Proof of two divisibility properties of binomial coefficients conjectured by Z.-W. Sun

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Abstract

For all positive integers \(n\), we prove the following divisibility properties:

\[
(2^n + 3) \binom{2n}{n} \left| \binom{6n}{3n}\binom{3n}{n} \right| \text{ and } (10^n + 3) \binom{3n}{n} \left| 21\binom{15n}{5n}\binom{5n}{n}. \right.
\]

This confirms two recent conjectures of Z.-W. Sun. Some similar divisibility properties are given. Moreover, we show that, for all positive integers \(m\) and \(n\), the product \(am\binom{am+bn-1}{am} \binom{an+bn}{an}\) is divisible by \(m + n\). In fact, the latter result can be further generalized to the \(q\)-binomial coefficients and \(q\)-integers case, which generalizes the positivity of \(q\)-Catalan numbers. We also propose several related conjectures.

Keywords: congruences, binomial coefficients, \(p\)-adic order, \(q\)-Catalan numbers, reciprocal and unimodal polynomials

1 Introduction

In [18, 19], Z.-W. Sun proved some divisibility properties of binomial coefficients, such as

\[
2(2^n + 1) \binom{2n}{n} \left| \binom{6n}{3n}\binom{3n}{n} \right|, \tag{1.1}
\]

\[
(10^n + 1) \binom{3n}{n} \left| \binom{15n}{5n}\binom{5n-1}{n-1} \right. \tag{1.2}
\]
Some similar divisibility results were later obtained by Guo [10] and Guo and Krattenthaler [11]. A generalization of (1.1) was recently given by Sepanski [15]. It is worth mentioning that Bober [6] has completely described when ratios of factorial products of the form
\[
\frac{(a_1n)! \cdots (a_kn)!}{(b_1n)! \cdots (b_{k+1}n)!}
\]
with \(a_1 + \cdots + a_k = b_1 + \cdots + b_{k+1}\) are always integers.

Let
\[
S_n = \frac{(6n^3)^n}{2(2n+1)^{2n}}\quad \text{and} \quad t_n = \frac{(15n^5)^{n-1}}{(10n+1)^{3n}}.
\]

In this paper we first prove the following two results conjectured by Z.-W. Sun [18, 19].

**Theorem 1.1** (see [18, Conjecture 3(i)]) Let \(n\) be a positive integer. Then
\[
3S_n \equiv 0 \pmod{2^n + 3}.
\]

**Theorem 1.2** [19, Conjecture 1.3] Let \(n\) be a positive integer. Then
\[
21t_n \equiv 0 \pmod{10^n + 3}.
\]

We shall also give more congruences for \(S_n\) and \(t_n\) as follows.

**Theorem 1.3** Let \(n\) be a positive integer. Then
\[
\begin{align*}
105S_n & \equiv 0 \pmod{2^n + 5}, \\
315S_n & \equiv 0 \pmod{2^n + 7}, \\
6435S_n & \equiv 0 \pmod{2^n + 9}, \\
3003t_n & \equiv 0 \pmod{2^n + 1}, \\
88179t_n & \equiv 0 \pmod{10^n + 7}, \\
43263t_n & \equiv 0 \pmod{10^n + 9}.
\end{align*}
\]

Let \(Z\) denote the set of integers. Another result in this paper is the following.

**Theorem 1.4** Let \(a, b, m, n\) be positive integers. Then
\[
abm\left(\begin{array}{c}am + bn \\mod{m + n} \\
am \\mod{m + n} \\
an \\mod{m + n}
\end{array}\right) = \frac{am}{m + n}\left(\begin{array}{c}am + bm - 1 \\mod{m + n} \\
am \\mod{m + n} \\
an \\mod{m + n}
\end{array}\right) \in Z.
\]

Letting \(a = b = 1\) in (1.10), we get the following result, of which a combinatorial interpretation was given by Gessel [9, Section 7].

**Corollary 1.5** Let \(m, n\) be positive integers. Then
\[
m\left(\frac{2m}{m + n}\right)\left(\frac{2n}{m + n}\right) \in Z.
\]

In the next section, we give some lemmas. The proofs of Theorems 1.1–1.3 will be given in Sections 3–5 respectively. A proof of the \(q\)-analogue of Theorem 1.4 will be given in Section 6. We close our paper with some further remarks and open problems in Section 7.
2 Some lemmas

For the $p$-adic order of $n!$, there is a known formula

$$\operatorname{ord}_p n! = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor,$$  \hspace{1cm} (2.1)

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding $x$. In this section, we give some results on the floor function $\lfloor x \rfloor$.

