Refinements of Van Hamme's (E.2) and (F.2) supercongruences and two supercongruences by Swisher

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Abstract. In 1997, Van Hamme proposed 13 supercongruences on truncated hypergeometric series. Van Hamme's (B.2) supercongruence was first confirmed by Mortenson and received a WZ proof by Zudilin later. In 2012, using the WZ method again, Sun extended Van Hamme's (B.2) supercongruence to the modulus p^4 case, where p is an odd prime. In this paper, by using a more general WZ pair, we generalize Hamme's (E.2) and (F.2) supercongruences, as well as two supercongruences by Swisher, to the modulus p^4 case. Our generalizations of these supercongruences are related to Euler polynomials. We also put forward a relevant conjecture on a q-supercongruence for further study.

Keywords: supercongruence; Legendre symbol; Euler number; Euler polynomials; q-congruence

AMS Subject Classifications: 33C20; 11B75; 11B65; 33E50

1. Introduction

In 1914, Ramanujan [16] listed a number of fast convergent series for $1/\pi$. Although Bauer's formula [1] is not listed in [16], it gives such an example:

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3}=\frac{2}{\pi}, \tag{1.1}$$

where the Pochhammer symbol is defined as $(a)_k=a(a+1)\cdots(a+k-1)$ for $k\in\mathbb{Z}^+$ and $(a)_0=1$. Ramanujan's formulas for $1/\pi$ became famous in 1980's when they were discovered to provide efficient algorithms for evaluating decimal digits of π (see [2]).

In 1997, Van Hamme [23] observed that 13 Ramanujan-type series have neat p-adic

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analogues, such as

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3}\equiv p(-1)^{(p-1)/2}\pmod{p^3},\tag{1.2}$$

$$\sum_{k=0}^{(p-1)/3}(-1)^k(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3}\equiv p\pmod{p^3}\quad\text{for } p\equiv 1\pmod 3,\tag{1.3}$$

$$\sum_{k=0}^{(p-1)/4}(-1)^k(8k+1)\frac{(\frac{1}{4})_k^3}{k!^3}\equiv p(-1)^{(p-1)/4}\pmod{p^3}\text{ for } p\equiv 1\pmod 4,\tag{1.4}$$

(tagged (B.2), (E.2), and (F.2) in his list, respectively), where p is an odd prime. Note that we may compute the sum in (1.2) for k up to $p-1$, since the p-adic order of $(\frac{1}{2})_k/k!$ is 1 for k satisfying $(p+1)/2\leqslant k\leqslant p-1$. Supercongruences of this kind are known as Ramanujan-type supercongruences nowadays. The supercongruence (1.2) was first confirmed by Mortenson [15] employing a ${}_6F_5$ transformation and the p-adic Gamma function in 2008, and reproved by Zudilin [27] with the help of the WZ (Wilf–Zeilberger [25,26]) method, and by Long [12] using hypergeometric identities. Swisher [21] utilized Long's method to prove (1.3) and (1.4). Almost at the same time, He [10] also applied Long's method to present a generalization of (1.3) and (1.4).

In 2012, making use of the WZ method again, Sun [19] proved the following refinement of (1.2): for any prime $p>3$,

$$\sum_{k=0}^{m}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3}\equiv p(-1)^{(p-1)/2}+p^3E_{p-3}\pmod{p^4},\tag{1.5}$$

where $m=p-1$ or $(p-1)/2$, and E_{p-3} is the $(p-3)$th Euler number, which may be defined as

$$\sum_{n=0}^{\infty}E_n\frac{x^n}{n!}=\frac{2}{e^x+e^{-x}}.$$

Recall that the Euler polynomials $E_n(x)$ can be defined by the following generating function:

$$\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}=\frac{2e^{xt}}{e^t+1}.$$

It is easy to see that $E_{p-3}=2^{p-3}E_{p-3}(\frac{1}{2})\equiv\frac{1}{4}E_{p-3}(\frac{1}{2})\pmod p$ for any prime $p>3$.

In this paper, we shall establish the following refinements of (1.3) and (1.4).

Theorem 1.1. *Let $p\equiv 1\pmod 3$ be a prime. Then*

$$\sum_{k=0}^{M}(-1)^k(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3}\equiv p+\frac{p^3}{9}E_{p-3}(\tfrac{1}{3})\pmod{p^4},\tag{1.6}$$

where $M=(p-1)/3$ or $p-1$.

Theorem 1.2. *Let $p \equiv 1 \pmod 4$ be a prime. Then*

$$\sum_{k=0}^{M}(-1)^k(8k+1)\frac{(\frac{1}{4})_k^3}{k!^3} \equiv p(-1)^{(p-1)/4} + \frac{p^3}{16}E_{p-3}(\tfrac{1}{4}) \pmod{p^4}, \tag{1.7}$$

where $M = (p-1)/4$ or $p-1$.

In 2015, Swisher [21, (1), (2)] proved the following supercongruences, which are very similar to (1.3) and (1.4):

$$\sum_{k=0}^{(2p-1)/3}(-1)^k(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3} \equiv -2p \pmod{p^3} \quad \text{for } p \equiv 2 \pmod 3, \tag{1.8}$$

$$\sum_{k=0}^{(3p-1)/4}(-1)^k(8k+1)\frac{(\frac{1}{4})_k^3}{k!^3} \equiv 3p(-1)^{(3p-1)/4} \pmod{p^3} \text{ for } p \equiv 3 \pmod 4, \tag{1.9}$$

where p is a prime.

