A Chung–Feller theorem for lattice paths with respect to cyclically shifting boundaries

Victor J. W. Guo[1*] and Xiao-Xin Wang[2]

[1]School of Mathematical Sciences, Huaiyin Normal University, Huai'an, Jiangsu 223300, People's Republic of China

[2]Department of Mathematics, East China Normal University, Shanghai 200241, People's Republic of China

[1]jwguo@hytc.edu.cn, [2]1519783201@qq.com

Abstract. Irving and Rattan gave a formula for counting lattice paths dominated by a cyclically shifting piecewise linear boundary of varying slope. Their main result may be considered as a deep extension of well-known enumerative formulas concerning lattice paths from $(0,0)$ to (kn,n) lying under the line $x=ky$ (e.g. the Dyck paths when $k=1$). On the other hand, the classical Chung–Feller theorem tells us that the number of lattice paths from $(0,0)$ to (n,n) with exactly $2k$ steps above the line $x=y$ is independent of k, and is therefore the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. In this paper, we study the number of lattice path boundary pairs $(P,\mathbf{a})$ with k flaws, where P is a lattice path from $(0,0)$ to (n,m), $\mathbf{a}$ is a weak m-part composition of n, and a flaw is a horizontal step of P above the boundary $\partial\mathbf{a}$. We prove bijectively, for a given $\mathbf{a}$, that summing these numbers over all cyclic shifts of the boundary $\partial\mathbf{a}$ is equal to $\binom{n+m}{m-1}$. That is, we generalize the Irving–Rattan formula to a Chung–Feller type theorem. We also give a refinement of this result by taking the number of double ascents of lattice paths into account.

Keywords: Chung–Feller theorem; lattice paths; piecewise linear boundary; Catalan numbers; double ascent

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1. Introduction

We follow the terminology and notation of Irving and Rattan [11]. A *lattice path* is a path in the integer lattice $\mathbb{Z}\times\mathbb{Z}$ with unit steps up $(0,1)$ and to the right $(1,0)$. Let $\mathbf{a}=(a_0,\dots,a_{m-1})$ be a weak m-part composition of n (i.e., $a_0+\cdots+a_{m-1}=n$ and $a_0,\dots,a_{m-1}\geqslant 0$). The piecewise linear boundary curve $\partial\mathbf{a}$ is defined by

$$x=a_i(y-i)+\sum_{j=0}^{i-1}a_j,\quad \text{for } y\in[i,i+1].$$

A path is said to be *dominated* by $\mathbf{a}$ if it lies weakly under $\partial\mathbf{a}$. For example, the boundary $\partial\mathbf{a}$ for $\mathbf{a}=(3,1,2)$ is shown in Figure 1, together with a path it dominates.

*Corresponding author.

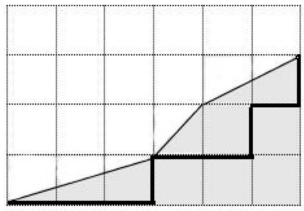

Figure 1: A path dominated by $\mathbf{a} = (3, 1, 2)$.

For a fixed $\mathbf{a}$, let $D(\mathbf{a})$ denote the number of paths from $(0, 0)$ to (n, m) dominated by $\mathbf{a}$. It is well known [7, Exercise 5.3.5] that, when all parts of $\mathbf{a}$ are equal, $D(\mathbf{a})$ is a generalized Catalan number. Namely, we have

$$D(\underbrace{a, \ldots, a}_{m \text{ copies}}) = \frac{1}{(a+1)m+1}\binom{(a+1)m+1}{m}, \tag{1.1}$$

where the $a = 1$ case corresponds to the famous Dyck paths enumerated by the usual Catalan numbers. It is easy to understand that for general $\mathbf{a}$ no simple formula for $D(\mathbf{a})$ is known, although the Kreweras dominance theorem [7, Section 5.4.6] gives a determinant expression.