Lemma 2.1 For any real number $x$, we have

$$[6x] + [x] \geq [3x] + 2 [2x],$$ \hspace{1cm} (2.2)

$$[15x] + [2x] \geq [10x] + [4x] + [3x].$$ \hspace{1cm} (2.3)

Proof. See [6, Theorem 1.1] and one of the 52 sporadic step functions given in [6, Table 2, line# 32]. \hfill $\Box$

Lemma 2.2 Let $m$ and $n$ be two positive integers such that $m|2n + 3$ and $m \geq 5$. Then

$$\left\lfloor \frac{6n}{m} \right\rfloor + \left\lfloor \frac{n}{m} \right\rfloor = \left\lfloor \frac{3n}{m} \right\rfloor + 2 \left\lfloor \frac{2n}{m} \right\rfloor + 1.$$ \hspace{1cm} (2.4)

Proof. Let $\{x\} = x - \lfloor x \rfloor$ be the fractional part of $x$. Then (2.4) is equivalent to

$$\left\{ \frac{6n}{m} \right\} + \left\{ \frac{n}{m} \right\} = \left\{ \frac{3n}{m} \right\} + 2 \left\{ \frac{2n}{m} \right\} - 1.$$ \hspace{1cm} (2.5)

Now suppose that $m|2n + 3$ and $m \geq 5$. We have

$$\left\{ \frac{2n}{m} \right\} = \frac{m - 3}{m} > \frac{1}{5}, \text{ and } \left\lfloor \frac{2n}{m} \right\rfloor = \frac{2n + 3}{m} - 1 \equiv 0 \pmod{2}.$$  \hspace{1cm}

It follows that

$$\left\{ \frac{6n}{m} \right\} = \begin{cases} \frac{2m - 9}{m}, & \text{if } m = 5, 7, \\ \frac{m - 9}{m}, & \text{if } m \geq 9, \end{cases}$$

$$\left\{ \frac{n}{m} \right\} = \frac{m - 3}{2m},$$

$$\left\{ \frac{3n}{m} \right\} = \begin{cases} \frac{3m - 9}{2m}, & \text{if } m = 5, 7, \\ \frac{m - 9}{2m}, & \text{if } m \geq 9. \end{cases}$$

Therefore, the identity (2.5) is true for any positive integer $m \geq 5$. \hfill $\Box$
Lemma 2.3 Let \( m \) and \( n \) be two positive integers such that \( m|10n + 3 \) and \( m \geq 9 \). Then
\[
\left\lfloor \frac{15n}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor = \left\lfloor \frac{10n}{m} \right\rfloor + \left\lfloor \frac{4n}{m} \right\rfloor + \left\lfloor \frac{3n}{m} \right\rfloor + 1. \tag{2.6}
\]

Proof. It is easy to see that (2.6) is equivalent to
\[
\left\{ \frac{15n}{m} \right\} + \left\{ \frac{2n}{m} \right\} = \left\{ \frac{10n}{m} \right\} + \left\{ \frac{4n}{m} \right\} + \left\{ \frac{3n}{m} \right\} - 1. \tag{2.7}
\]
Now suppose that \( m|10n + 3 \) and \( m \geq 9 \). We have
\[
\left\{ \frac{10n}{m} \right\} = \frac{m - 3}{m} \geq \frac{2}{3}, \quad \text{and} \quad A := \left\lfloor \frac{10n}{m} \right\rfloor = \frac{10n + 3}{m} - 1 \equiv 0, 2, 6, 8 \pmod{10}.
\]
It is easy to check that
\[
\left\{ \frac{15n}{m} \right\} = \frac{m - 9}{2m},
\]
\[
\left( \left\{ \frac{2n}{m} \right\}, \left\{ \frac{4n}{m} \right\}, \left\{ \frac{3n}{m} \right\} \right) = \begin{cases} 
\left( \frac{2m - 6}{10m}, \frac{4m - 12}{10m}, \frac{3m - 9}{10m} \right), & \text{if } A \equiv 0 \pmod{10}, \\
\left( \frac{6m - 6}{10m}, \frac{2m - 12}{10m}, \frac{9m - 9}{10m} \right), & \text{if } A \equiv 2 \pmod{10}, \\
\left( \frac{4m - 6}{10m}, \frac{8m - 12}{10m}, \frac{m - 9}{10m} \right), & \text{if } A \equiv 6 \pmod{10}, \\
\left( \frac{8m - 6}{10m}, \frac{6m - 12}{10m}, \frac{7m - 9}{10m} \right), & \text{if } A \equiv 8 \pmod{10},
\end{cases}
\]
and so the identity (2.7) holds. \( \square \)