In this paper, we shall give the following refinements of (1.8) and (1.9).

Theorem 1.3. *Let $p \equiv 2 \pmod 3$ be a prime. Then*

$$\sum_{k=0}^{M}(-1)^k(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3} \equiv -2p + \frac{8p^3}{9}E_{p-3}(\tfrac{1}{3}) \pmod{p^4}, \tag{1.10}$$

where $M = (2p-1)/3$ or $p-1$.

Theorem 1.4. *Let $p \equiv 3 \pmod 4$ be a prime. Then*

$$\sum_{k=0}^{M}(-1)^k(8k+1)\frac{(\frac{1}{4})_k^3}{k!^3} \equiv 3p(-1)^{(3p-1)/4} + \frac{27p^3}{16}E_{p-3}(\tfrac{1}{4}) \pmod{p^4}, \tag{1.11}$$

where $M = (3p-1)/4$ or $p-1$.

For any p-adic integer x, let $\langle x\rangle_p$ stand for the least non-negative residue of x modulo p. We shall prove Theorems 1.1–1.4 by establishing the following more general result.

Theorem 1.5. *Let $p > 3$ be a prime and let α be a p-adic integer. Then*

$$\sum_{k=0}^{p-1}(-1)^k(2k+\alpha)\frac{(\alpha)_k^3}{(1)_k^3} \equiv (-1)^{\langle-\alpha\rangle_p}(\alpha+\langle-\alpha\rangle_p)+(\alpha+\langle-\alpha\rangle_p)^3E_{p-3}(\alpha) \pmod{p^4}, \tag{1.12}$$

and

$$\sum_{k=0}^{\langle-\alpha\rangle_p}(-1)^k(2k+\alpha)\frac{(\alpha)_k^3}{(1)_k^3} \equiv (-1)^{\langle-\alpha\rangle_p}(\alpha+\langle-\alpha\rangle_p)+(\alpha+\langle-\alpha\rangle_p)^3E_{p-3}(\alpha) \pmod{p^4}. \tag{1.13}$$

Note that the case $\alpha=-1/2$ of (1.13) was proved by the first author and Liu [8, Theorem 1.1]. The modulus p^3 case of Theorem 1.5 was obtained by the second author and Sun [24, Theorem 1.3]. Moreover, q-analogues of this result modulo p^3 for $\alpha=r/d$ being a rational p-adic integer and $p\equiv \pm r \pmod{d}$ were given by the first author [6, Theorems 1.5 and 1.6] and the first author and Zudilin [9, Theorems 4.9 and 4.10].

The paper is organized as follows. In the next section, we give some preliminary results. The proof of Theorem 1.5 will be given in Section 3. Finally, in Section 4, we present an open problem on a q-supercongruence for further study.

2. Preliminary Results

We first recall some basic properties of the harmonic numbers $H_n^{(m)}=\sum_{k=1}^n 1/k^m$ and the Euler polynomials.

Lemma 2.1 (Lehmer [11])**.** *For any prime $p>3$,*

$$H_{p-1}\equiv 0 \pmod{p^2} \quad \text{and} \quad H_{p-1}^{(2)}\equiv H_{(p-1)/2}^{(2)}\equiv 0 \pmod{p}.$$

Lemma 2.2 ([13])**.** *Let $x\in\mathbb{C}$ and $n,m\in\mathbb{N}$. Then*

$$\begin{aligned}
E_{2n}(0)&=E_{2n}(1)=0,\\
E_n(1-x)&=(-1)^nE_n(x),\\
\sum_{k=1}^{n}(-1)^k k^m&=\frac{(-1)^n}{2}\left(E_m(n+1)+(-1)^nE_m(0)\right).
\end{aligned}$$

In order to prove Theorem 1.5, we need to establish another six lemmas. For the sake of convenience, we shall always assume that $\alpha+\langle -\alpha\rangle_p=pt$ from now on.

Lemma 2.3. *If $\alpha\equiv 0 \pmod{p}$, then Theorem 1.5 holds.*

Proof. It is easy to see that

$$\begin{aligned}
\sum_{k=0}^{p-1}(-1)^k(2k+pt)\frac{(pt)_k^3}{(1)_k^3}&=pt+\sum_{k=1}^{p-1}(-1)^k(2k+pt)\frac{(pt)_k^3}{(1)_k^3}\\
&\equiv pt+2p^3t^3\sum_{k=1}^{p-1}(-1)^k k\frac{(1+pt)_{k-1}^3}{(1)_k^3}\\
&\equiv pt+2p^3t^3\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\\
&=pt+2p^3t^3\left(\frac{1}{2}H_{(p-1)/2}^{(2)}-H_{p-1}^{(2)}\right) \pmod{p^4}.
\end{aligned}$$

