \left[\begin{matrix} f, bf/a, fq/ab \\ fq^2/d, df^2q/a^2 \end{matrix}; q^2, q^2 \right] \\
& = \frac{(aq, f/a; q)_\infty (aq^2/bd, abq/d, bdf/a, dfq/ab; q^2)_\infty}{(aq/d, df/a; q)_\infty (aq^2/b, abq, bf/a, fq/ab; q^2)_\infty}. \tag{2.1}
\end{aligned}$$

Note that $(q^{-2n}; q^2)_\infty = 0$ for any non-negative integer n . Putting $d = q^{-2n}$ in (2.1), we get the following identity:

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{1 - aq^{3k}}{1 - a} \frac{(a, b, q/b; q)_k (q^{-2n}, f, a^2q^{1+2n}f; q^2)_k}{(aq^{1+2n}, aq/f, q^{-2n}f/a; q)_k (q^2, aq^2/b, abq; q^2)_k} q^k \\
& = \frac{(aq, f/a; q)_\infty (aq^{2+2n}/b, abq^{1+2n}, q^{-2n}bf/a, q^{1-2n}f/ab; q^2)_\infty}{(aq^{1+2n}, q^{-2n}f/a; q)_\infty (aq^2/b, abq, bf/a, fq/ab; q^2)_\infty} \\
& = \frac{(aq; q)_{2n} (q^{-2n}bf/a, q^{1-2n}f/ab; q^2)_n}{(q^{-2n}f/a; q)_{2n} (aq^2/b, abq; q^2)_n} \\
& = \frac{(aq; q)_{2n} (aq^2/bf, abq/f; q^2)_n}{(aq/f; q)_{2n} (aq^2/b, abq; q^2)_n}.
\end{aligned}$$

Further, letting $n \rightarrow \infty$ in the above identity, we arrive at the following formula:

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{1 - aq^{3k}}{1 - a} \frac{(a, b, q/b; q)_k (f; q^2)_k}{(aq/f; q)_k (q^2, aq^2/b, abq; q^2)_k} q^{(k^2+k)/2} \left(\frac{a}{f}\right)^k \\
& = \frac{(aq, aq^2, aq^2/bf, abq/f; q^2)_\infty}{(aq/f, aq^2/f, aq^2/b, abq; q^2)_\infty}, \tag{2.2}
\end{aligned}$$

which was already observed by Gu and Wang [3, eq. (2.2)].

Applying (2.2), we are able to give the following parametric generalization of Theorem 1.1.

Theorem 2.1. *Let $d > 1$ be an odd integer, $r = (d + 1)/2$, and a an indeterminate. Let n be a positive integer satisfying $n \equiv -1 \pmod{2d}$, and let $0 \leq s \leq f(d, n)$. Then, modulo $(1 - aq^{(r-1)n})(a - q^{(r-1)n})(1 - aq^{rn})(1 - aq^{(d+r-1)n})$,*

$$\sum_{k=s}^M \frac{1 - aq^{3dk+r}}{1 - q} \frac{(q^r/a; q^d)_{k-2s} (aq^r; q^d)_{k+2s} (aq^{d-r}; q^d)_k (aq^r; q^{2d})_k}{(q^d; q^d)_k (q^{2d}; q^{2d})_{k-s} (a^2q^{2d}; q^{2d})_{k+s} (q^{d+2r}; q^{2d})_k} q^{d(k^2+k)/2} \equiv 0, \tag{2.3}$$

where $M = (2n + 2 - d)(d - 1)/(2d)$.

Proof. Replacing k by $k + s$, we can rewrite the left-hand side of (2.3) as

$$\begin{aligned}
& \sum_{k=0}^{M-s} \frac{1 - aq^{3dk+3ds+r}}{1 - q} \frac{(q^r/a; q^d)_{k-s} (aq^r; q^d)_{k+3s} (aq^{d-r}; q^d)_{k+s} (aq^r; q^{2d})_{k+s}}{(q^d; q^d)_{k+s} (q^{2d}; q^{2d})_k (a^2q^{2d}; q^{2d})_{k+2s} (q^{d+2r}; q^{2d})_{k+s}} q^{d((k+s)^2+k+s)/2} \\
& = \frac{(q^r/a; q^d)_{-s} (aq^r; q^d)_{3s} (aq^{d-r}; q^d)_s (aq^r; q^{2d})_s}{(q^d; q^d)_s (a^2q^{2d}; q^{2d})_{2s} (q^{d+2r}; q^{2d})_s} q^{d(s^2+s)/2}
\end{aligned}$$

$$\times \sum_{k=0}^{M-s} \frac{1 - aq^{3dk+3ds+r}}{1 - q} \frac{(q^{r-ds}/a, aq^{r+3ds}, aq^{d+ds-r}; q^d)_k (aq^{r+2ds}; q^{2d})_k}{(q^{d+ds}; q)_k (q^{2d}, a^2q^{2d+4ds}, q^{d+2r+2ds}; q^{2d})_k} q^{d(k^2+2ks+k)/2},$$

where $M = (2n + 2 - d)(d - 1)/(2d)$.