3 Proofs of Theorem 1.1

First Proof. Let \( \gcd(a, b) \) denote the greatest common divisor of two integers \( a \) and \( b \). For any positive integer \( n \), since \( \gcd(2n + 3, 4n + 2) = 1 \), to prove Theorem 1.1, it is enough to show that
\[
(2n + 3) \left| \frac{3^6(3n)_{3n}}{(2n)_{2n}} \right. \tag{3.1}
\]
By (2.1), for any odd prime \( p \), the \( p \)-adic order of
\[
\frac{\binom{6n}{3n}\binom{3n}{n}}{(2n + 3)\binom{2n}{n}} = \frac{(2n + 2)!(6n)!(n)!}{(2n + 3)!(3n)!(2n)!^2}
\]
is given by
\[
\sum_{i=1}^{\infty} \left( \left\lfloor \frac{2n+2}{p^i} \right\rfloor - \left\lfloor \frac{2n+3}{p^i} \right\rfloor - \left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{3n}{p^i} \right\rfloor + 2 \left\lfloor \frac{2n}{p^i} \right\rfloor \right). \tag{3.2}
\]

Note that
\[
\left\lfloor \frac{2n+2}{p^i} \right\rfloor - \left\lfloor \frac{2n+3}{p^i} \right\rfloor = \begin{cases} 
-1, & \text{if } p^i | 2n+3, \\
0, & \text{otherwise.}
\end{cases}
\]
By Lemmas 2.1 and 2.2, for \( p \geq 5 \), the summation (3.2) is clearly greater than or equal to 0. For \( p = 3 \), we have (3.2) \( \geq -1 \) because if the positive integer \( i \) satisfies \( 3^i | 2n+3 \) and \( 3^i < 5 \) then we must have \( i = 1 \). This proves that
\[
\frac{3 \binom{6n}{3n}}{(2n+3) \binom{2n}{n}}
\]
is always an integer. Hence (3.1) holds. \( \square \)

**Second Proof (provided by T. Amdeberhan and V.H. Moll).** Replacing \( n \) by \( n+1 \) in (1.1), we see that (after some rearrangement)
\[
\frac{\binom{6n+6}{3n+3}}{2(2n+3) \binom{2n+2}{n+1}} = \frac{6(6n+5)(6n+1)S_n}{(n+1)(2n+3)} \in \mathbb{Z}.
\]
Hence, \( (2n+3) | 6(6n+5)(6n+1)S_n \). Since \( \gcd(2n+3,2) = \gcd(2n+3,6n+5) = \gcd(2n+3,6n+1) = 1 \), we must have \( (2n+3) | 3S_n \). \( \square \)

**Remark.** Z.-W. Sun [18, Conjecture 3(i)] also conjectured that \( S_n \) is odd if and only if \( n \) is a power of 2. After reading a previous version of this paper, Quan-Hui Yang told me that it is easy to show that \( \text{ord}_2((6n)!n!(3n)!(2n)!^2) \) equals the number of 1’s in the binary expansion of \( n \) by noticing
\[
\text{ord}_2(6n)! = 3n + \text{ord}_2(3n)!, \quad \text{ord}_2(2n)! = n + \text{ord}_2(n!),
\]
and using Legendre’s theorem. T. Amdeberhan and V.H. Moll also pointed out this.

### 4 Proof of Theorem 1.2

For any positive integer \( n \), since \( \gcd(10n+3,10n+1) = 1 \), to prove Theorem 1.2, it is enough to show that
\[
(10n+3) \left| \frac{21 \binom{15n}{5n} \binom{5n-1}{n-1}}{\binom{3n}{n}} \right. \tag{4.1}
\]
Furthermore, since \( \gcd(10n+3, 5) = 1 \) and \( \binom{5n}{n} = 5\binom{5n-1}{n-1} \), one sees that (4.1) is equivalent to

\[
(10n + 3) \divides \frac{21\binom{15n}{5n}}{\binom{3n}{n}}. \tag{4.2}
\]

By (2.1), for any odd prime \( p \), the \( p \)-adic order of

\[
\frac{\binom{15n}{5n}}{\binom{3n}{n}} = \frac{(10n + 2)!(15n)!(2n)!}{(10n + 3)!(10n)!(4n)!(3n)!}
\]

is given by

\[
\sum_{i=1}^{\infty} \left( \left\lfloor \frac{10n + 2}{p^i} \right\rfloor + \left\lfloor \frac{15n}{p^i} \right\rfloor + \left\lfloor \frac{2n}{p^i} \right\rfloor - \left\lfloor \frac{10n + 3}{p^i} \right\rfloor - \left\lfloor \frac{4n}{p^i} \right\rfloor - \left\lfloor \frac{3n}{p^i} \right\rfloor \right). \tag{4.3}
\]