Then, by Lemma 2.1,

$$\sum_{k=0}^{p-1}(-1)^k(2k+pt)\frac{(pt)_k^3}{(1)_k^3}\equiv pt \pmod{p^4}.$$

Now, it suffices to show $E_{p-3}(pt)\equiv 0 \pmod p$. In fact, by Lemma 2.2 we have

$$E_{p-3}(pt)\equiv E_{p-3}(0)=0 \pmod p.$$

This proves (1.12). Further, the proof of (1.13) is trivial. $\square$

Lemma 2.4. *For any prime $p>3$ and $\alpha\in\mathbb{Z}_p$ with $\alpha\not\equiv 0 \pmod p$, we have the following congruence modulo p^4:*

$$\frac{(\alpha)_{2p-1}}{(1)_{p-1}^2}\equiv\begin{cases} pt, & \text{if } \langle-\alpha\rangle_p=p-1,\\ -\dfrac{p^2t(t+1)}{\langle-\alpha\rangle_p+1}\left(1+2pH_{\langle-\alpha\rangle_p}+\dfrac{p(t+2)}{\langle-\alpha\rangle_p+1}\right), & \text{otherwise.}\end{cases}$$

Proof. If $\langle-\alpha\rangle_p=p-1$, then

$$\begin{aligned}\frac{(\alpha)_{2p-1}}{(1)_{p-1}^2}&=\frac{(1+p(t-1))_{2p-1}}{(1)_{p-1}^2}=\frac{pt(1+p(t-1))_{p-1}(1+pt)_{p-1}}{(1)_{p-1}^2}\\&\equiv pt\left(1+p(t-1)H_{p-1}+\frac{p^2(t-1)^2}{2}(H_{p-1}^2-H_{p-1}^{(2)})\right)\\&\quad\times\left(1+ptH_{p-1}+\frac{p^2t^2}{2}(H_{p-1}^2-H_{p-1}^{(2)})\right)\\&\equiv pt \pmod{p^4},\end{aligned}$$

where we have used Lemma 2.1.

Below, we assume that $1\leqslant\langle-\alpha\rangle_p\leqslant p-2$ and let $a=\langle-\alpha\rangle_p$. Since

$$H_{p-2-a}=\sum_{j=1}^{p-2-a}\frac{1}{j}=\sum_{j=a+2}^{p-1}\frac{1}{p-j}\equiv-(H_{p-1}-H_{a+1})\equiv H_{a+1} \pmod p,$$

we have

$$\begin{aligned}\frac{(\alpha)_{2p-1}}{(1)_{p-1}^2}&=\frac{(-a+pt)_{2p-1}}{(1)_{p-1}^2}=\frac{p^2t(t+1)(-a+pt)_a(1+pt)_{p-1}(1+p(t+1))_{p-2-a}}{(1)_{p-1}^2}\\&\equiv p^2t(t+1)\frac{(-a)_a(1)_{p-2-a}(1-ptH_a+p(t+1)H_{p-2-a})}{(1)_{p-1}}\\&=-p^2t(t+1)\frac{(1)_a(1-ptH_a+p(t+1)H_{p-2-a})}{(1-p)_{a+1}}\\&\equiv-\frac{p^2t(t+1)(1-ptH_a+p(t+1)H_{p-2-a}+pH_{a+1})}{a+1}\\&\equiv-\frac{p^2t(t+1)}{a+1}\left(1+2pH_a+\frac{p(t+2)}{a+1}\right) \pmod{p^4}.\end{aligned}$$

This concludes the proof. $\square$

Lemma 2.5. *For any positive integer n, we have the following identities:*

$$\sum_{k=1}^{n} \frac{(-1)^k}{k^2\binom{n}{k}} = H_n^{(2)} + 2\sum_{k=1}^{n} \frac{(-1)^k}{k^2}, \tag{2.1}$$

$$\sum_{k=1}^{n} \frac{(-1)^k}{k}\binom{n}{k} = -H_n, \tag{2.2}$$

$$\sum_{k=1}^{n} \frac{(-1)^k}{k^2}\binom{n}{k} = -\frac{1}{2}(H_n^{(2)} + H_n^2), \tag{2.3}$$

$$\sum_{k=1}^{n} \frac{(-1)^k}{k}\binom{n}{k} H_k = -H_n^{(2)}. \tag{2.4}$$

Proof. We think these identities should be known. For example, the identity (2.2) was recorded in [5]. A q-analogue of (2.2) was given by Van Hamme [22]. The identity (2.3) is a special case of a result due to Dilcher [4]. Of course, all of these identities can be automatically found and proved by using the package `Sigma` in Mathematica (cf. [17]). $\square$

Lemma 2.6. *For any prime p and $\alpha \in \mathbb{Z}_p$ with $\alpha \not\equiv 0 \pmod{p}$, we have*

$$\frac{(\alpha)_p^2}{(1)_{p-1}^2} \sum_{k=1}^{\langle -\alpha\rangle_p} (-1)^k \frac{(\alpha)_{p+k-1}}{(1)_{p-k}(\alpha)_k^2} \equiv (-1)^{\langle -\alpha\rangle_p+1} p^3 t^3 \left(H_{\langle -\alpha\rangle_p}^{(2)} + 2\sum_{k=1}^{\langle -\alpha\rangle_p} \frac{(-1)^k}{k^2} \right) \pmod{p^4}.$$