Irving and Rattan [11] gave a surprisingly simple enumerative formula for paths dominated by *all cyclic shifts* of an arbitrary composition. To be precise, for any integer j, let $\mathbf{a}^{\langle j\rangle}$ denote the j-th shift of $\mathbf{a}$, namely,

$$\mathbf{a}^{\langle j\rangle} = (a_{-j}, a_{-j+1}, \ldots, a_{-j+m-1}),$$

where the indices are understood modulo m. Irving and Rattan [11, Corollary 2] proved that, for any weak m-part composition of n,

$$D(\mathbf{a}) + D(\mathbf{a}^{\langle 1\rangle}) + \cdots + D(\mathbf{a}^{\langle m-1\rangle}) = \binom{n+m}{m-1}. \tag{1.2}$$

For example, $D(1, 2, 3) = 18$, $D(3, 1, 2) = D(2, 3, 1) = 9$ and $\binom{6+3}{3-1} = 36$. Fig. 2 gives the boundaries $\partial\mathbf{a}$, $\partial\mathbf{a}^{\langle 1\rangle}$, and $\partial\mathbf{a}^{\langle 2\rangle}$ for $\mathbf{a} = (1, 2, 3)$ and $(2, 0, 4)$. It is clear that, when all parts of $\mathbf{a}$ are equal to a, the formula (1.2) reduces to (1.1). It is worth mentioning that Irving and Rattan [11, Theorem 1] gave a more general result, which was essentially equivalent to a conjecture by Tamm [22], and also generalized some recent work [2, 8] on the enumeration of lattice paths. Moreover, Nakamigawa and Tokushige [19] gave a generalization of [11, Theorem 1] via a new cycle lemma.

A *lattice path boundary pair (LPBP)* is an ordered pair $(P, \mathbf{a})$, where P is a lattice path from $(0, 0)$ to (n, m) and $\mathbf{a}$ is a weak m-part composition of n. We say that a LPBP $(P, \mathbf{a})$ has k flaws, if there are exactly k horizontal steps of P lying weakly above (or equivalently, to the left of) the boundary $\partial\mathbf{a}$. Thus, the path P is dominated by $\mathbf{a}$ if and

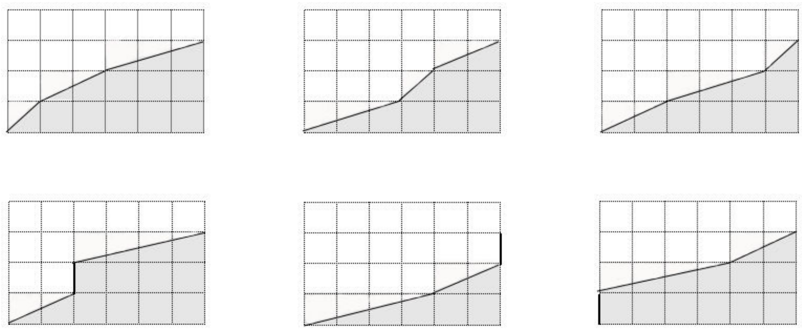

Figure 2: The boundaries $\partial(1,2,3)$, $\partial(3,1,2)$, $\partial(2,3,1)$, $\partial(2,0,4)$, $\partial(4,2,0)$ and $\partial(0,4,2)$.

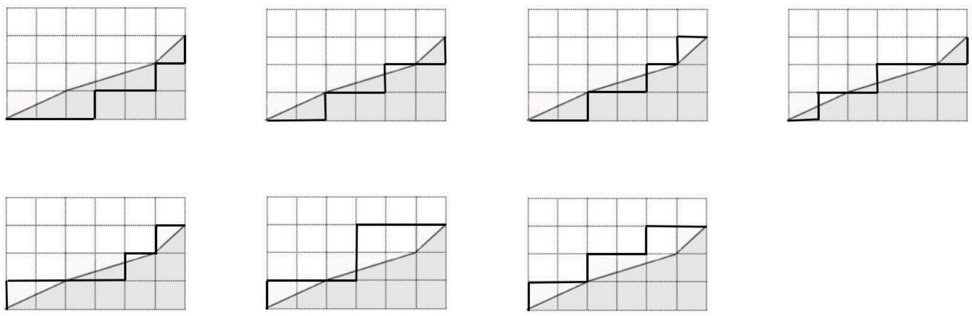

Figure 3: 7 LPBPs with $0, 1, \ldots, 6$ flaws, respectively.

only if $(P, \mathbf{a})$ has no flaws. For $\mathbf{a} = (2,3,1)$, seven LPBPs with different numbers of flaws are given in Figure 3.