Letting $q \mapsto q^d$, $a \mapsto aq^{r+3ds}$, $b = q^{r-ds}/a$, and $f = aq^{r+2ds}$ in (2.2), we have

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{1 - aq^{3dk+3ds+r}}{1 - aq^{r+3ds}} \frac{(aq^{r+3ds}, q^{r-ds}/a, aq^{d+ds-r}; q^d)_k (aq^{r+2ds}; q^{2d})_k}{(q^{d+ds}; q)_k (q^{2d}, a^2q^{2d+4ds}, q^{d+2r+2ds}; q^{2d})_k} q^{d(k^2+2ks+k)/2} \\ &= \frac{(aq^{d+r+3ds}, aq^{2d+r+3ds}, aq^{2d-r+2ds}, q^{d+r}/a; q^{2d})_{\infty}}{(q^{d+ds}, q^{2d+ds}, a^2q^{2d+4ds}, q^{d+2r+2ds}; q^{2d})_{\infty}}, \end{aligned}$$

or equivalently,

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{1 - aq^{3dk+3ds+r}}{1 - q} \frac{(aq^{r+3ds}, q^{r-ds}/a, aq^{d+ds-r}; q^d)_k (aq^{r+2ds}; q^{2d})_k}{(q^{d+ds}; q)_k (q^{2d}, a^2q^{2d+4ds}, q^{d+2r+2ds}; q^{2d})_k} q^{d(k^2+2ks+k)/2} \\ &= \frac{(aq^{d+r+3ds}, aq^{r+3ds}, aq^{2d-r+2ds}, q^{d+r}/a; q^{2d})_{\infty}}{(1 - q)(q^{d+ds}, q^{2d+ds}, a^2q^{2d+4ds}, q^{d+2r+2ds}; q^{2d})_{\infty}} \\ &= \frac{(aq^{r+3ds}; q^d)_{\infty} (aq^{2d-r+2ds}, q^{d+r}/a; q^{2d})_{\infty}}{(1 - q)(q^{d+ds}, q^{2d+ds}, a^2q^{2d+4ds}, q^{d+2r+2ds}; q^{2d})_{\infty}}. \end{aligned} \quad (2.4)$$

Keep in mind that $f(d, n)$ is given by (1.5) and $0 \leq s \leq f(d, n)$. For $a = q^{-(r-1)n}$, we can check that $(aq^{r+3ds}; q^d)_k = (q^{r+3ds-(r-1)n}; q^d)_k = 0$ for $k > ((r-1)n-r)/d-3s$, and that $(aq^{d+r+3ds}; q^d)_{\infty} = (q^{d+r+3ds-(r-1)n}; q^d)_{\infty} = 0$. Thus, from (2.4) we can deduce the following identity:

$$\begin{aligned} & \sum_{k=0}^{M-s} \frac{1 - q^{3dk+3ds+r-(r-1)n}}{1 - q} \frac{(q^{r-ds+(r-1)n}, q^{r+3ds-(r-1)n}, q^{d+ds-r-(r-1)n}; q^d)_k}{(q^{d+ds}; q)_k} \\ & \times \frac{(q^{r+2ds-(r-1)n}; q^{2d})_k q^{d(k^2+2ks+k)/2}}{(q^{2d}, q^{2d+4ds-2(r-1)n}, q^{d+2r+2ds}; q^{2d})_k} = 0, \end{aligned}$$

where we have utilized the inequalities $0 \leq ((r-1)n-r)/d-3s \leq (2n+2-d)(d-1)/(2d) - s = M-s$. This proves that the left-hand side of (2.3) is equal to 0 when $a = q^{-(r-1)n}$. Namely, the q -congruence (2.3) holds modulo $1 - aq^{(r-1)n}$.

For $a = q^{(r-1)n}$, we can check that $(q^{r-ds}/a; q^d)_k = (q^{r-ds-(r-1)n}) = 0$ for $k > ((r-1)n-r)/d-s$, and that $(q^{d+r}/a; q^{2d})_{\infty} = (q^{d+r-(r-1)n}; q^{2d})_{\infty} = 0$. Hence, from (2.4) we conclude the following identity:

$$\begin{aligned} & \sum_{k=0}^{M-s} \frac{1 - q^{3dk+3ds+r+(r-1)n}}{1 - q} \frac{(q^{r-ds-(r-1)n}, q^{r+3ds+(r-1)n}, q^{d+ds-r+(r-1)n}; q^d)_k}{(q^{d+ds}; q)_k} \\ & \times \frac{(q^{r+2ds+(r-1)n}; q^{2d})_k q^{d(k^2+2ks+k)/2}}{(q^{2d}, q^{2d+4ds+2(r-1)n}, q^{d+2r+2ds}; q^{2d})_k} = 0, \end{aligned}$$

where we have applied $0 \leq ((r-1)n-r)/d-s \leq M-s$. This implies that the left-hand side of (2.3) is equal to 0 when $a = q^{(r-1)n}$. Namely, the q -congruence (2.3) holds modulo $a - q^{(r-1)n}$.