Proof. Similarly as before, let $a = \langle -\alpha\rangle_p$. It is easy to see that

$$\begin{aligned} \frac{(\alpha)_p^3}{(1)_{p-1}^3} &= \frac{(pt)^3(\alpha)_a^3(1+pt)_{p-1-a}^3}{(1)_{p-1}^3} \\ &\equiv \frac{(pt)^3(-a)_a^3(1)_{p-1-a}^3}{(1)_{p-1}^3} \\ &= \frac{(pt)^3(-1)^a(-a)_a^3}{(1-p)_a^3} \\ &\equiv p^3t^3 \pmod{p^4}. \end{aligned} \tag{2.5}$$

For any positive integer k, there hold

$$(\alpha)_{p+k-1} = (\alpha)_p(\alpha+p)_{k-1},$$

and

$$(1)_{p-k} = (-1)^{k-1}\frac{(1)_{p-1}}{(1-p)_{k-1}}.$$

Then, by (2.1) we obtain

$$
\begin{aligned}
\frac{(\alpha)_p^2}{(1)_{p-1}^2}\sum_{k=1}^{a}(-1)^k\frac{(\alpha)_{p+k-1}}{(1)_{p-k}(\alpha)_k^2} &= -\frac{(\alpha)_p^3}{(1)_{p-1}^3}\sum_{k=1}^{a}\frac{(\alpha+p)_{k-1}(1-p)_{k-1}}{(\alpha)_k^2}\\
&\equiv -p^3t^3\sum_{k=1}^{a}\frac{(1)_{k-1}}{(-a+k-1)^2(-a)_{k-1}}\\
&= -p^3t^3\sum_{k=0}^{a-1}\frac{(-1)^k}{(-a+k)^2\binom{a}{k}}\\
&= (-1)^{a+1}p^3t^3\sum_{k=1}^{a}\frac{(-1)^k}{k^2\binom{a}{k}}\\
&= (-1)^{a+1}p^3t^3\left(H_a^{(2)}+2\sum_{k=1}^{a}\frac{(-1)^k}{k^2}\right) \pmod{p^4},
\end{aligned}
$$

as desired. □

Lemma 2.7. *For any prime $p>3$ and $\alpha\in\mathbb{Z}_p$, we have*

$$
\frac{(\alpha)_p^2(\alpha)_{p+a}}{(1)_{p-1}^2(1)_{p-a-1}(\alpha)_{a+1}^2}\equiv pt+p^2t(t+1)H_a+\frac{p^3t(t+1)^2}{2}H_a^2+\frac{p^3t(t^2+4t+1)}{2}H_a^{(2)} \pmod{p^4}, \tag{2.6}
$$

where $a=\langle-\alpha\rangle_p$.

Proof. As in (2.5), the left-hand side of (2.6) is equal to

$$
\begin{aligned}
&(-1)^a\frac{(\alpha)_p^3(\alpha+p)_a(1-p)_a}{(1)_{p-1}^3(pt)^2(\alpha)_a^2}\\
&\quad= (-1)^a pt\frac{(1+pt)_{p-1-a}^3(\alpha+p)_a(1-p)_a}{(1)_{p-1}}\\
&\quad= pt\frac{(1+pt)_{p-1}^3(1-pt)_a(1-p)_a}{(1)_{p-1}^3(1-p(t+1))_a^2}.
\end{aligned}
$$

It is routine to check that for any $u\in p\mathbb{Z}_p$ and $k\in\mathbb{N}$,

$$
(1+u)_k\equiv(1)_k\left(1+uH_k+u^2\sum_{1\leqslant i<j\leqslant k}\frac{1}{ij}\right)=(1)_k\left(1+uH_k+\frac{u^2}{2}(H_k^2-H_k^{(2)})\right) \pmod{p^3}.
$$

Thus, in view of Lemma 2.1, we arrive at

$$
\frac{(1+pt)_{p-1}^3}{(1)_{p-1}^3}\equiv\left(1+ptH_{p-1}+\frac{p^2t^2}{2}H_{(p-1)/2}\right)^3\equiv 1 \pmod{p^3},
$$

and

$$\begin{aligned}
&\frac{(1-pt)_a(1-p)_a}{(1-p(t+1))_a^2}\\
&\quad\equiv \frac{\left(1-ptH_a+\frac{p^2t^2}{2}(H_a^2-H_a^{(2)})\right)\left(1-pH_a+\frac{p^2}{2}(H_a^2-H_a^{(2)})\right)}{\left(1-p(t+1)H_a+\frac{p^2(t+1)^2}{2}(H_a^2-H_a^{(2)})\right)^2}\\
&\quad\equiv 1+p(t+1)H_a+\frac{p^2(t+1)^2}{2}H_a^2+\frac{p^2(t^2+4t+1)}{2}H_a^{(2)} \pmod{p^3}.
\end{aligned}$$