The number of all LPBPs $(P, \mathbf{a})$ with k flaws is denoted by $D_k(\mathbf{a})$. The Chung–Feller theorem tells us that, for $\mathbf{a} = (1, \ldots, 1)$ with n copies of 1's, the number $D_k(\mathbf{a})$ is independent of k, and is therefore the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. The Chung-Feller theorem was proved by MacMahon [17] in 1909. Chung and Feller [4] reproved this theorem by using an analytic method in 1949. Narayana [20] proved the Chung-Feller theorem by more combinatorial methods (using cycle permutation). Eu et al. [5] gave a refinement of the Chung–Feller theorem and proved that $D_k(d, \ldots, d)$ is independent of k. Eu et al. [6] gave a strengthening of the Chung–Feller theorem together with a weighted version for Schröder paths. Chen [3] revisited this theorem by establishing a simple bijection. The Chung-Feller theorem was also proved bijectively by Callan [1] and Jewett and Ross [12]. Liu et al. [14] proved Chung–Feller theorems for Dyck paths and Motzkin paths by using a unified algebra approach. Ma and Yeh [15] proved Chung-Feller theorems for three kinds of different rooted lattice paths. Ma and Yeh [16] also studied refinements of (n, m)-Dyck paths by using four parameters, namely the peak, valley, double descent and double ascent. Huq [10] developed many generalized versions of the Chung–Feller theorem for lattice paths. Motivated by the Chung–Feller theorem, Li et al. [16] and Huang et al. [9] studied uniform partitions of certain sequences and rooted lattice paths, respectively. Mohanty's book [18] includes the Chung-Feller theorem as an entire section, and a refinement of this theorem can also be found in Narayana's book [21].

The aim of this paper is to give a Chung-Feller theorem for LPBPs. We shall give a bijective proof of this theorem. Our proof also leads to a refinement of (1.2) (see Theorem 2.3). Some combinatorial results in [15] are proved algebraically. We do not know whether there are similar algebraic proofs of the main results in this paper.

2. The main results

In this section, we give the Chung-Feller theorem for LPBPs.

Theorem 2.1. *Let* $\mathbf{a}$ *be a weak* m*-part composition of* n*, and let* $0 \leqslant k \leqslant n$*. Then*

$$D_k(\mathbf{a}) + D_k(\mathbf{a}^{\langle 1\rangle}) + \cdots + D_k(\mathbf{a}^{\langle m-1\rangle}) = \binom{n+m}{m-1}. \tag{2.1}$$

It is clear that (2.1) is a generalization of the classical Chung-Feller theorem and the Irving–Rattan formula (1.2). As already mentioned in the introduction, Equation (2.1) in the case where all parts of $\mathbf{a}$ are equal to d is due to Eu-Fu-Yeh [5, Corollary 3.2].

Corollary 2.2 (Eu–Fu–Yeh)**.** *The number of lattice paths from* $(0,0)$ *to* (dm, m) *that contains* k *horizontal steps lying weakly above the line* $x = dy$ *is always* $\frac{1}{dm+1}\binom{dm+m}{m}$ *for* $0 \leqslant k \leqslant dm$*.*

Consider a LPBP $(P, \mathbf{a})$. If a joint node in a lattice path P is formed by an up step followed by an up step, then we call this node a double ascent of P or $(P, \mathbf{a})$. It is clear that, if a lattice path P from $(0,0)$ to (n,m) has j double ascents and is ended with a horizontal step, then it has $m-j$ up-right corners (called in [11]) or NE-turns (called in [13]); while if P is ended with an up step, then it has $m-j-1$ up-right corners. The number of all LPBPs $(P, \mathbf{a})$ with k flaws and j double ascents is denoted by $D_{k,j}(\mathbf{a})$. Ma and Yeh [16, Theorem 4.2] proved that, for $0 \leqslant k \leqslant n$ and $0 \leqslant j \leqslant n-1$,

$$D_{k,j}(\underbrace{1,\ldots,1}_{n \text{ copies}}) = \frac{1}{j+1}\binom{n-1}{j}\binom{n}{j}. \tag{2.2}$$

We have a generalization of (2.2) as follows.