For $a = q^{-rn}$, we can verify that $(aq^{d+ds-r}; q^d)_k = (q^{d+ds-r-rn}; q^d) = 0$ for $k > (rn+r)/d-s-1$, and that $(aq^{2d-r+2ds}; q^{2d})_\infty = (q^{2d-r+2ds-rn}; q^{2d})_\infty = 0$. Therefore, the identity (2.4) leads to

$$\begin{aligned} & \sum_{k=0}^{M-s} \frac{1 - q^{3dk+3ds+r-rn}}{1 - q} \frac{(q^{r-ds+rn}, q^{r+3ds-rn}, q^{d+ds-r-rn}; q^d)_k}{(q^{d+ds}; q)_k} \\ & \times \frac{(q^{r+2ds-rn}; q^{2d})_k q^{d(k^2+2ks+k)/2}}{(q^{2d}, q^{2d+4ds-2rn}, q^{d+2r+2ds}; q^{2d})_k} = 0, \end{aligned}$$

where we have applied $0 \leq (rn+r)/d-s-1 \leq M-s$. This means that the left-hand side of (2.3) is equal to 0 when $a = q^{-rn}$. Namely, the q -congruence (2.3) holds modulo $1 - aq^{rn}$.

Finally, for $a = q^{-(d+r-1)n}$, we can check that $(aq^{r+3ds}; q^d)_k = (q^{r+3ds-(d+r-1)n}; q^d)_k = 0$ for $k > ((d+r-1)n-r)/d-3s$, and that $(aq^{d+r+3ds}; q^d)_\infty = (q^{d+r+3ds-(d+r-1)n}; q^d)_\infty = 0$. Thereby, in this case the identity (2.4) reduces to

$$\begin{aligned} & \sum_{k=0}^{M-s} \frac{1 - q^{3dk+3ds+r-(d+r-1)n}}{1 - q} \frac{(q^{r-ds+(d+r-1)n}, q^{r+3ds-(d+r-1)n}, q^{d+ds-r-(d+r-1)n}; q^d)_k}{(q^{d+ds}; q)_k} \\ & \times \frac{(q^{r+2ds-(d+r-1)n}; q^{2d})_k q^{d(k^2+2ks+k)/2}}{(q^{2d}, q^{2d+4ds-2(d+r-1)n}, q^{d+2r+2ds}; q^{2d})_k} = 0, \end{aligned}$$

where we have used $0 \leq ((d+r-1)n-r)/d-3s \leq M-s$. This indicates that the left-hand side of (2.3) is equal to 0 when $a = q^{-(d+r-1)n}$. Namely, the q -congruence (2.3) holds modulo $1 - aq^{(d+r-1)n}$.

Noticing that $(1 - aq^{(r-1)n})$, $(a - q^{(r-1)n})$, $(1 - aq^{rn})$, and $(1 - aq^{(d+r-1)n})$ are pairwise coprime polynomials in q , we complete the proof of the theorem. \square

Proof of Theorem 1.1. When $a = 1$, the modulus $(1 - aq^{(r-1)n})(a - q^{(r-1)n})(1 - aq^{rn})(1 - aq^{(d+r-1)n})$ of (2.3) has the factor $\Phi_n(q)^4$. Since $n \equiv -1 \pmod{2d}$ and $r = (d+1)/2$, we know that the minimum positive integer k such that $(q^{d+2r}; q^{2d})_k = (q^{2d+1}; q^{2d})_k$ is divisible by $\Phi_n(q)$ is $((2d-1)n-1)/(2d)$, which is greater than $M = (2n+2-d)(d-1)/(2d)$. This means that $(q^d; q^d)_k (q^{2d}; q^{2d})_{k-s} (q^{d+2r}; q^{2d})_k$ is always coprime with $\Phi_n(q)$ for $s \leq k \leq M$. In addition, for $0 \leq s \leq f(d, n)$ and $k \leq M$, we can easily check that $k+s \leq n-1$ and so $(q^{2d}; q^{2d})_{k+s}$ is also coprime with $\Phi_n(q)$. Finally, the minimum positive k such that $(q^{(d-1)/2}; q^d)_k$ is divisible by $\Phi_n(q)$ is $((d+1)n+d+1)/(2d)$, and so the denominator of the fraction

$$(q^r; q^d)_{-s} = \frac{1}{(1 - q^{r-d})(1 - q^{r-2d}) \cdots (1 - q^{r-ds})}$$

is coprime with $\Phi_n(q)$ for $0 \leq s \leq f(d, n)$. Hence, the q -supercongruence (1.6) immediately follows from (2.3) by taking $a = 1$. \square

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