Combining the above congruences, we are led to (2.6). □

Lemma 2.8. *For any prime $p>3$ and $\alpha\in\mathbb{Z}_p$ with $a=\langle-\alpha\rangle_p\leqslant p-2$, we have*

$$\begin{aligned}
&\frac{(\alpha)_p^2}{(1)_{p-1}^2}\sum_{k=a+2}^{p-1}(-1)^k\frac{(\alpha)_{p+k-1}}{(1)_{p-k}(\alpha)_k^2}\\
&\quad\equiv (-1)^a p^2 t(t+1)\left(H_a-\frac{(-1)^a}{a+1}\right)+(-1)^a p^3 t(t+1)\\
&\quad\times\left(\frac{t+1}{2}H_a^2+\frac{3t+1}{2}H_a^{(2)}-\frac{(-1)^a 2}{a+1}H_a-\frac{(-1)^a(t+2)}{(a+1)^2}\right) \pmod{p^4}. \qquad (2.7)
\end{aligned}$$

Proof. If $a=p-2$, then the left-hand side of (2.7) is equal to 0. Meanwhile, modulo p^4, the right-hand side of (2.7) is congruent to

$$\begin{aligned}
&-p^2t(t+1)H_{p-1}-p^3t(t+1)\Bigg(\frac{t+1}{2}\left(H_{p-1}-\frac{1}{p-1}\right)^2+\frac{3t+1}{2}\left(H_{p-1}^{(2)}-\frac{1}{(p-1)^2}\right)\\
&+\frac{2}{p-1}\left(H_{p-1}-\frac{1}{p-1}\right)+\frac{t+2}{(p-1)^2}\Bigg)\\
&\equiv -\frac{p^3t(t+1)^2}{2}+\frac{p^3t(t+1)(3t+1)}{2}+2p^3t(t+1)-p^3t(t+1)(t+2)\\
&\equiv 0,
\end{aligned}$$

where we have applied Lemma 2.1.

In what follows, we suppose that $a\leqslant p-3$. It is easy to see that

$$\begin{aligned}
&\frac{(\alpha)_p^2}{(1)_{p-1}^2}\sum_{k=a+2}^{p-1}(-1)^k\frac{(\alpha)_{p+k-1}}{(1)_{p-k}(\alpha)_k^2}\\
&\quad=-\frac{(\alpha)_p^3(\alpha+p)_a(\alpha+a+p)}{(1)_{p-1}^3(\alpha+a)^2(\alpha)_a^2}\sum_{k=a+2}^{p-1}\frac{(1+\alpha+a+p)_{k-a-2}(1-p)_{k-1}}{(1+\alpha+a)_{k-a-1}^2}\\
&\quad=-\frac{(\alpha)_p^3(\alpha+p)_a(\alpha+a+p)}{(1)_{p-1}^3(\alpha+a)^2(\alpha)_a^2}\sum_{k=1}^{p-a-2}\frac{(1+\alpha+a+p)_{k-1}(1-p)_{a+k}}{(1+\alpha+a)_k^2}. \qquad (2.8)
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
&-\frac{(\alpha)_p^3(\alpha+p)_a(\alpha+a+p)}{(1)_{p-1}^3(\alpha+a)^2(\alpha)_a^2}\\
&\quad=-p^2t(t+1)(-1)^a\frac{(-a+pt)_a(1+pt)_{p-1}^3(-a+p(t+1))_a(1)_a}{(1)_{p-1}^3(1-p(t+1))_a^3}\\
&\quad\equiv-p^2t(t+1)(-1)^a\frac{(-a)_a^2}{(1)_a^2}\left(1+3ptH_{p-1}-p(2t+1)H_a+3p(t+1)H_a\right)\\
&\quad\equiv(-1)^{a+1}p^2t(t+1)(1+p(t+2)H_a) \pmod{p^4}.
\end{aligned}\tag{2.9}
$$

In addition,

$$
\begin{aligned}
&\sum_{k=1}^{p-a-2}\frac{(1+\alpha+a+p)_{k-1}(1-p)_{a+k}}{(1+\alpha+a)_k^2}\\
&\quad=\sum_{k=1}^{p-a-2}\frac{(1+p(t+1))_{k-1}(1-p)_{a+k}}{(1+pt)_k^2}\\
&\quad\equiv(1)_a\sum_{k=1}^{p-a-2}\frac{(a+1)_k}{k(1)_k}\left(1+p(t+1)H_{k-1}-pH_{a+k}-2ptH_k\right)\\
&\quad\equiv\sum_{k=1}^{p-a-2}\frac{(-1)^k\binom{p-a-1}{k}}{k}\left(1+p(t+1)H_k-\frac{p(t+1)}{k}-pH_a-2ptH_k\right)\\
&\quad\equiv\sum_{k=1}^{p-a-1}\frac{(-1)^k\binom{p-a-1}{k}}{k}\left(1+p(1-t)H_k-\frac{p(t+1)}{k}-pH_a\right)\\
&\qquad+(-1)^a\left(\frac{1}{a+1}+\frac{p}{(a+1)^2}\right)(1+p(t+1)H_{a+1}-pH_a-2ptH_a) \pmod{p^2}.
\end{aligned}\tag{2.10}
$$