Theorem 2.3. *Let* $\mathbf{a}$ *be a weak* m*-part composition of* n*. Let* $0 \leqslant k \leqslant n$ *and* $0 \leqslant j \leqslant m-1$*. Then*

$$D_{k,j}(\mathbf{a}) + D_{k,j}(\mathbf{a}^{\langle 1\rangle}) + \cdots + D_{k,j}(\mathbf{a}^{\langle m-1\rangle}) = \frac{m}{n+1}\binom{n+1}{m-j}\binom{m-1}{j}. \tag{2.3}$$

It is easy to see that, when $m = n$ and all parts of $\mathbf{a}$ are equal to 1, the formula (2.3) reduces to (2.2). On the other hand, summing (2.3) over j from 0 to $m-1$, we obtain (2.1).

Letting $n = dm$ and $a_0 = \cdots = a_m = d$ in (2.3), we obtain the following formula:

$$D_{k,j}(\underbrace{d,\ldots,d}_{m \text{ copies}}) = \frac{1}{dm+1}\binom{dm+1}{m-j}\binom{m-1}{j},$$

which is a generalization of Ma and Yeh's formula (2.2), and can also be restated as refinement of Corollary 2.2 due to Eu–Fu–Yeh as follows.

Corollary 2.4. *The number of lattice paths from* $(0,0)$ *to* (dm, m) *that contains* k *horizontal steps lying weakly above the line* $x = dy$ *and that have* j *double ascents, is always* $\frac{1}{dm+1}\binom{dm+1}{m-j}\binom{m-1}{j}$ *for* $0 \leqslant k \leqslant dm$.

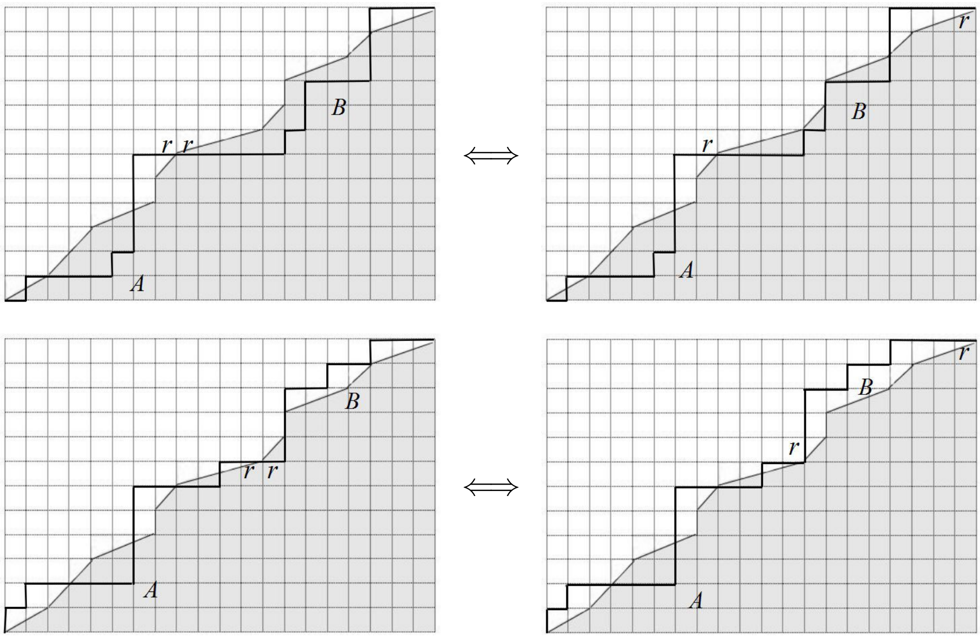

Figure 4: The bijection θ for the first case.

3. Proof of Theorem 2.3

Ma and Yeh [16] gave algebraic and bijective proofs of (2.2). Our proof of (2.3) is bijective and is motivated by Ma and Yeh's bijective proof of (2.2). It is easy to see that a lattice path from $(0,0)$ to (n, m) can be interpreted as a permutation of the word $u^m r^n$, where $u = (0,1)$ and $r = (1,0)$ denote a step up and a step to the right, respectively. Any such path has j double ascents if and only if the m u's are decomposed into $m-j$ segments in the correponding permutation of $u^m r^n$. There are $\binom{m-1}{m-j-1}$ ways to decompose u^m into $m-j$ segments and $\binom{n+1}{m-j}$ ways to insert them into the word r^n. Hence, the number of lattice paths from $(0,0)$ to (n,m) with j double ascents is equal to $\binom{n+1}{m-j}\binom{m-1}{j}$. It suffices to show that, for $0 \leqslant k \leqslant n$, the left-hand side of (2.3) is independent of k.