With the help of Lemma 2.5, we conclude that

$$
\begin{aligned}
&\sum_{k=1}^{p-a-1}\frac{(-1)^k\binom{p-a-1}{k}}{k}\left(1+p(1-t)H_k-\frac{p(t+1)}{k}-pH_a\right)\\
&\quad=-H_{p-1-a}+pH_aH_{p-1-a}-p(1-t)H_{p-1-a}^{(2)}+\frac{p(t+1)}{2}(H_{p-1-a}^{(2)}+H_{p-1-a}^2)\\
&\quad\equiv-H_a-pH_a^{(2)}+pH_a^2+p(1-t)H_a^{(2)}+\frac{p(t+1)}{2}(H_a^2-H_a^{(2)}) \pmod{p^2}.
\end{aligned}\tag{2.11}
$$

Combining (2.8)–(2.11), we finish the proof. □

3. Proof of Theorem 1.5

As in Section 2, we write $a = \langle -\alpha \rangle_p$ and $\alpha + a = pt$. For non-negative integers n and k, we define

$$F(n,k) = (-1)^{n+k} \frac{(2n+\alpha)(\alpha)_n^2(\alpha)_{n+k}}{(1)_n^2(1)_{n-k}(\alpha)_k^2},$$

and

$$G(n,k) = (-1)^{n+k} \frac{(\alpha)_n^2(\alpha)_{n+k-1}}{(1)_{n-1}^2(1)_{n-k}(\alpha)_k^2},$$

where we assume that $1/(1)_m = 0$ if $m < 0$. Then we can easily verify that

$$F(n,k-1) - F(n,k) = G(n+1,k) - G(n,k). \tag{3.1}$$

Proof of (1.12). Summing both sides of (3.1) over n from 0 to $p-1$, we get

$$\sum_{n=0}^{p-1} F(n,k-1) - \sum_{n=0}^{p-1} F(n,k) = G(p,k) - G(0,k) = G(p,k).$$

Further, summing the above identity over k from 1 to $p-1$, we obtain

$$\sum_{n=0}^{p-1} F(n,0) = F(p-1,p-1) + \sum_{k=1}^{p-1} G(p,k).$$

Namely, we have

$$\sum_{k=0}^{p-1} (-1)^k (2k+\alpha) \frac{(\alpha)_k^3}{(1)_k^3} = \frac{(\alpha)_{2p-1}}{(1)_{p-1}^2} - \frac{(\alpha)_p^2}{(1)_{p-1}^2} \sum_{k=1}^{p-1} (-1)^k \frac{(\alpha)_{p+k-1}}{(1)_{p-k}(\alpha)_k^2}. \tag{3.2}$$

We consider three cases. If $a = 0$, then by Lemma 2.3, the congruence (1.12) holds in this case. If $a = p-1$, then by (3.2),

$$\sum_{k=0}^{p-1} (-1)^k (2k+\alpha) \frac{(\alpha)_k^3}{(1)_k^3} = \frac{(\alpha)_{2p-1}}{(1)_{p-1}^2} - \frac{(\alpha)_p^2}{(1)_{p-1}^2} \sum_{k=1}^{a} (-1)^k \frac{(\alpha)_{p+k-1}}{(1)_{p-k}(\alpha)_k^2}.$$

With the help of Lemmas 2.4 and 2.6, we obtain

$$\begin{aligned}\sum_{k=0}^{p-1} (-1)^k (2k+\alpha) \frac{(\alpha)_k^3}{(1)_k^3} &\equiv pt + p^3t^3 \left(H_{p-1} + 2\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \right) \\ &= pt + p^3t^3 \left(H_{p-1} + H_{(p-1)/2}^{(2)} - 2H_{p-1}^{(2)} \right) \\ &\equiv pt \pmod{p^4},\end{aligned}$$

where we have utilized (2.1) in the last step. Then the desired result follows from the fact that

$$E_{p-3}(\alpha) \equiv E_{p-3}(1) = E_{p-3}(0) = 0 \pmod{p}.$$

If $1 \leqslant a \leqslant p-2$, then, in light of (3.2), the left-hand side of (1.12) is equal to

$$\begin{aligned}
&\frac{(\alpha)_{2p-1}}{(1)_{p-1}^2} - \frac{(\alpha)_p^2}{(1)_{p-1}^2}\sum_{k=1}^{a}(-1)^k \frac{(\alpha)_{p+k-1}}{(1)_{p-k}(\alpha)_k^2} - (-1)^a \frac{(\alpha)_p^2(\alpha)_{p+a}}{(1)_{p-1}^2(1)_{p-a-1}(\alpha)_{a+1}^2} \\
&\quad - \frac{(\alpha)_p^2}{(1)_{p-1}^2}\sum_{k=a+2}^{p-1}(-1)^k \frac{(\alpha)_{p+k-1}}{(1)_{p-k}(\alpha)_k^2}.
\end{aligned} \tag{3.3}$$

By Lemmas 2.4, 2.6–2.8, modulo p^4, (3.3) is congruent to

$$\begin{aligned}
&-\frac{p^2t(t+1)}{a+1}\left(1+2pH_a+\frac{p(t+2)}{a+1}\right) + (-1)^a p^3t^3\left(H_a^{(2)}+2\sum_{k=1}^{a}\frac{(-1)^k}{k^2}\right) \\
&+(-1)^a\left(pt+p^2t(t+1)H_a+\frac{p^3t(t+1)^2}{2}H_a^2+\frac{p^3t(t^2+4t+1)}{2}H_a^{(2)}\right) \\
&-(-1)^a p^2t(t+1)\left(H_a-\frac{(-1)^a}{a+1}\right)-(-1)^a p^3t(t+1) \\
&\times\left(\frac{t+1}{2}H_a^2+\frac{3t+1}{2}H_a^{(2)}-\frac{(-1)^a 2}{a+1}H_a-\frac{(-1)^a(t+2)}{(a+1)^2}\right) \\
&\quad = (-1)^a pt + (-1)^a 2p^3t^3\sum_{k=1}^{a}\frac{(-1)^k}{k^2}.
\end{aligned}$$