We will construct a new bijection

$$\theta\colon \biguplus_{i=0}^{m-1} \mathcal{D}_k(\mathbf{a}^{\langle i\rangle}) \to \biguplus_{i=0}^{m-1} \mathcal{D}_{k+1}(\mathbf{a}^{\langle i\rangle}),$$

which keeps the number of double ascents unchanged, to prove this result.

Let $0 \leqslant k \leqslant n-1$ and $(P, \mathbf{a}^{\langle i\rangle})$ be a LPBP in $\mathcal{D}_k(\mathbf{a}^{\langle i\rangle})$ with j double ascents. Then $P = A'rB$, where r is the rightmost horizontal step intersecting the boundary $\partial\mathbf{a}^{\langle i\rangle}$ and also lying weakly under $\partial\mathbf{a}^{\langle i\rangle}$. Since $k \leqslant n-1$, such r must exist. We have two cases:

- If A' is empty or A' is ended by r (i.e., $A' = Ar$), then we define $P' = A'Br$ and $\theta(P, \mathbf{a}^{\langle i\rangle}) = (P', \mathbf{a}^{\langle i\rangle})$.

- If A' is ended by u (i.e., $A' = Au$) and contains d u's, then we define $P' = BrA'$ and $\theta(P, \mathbf{a}^{\langle i\rangle}) = (P', \mathbf{a}^{\langle i-d\rangle})$.

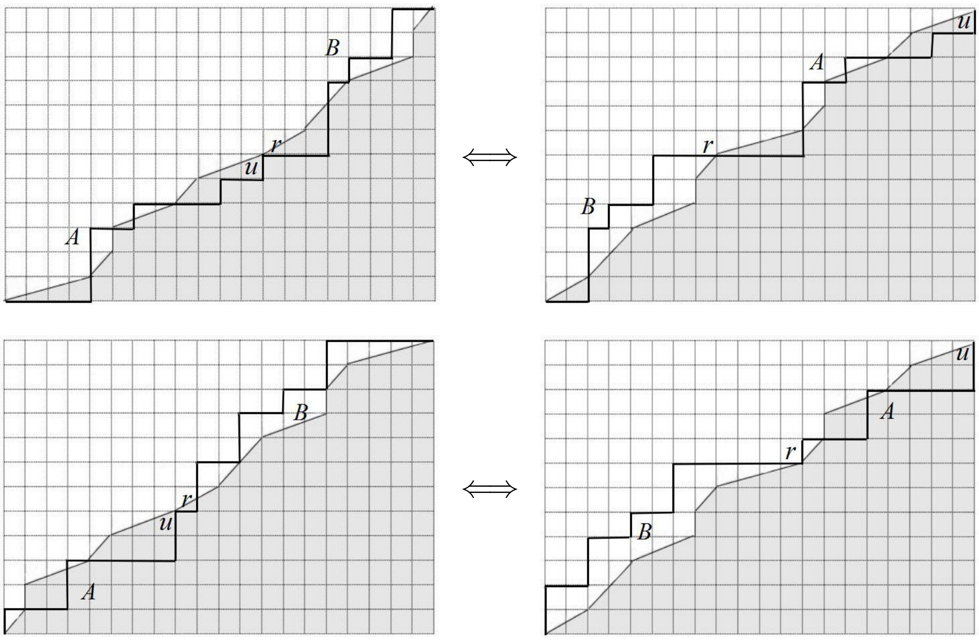

Figure 5: The bijection θ for the second case.

Since the number of vertical segments in the lattice path P' is the same as that in P, one sees that $\theta(P, \mathbf{a}^{\langle i\rangle})$ is in $\mathcal{D}_{k+1}(\mathbf{a})$ and has j double ascents.

To prove that the mapping θ is a bijection, we construct its inverse θ^{-1} as follows: Suppose that $(P', \mathbf{a}^{\langle h\rangle})$ be a LPBP in $\mathcal{D}_{k+1}(\mathbf{a}^{\langle h\rangle})$. We consider the following two cases:

- P' is ended with r. We write $P' = A'Br$, where $A' = Ar$ and this r is the rightmost flaw in $A'B$ intersecting $\partial\mathbf{a}^{\langle h\rangle}$ if such r exists, and we write $P' = Br$ $(A' = \emptyset)$ otherwise. Then $P = A'rB$ and $\theta^{-1}(P', \mathbf{a}^{\langle h\rangle}) = (P, \mathbf{a}^{\langle h\rangle})$.