In view of Lemma 2.2, we deduce that

$$\begin{aligned}
\sum_{k=1}^{a}\frac{(-1)^k}{k^2} &\equiv \sum_{k=1}^{a}(-1)^k k^{p-3} \\
&= \frac{(-1)^a}{2}\left(E_{p-3}(a+1)+(-1)^a E_{p-3}(0)\right) \\
&\equiv \frac{(-1)^a}{2}E_{p-3}(\alpha) \pmod{p}.
\end{aligned}$$

This proves (1.12). □

Proof of (1.13). It suffices to show that

$$\sum_{k=a+1}^{p-1}(-1)^k(2k+\alpha)\frac{(\alpha)_k^3}{(1)_k^3} \equiv 0 \pmod{p^4} \tag{3.4}$$

for $a \leqslant p-2$. In fact,

$$
\begin{aligned}
&\sum_{k=a+1}^{p-1}(-1)^k(2k-a+pt)\frac{(-a+pt)_k^3}{(1)_k^3}\\
&\quad=(-1)^{a+1}\sum_{k=0}^{p-a-2}(-1)^k(2k+a+2+pt)\frac{(-a+pt)_{k+a+1}^3}{(1)_{k+a+1}^3}\\
&\quad=\frac{(-1)^{a+1}(-a+pt)_{a+1}^3}{(1)_{a+1}^3}\sum_{k=0}^{p-a-2}(-1)^k(2k+a+2+pt)\frac{(1+p(t+1))_k^3}{(a+2)_k^3}.
\end{aligned}
$$

Since $(-a+pt)_{a+1}^3/(1)_{a+1}^3 \equiv 0 \pmod{p^3}$ and

$$
\begin{aligned}
&\sum_{k=0}^{p-a-2}(-1)^k(2k+a+2+pt)\frac{(1+p(t+1))_k^3}{(a+2)_k^3}\\
&\quad\equiv\sum_{k=0}^{p-a-2}\frac{2k+a+2}{\binom{p-a-2}{k}^3}\\
&\quad=\sum_{k=0}^{p-a-2}\frac{2(p-a-2-k)+a+2}{\binom{p-a-2}{k}^3}\\
&\quad\equiv-\sum_{k=0}^{p-a-2}\frac{2k+a+2}{\binom{p-a-2}{k}^3}\\
&\quad\equiv 0 \pmod{p},
\end{aligned}
$$

as desired. □

4. Concluding remarks and an open problem

For any odd prime p, let $\left(\frac{\cdot}{p}\right)$ be the Legendre symbol modulo p. Recently, using a WZ pair in [3], Mao [14] proved the following two supercongruences: for any prime $p>3$,

$$
\sum_{k=0}^{(p-1)/2}(-1)^k(6k+1)\frac{(\frac12)_k^3}{k!^3 8^k}\equiv p\left(\frac{-2}{p}\right)+\frac{p^3}{4}\left(\frac{2}{p}\right)E_{p-3} \pmod{p^4},
$$

which was originally conjectured by Sun in [18, Conjecture 5.1], and

$$
\sum_{k=0}^{p-1}(-1)^k(6k+1)\frac{(\frac12)_k^3}{k!^3 8^k}\equiv p\left(\frac{-2}{p}\right)+\frac{p^3}{16}E_{p-3}(\tfrac14) \pmod{p^4}, \tag{4.1}
$$

which was the $n=1$ case of a conjecture of Sun [20, (2.16)].

It is clear that, for any prime $p\equiv 1 \pmod 4$, we have $\left(\frac{-2}{p}\right)=(-1)^{(p-1)/4}$. Combining the two supercongruences (1.7) and (4.1), we immediately obtain the following result: for any prime $p\equiv 1 \pmod 4$,

$$\sum_{k=0}^{p-1}(-1)^k(6k+1)\frac{(\frac12)_k^3}{k!^3 8^k}\equiv\sum_{k=0}^{p-1}(-1)^k(8k+1)\frac{(\frac14)_k^3}{k!^3}\pmod{p^4}. \tag{4.2}$$

Let $[n]=(1-q^n)/(1-q)$ be the q-integer, $(x;q)_n=(1-x)(1-xq)\cdots(1-xq^{n-1})$ $(n\geqslant 0)$ the q-shifted factorial, and let the n-th cyclotomic polynomial be given by

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity. We believe that the following q-analogue of (4.2) should be true.