- P' is ended by u. We may write $P' = BrA' = BrAu$, where r is the leftmost flaw in $(P', \mathbf{a}^{\langle h\rangle})$ intersecting $\partial\mathbf{a}^{\langle h\rangle}$. Since $(P', \mathbf{a}^{\langle h\rangle})$ has at least one flaw, such r must exist. Assume that A' contains d u's, then we let $P = A'rB$ and $\theta^{-1}(P', \mathbf{a}^{\langle h\rangle}) = (P, \mathbf{a}^{\langle h+d\rangle})$.

Some examples for the bijection θ are given in Figures 4–6. □

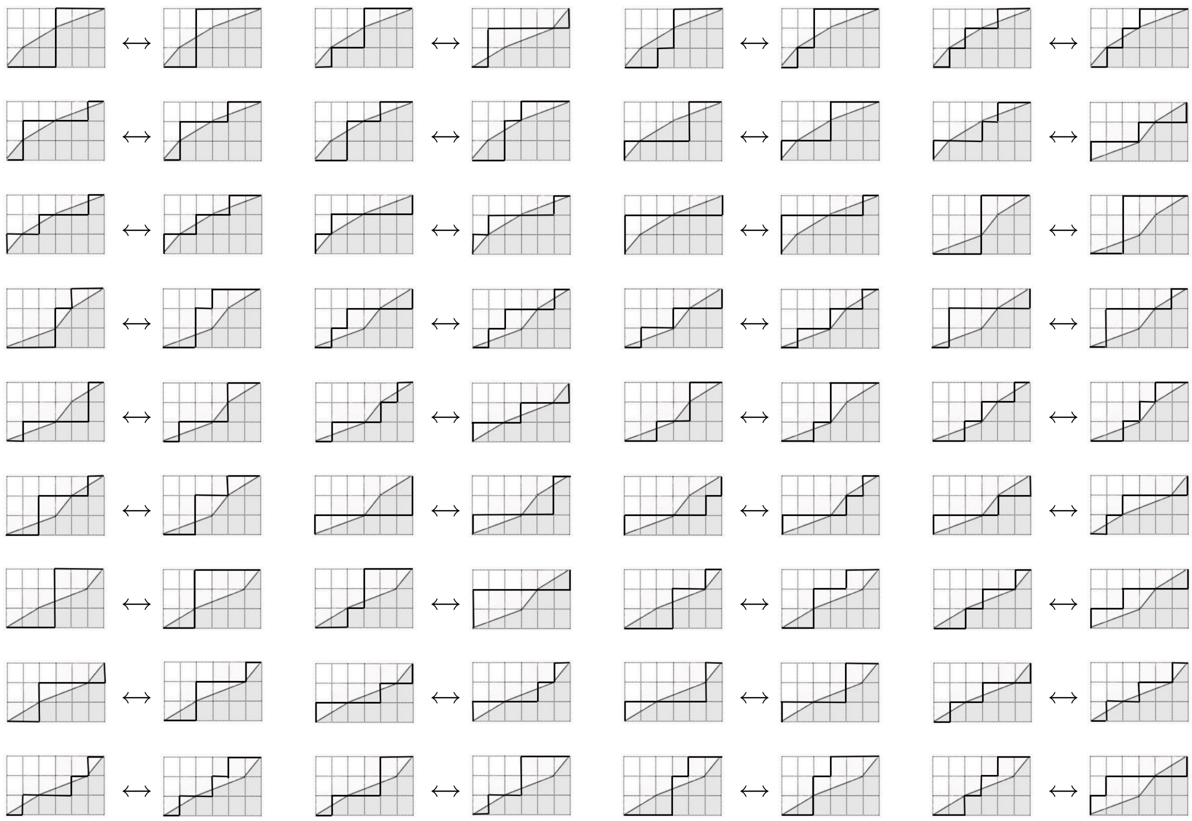

Figure 6: The bijection θ from $\biguplus_{j=0}^{2}\mathcal{D}_3(\mathbf{a}^{\langle j\rangle})$ to $\biguplus_{j=0}^{2}\mathcal{D}_4(\mathbf{a}^{\langle j\rangle})$, where $\mathbf{a} = (1, 2, 3)$.

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