Conjecture 4.1. *Let $n\equiv 1 \pmod 4$ be a positive integer. Then*

$$\sum_{k=0}^{n-1}(-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3}q^{3k^2}\equiv\sum_{k=0}^{n-1}(-1)^k[8k+1]\frac{(q;q^4)_k^3}{(q^4;q^4)_k^3}q^{2k^2+k}\pmod{[n]\Phi_n(q)^3}. \tag{4.3}$$

Note that the first author and Zudilin [9, Theorem 4.4] proved that, for any positive odd integer n,

$$\sum_{k=0}^{n-1}(-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3}q^{3k^2}\equiv(-q)^{(n-1)(n-3)/8}[n]\pmod{[n]\Phi_n(q)^2},$$

and they [9, Theorem 4.9] also showed that, for positive integers $n\equiv 1 \pmod 4$,

$$\sum_{k=0}^{n-1}(-1)^k[8k+1]\frac{(q;q^4)_k^3}{(q^4;q^4)_k^3}q^{2k^2+k}\equiv(-q)^{(n-1)(n-3)/8}[n]\pmod{[n]\Phi_n(q)^2}.$$

This means that the q-congruence (4.3) holds modulo $[n]\Phi_n(q)^2$. Nevertheless, Conjecture 4.1 seems still rather challenging. For a similar but more difficult conjecture, we refer the reader to [7, Conjecture 6.4].

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References

[1] G. Bauer, Von den Coefficienten der Reihen von Kugelfunctionen einer Variablen, J. Reine Angew. Math. 56 (1859), 101–121.

[2] J.M. Borwein and P.B. Borwein, Pi and the AGM, a study in analytic number theory and computational complexity, reprint of the 1987 original, Canadian Mathematical Society Series of Monographs and Advanced Texts, volume 4, John Wiley & Sons, Inc., New York, 1998.

[3] Y.G. Chen, X.Y. Xie, and B. He, On some congruences of certain binomial sums, Ramanujan J. (2016) 40, 237–244.

[4] K. Dilcher, Some q-series identities related to divisor functions, Discrete Math. 145 (1995), 83–93.

[5] H.W. Gould, Combinatorial Identities, Morgantown Printing and Binding Co., 1972.

[6] V.J.W. Guo, q-Analogues of the (E.2) and (F.2) supercongruences of Van Hamme, Ramanujan J. 49 (2019), 531–544.

[7] V.J.W. Guo, q-Supercongruences modulo the fourth power of a cyclotomic polynomial via creative microscoping, Adv. Appl. Math. 120 (2020), Art. 102078.

[8] V.J.W. Guo and J.-C. Liu, Some congruences related to a congruence of Van Hamme, Integral Transforms Spec. Funct. 31 (2020), 221–231.

[9] V.J.W. Guo and W. Zudilin, A q-microscope for supercongruences, Adv. Math. 346 (2019), 329–358.

[10] B. He, Some congruences on truncated hypergeometric series, Proc. Amer. Math. Soc. 143 (2015), 5173–5180.

[11] E. Lehmer, On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson, Ann. Math. 39 (1938), 350–360.

[12] L. Long, Hypergeometric evaluation identities and supercongruences, Pacific J. Math. 249 (2011), 405–418.

[13] W. Magnus, F. Oberhettinger and R.P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics (3rd edition), Springer, New York, 1966.

[14] G.-S. Mao, Proof of two supercongruences of truncated hypergeometric series ${}_4F_3$, Acta Math. Sin. (Engl. Ser.) 40 (2024), 1015–1028.

[15] E. Mortenson, A p-adic supercongruence conjecture of van Hamme, Proc. Amer. Math. Soc. 136 (2008), 4321–4328.

[16] S. Ramanujan, Modular equations and approximation to π, Quart. J. Math. 45 (1914), 350–372.

[17] C. Schneider, Symbolic summation assists combinatorics, Sém. Lothar. Combin. 56 (2007), B56b.

[18] Z.-W. Sun, Super congruences and Euler numbers, Sci. China Math. 54 (2011), 2509–2535.

[19] Z.-W. Sun, A refinement of a congruence result by van Hamme and Mortenson, Illinois J. Math. 56 (2012), 967–979.

[20] Z.-W. Sun, Open conjectures on congruences. J. Nanjing Univ. Math. Biquart. 36 (2019), 1–99.

[21] H. Swisher, On the supercongruence conjectures of van Hamme, Res. Math. Sci. 2 (2015), Art. 18.

[22] L. Van Hamme, Advanced problem 6407, Amer. Math. Monthly 40 (1982), 703–704.

[23] L. Van Hamme, Some conjectures concerning partial sums of generalized hypergeometric series, in: p-Adic functional analysis (Nijmegen, 1996), Lecture Notes in Pure and Appl. Math. 192, Dekker, New York, 1997, pp. 223–236.

[24] C. Wang and Z.-W. Sun, p-Adic analogues of hypergeometric identities and their applications, Nanjing Univ. J. Math. Biquarterly 41 (2024), no. 1, 34–56. (See also arXiv:1910.06856)

[25] H.S. Wilf and D. Zeilberger, An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities, Invent. Math. 108 (1992), 575–633.

[26] H.S. Wilf and D. Zeilberger, Rational function certification of multisum/integral/"q" identities, Bull. Amer. Math. Soc. (N.S.), 27 (1992), 148–153.

[27] W. Zudilin, Ramanujan-type supercongruences, J. Number Theory 129 (2009), 1848–1857.