Note that

\[
\left\lfloor \frac{10n + 2}{p^i} \right\rfloor - \left\lfloor \frac{10n + 3}{p^i} \right\rfloor = \begin{cases} -1 & \text{if } p^i | 10n + 3, \\ 0 & \text{otherwise}. \end{cases}
\]

By Lemmas 2.1 and 2.3, for \( p \geq 11 \), or \( p = 5 \), the summation (4.3) is clearly greater than or equal to 0. For \( p = 3, 7 \), we have (4.3) \( \geq -1 \) because there is at most one index \( i \geq 1 \) satisfying \( p^i | 10n + 3 \) and \( p^i < 9 \) in this case. This proves that

\[
\frac{21\binom{15n}{5n}}{\binom{3n}{n}}
\]

is always an integer. Namely, (4.2) is true.

5 Proof of Theorem 1.3

**Lemma 5.1** Let \( m \) and \( n \) be two positive integers. Then

\[
\left\lfloor \frac{6n}{m} \right\rfloor + \left\lfloor \frac{n}{m} \right\rfloor = \left\lfloor \frac{3n}{m} \right\rfloor + 2 \left\lfloor \frac{2n}{m} \right\rfloor + 1, \tag{5.1}
\]

if \( m | 2n + 5 \) and \( m \geq 9 \), or \( m | 2n + 7 \) and \( m \geq 11 \), or \( m | 2n + 9 \) and \( m \geq 15 \).

**Proof.** The proof is similar to that of Lemma 2.2. We only consider the case when \( m | 2n + 5 \) and \( m \geq 9 \). In this case, we have

\[
\left\lfloor \frac{2n}{m} \right\rfloor = \frac{m - 5}{m} > \frac{1}{3}, \quad \text{and} \quad \left\lfloor \frac{2n}{m} \right\rfloor = \frac{2n + 5}{m} - 1 \equiv 0 \pmod{2}.
\]
It follows that

\[
\begin{align*}
\{ \frac{6n}{m} \} &= \begin{cases} 
\frac{2m - 15}{m}, & \text{if } m = 9, 11, 13, \\
\frac{m - 15}{m}, & \text{if } m \geq 15,
\end{cases} \\
\{ \frac{n}{m} \} &= \frac{m - 5}{2m}, \\
\{ \frac{3n}{m} \} &= \begin{cases} 
\frac{3m - 15}{2m}, & \text{if } m = 9, 11, 13, \\
\frac{m - 15}{2m}, & \text{if } m \geq 15,
\end{cases}
\end{align*}
\]

and so

\[
\{ \frac{6n}{m} \} + \{ \frac{n}{m} \} = \{ \frac{3n}{m} \} + 2 \left\{ \frac{2n}{m} \right\} - 1.
\]

This proves (5.1). \(\square\)

**Lemma 5.2** Let \(m\) and \(n\) be two positive integers. Then

\[
\left\lfloor \frac{15n}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor = \left\lfloor \frac{10n}{m} \right\rfloor + \left\lfloor \frac{4n}{m} \right\rfloor + \left\lfloor \frac{3n}{m} \right\rfloor + 1, \tag{5.2}
\]

if \(m|2n + 1\) and \(m \geq 15\), or \(m|10n + 7\) and \(m \geq 21\), or \(m|10n + 9\) and \(m \geq 27\).

**Proof.** The proof is similar to that of Lemma 2.3. We only consider the case when \(m|10n + 9\) and \(m \geq 27\). In this case, we have

\[
\left\lfloor \frac{10n}{m} \right\rfloor = \frac{m - 9}{m} \geq \frac{2}{3} \quad \text{and} \quad A := \left\lfloor \frac{10n}{m} \right\rfloor = \frac{10n + 9}{m} - 1 \equiv 0, 2, 6, 8 \pmod{10}.
\]

It follows that

\[
\left\lfloor \frac{15n}{m} \right\rfloor = \frac{m - 27}{2m},
\]

\[
\left( \left\lfloor \frac{2n}{m} \right\rfloor, \left\lfloor \frac{4n}{m} \right\rfloor, \left\lfloor \frac{3n}{m} \right\rfloor \right) = \begin{cases} 
\left( \frac{2m - 18}{10m}, \frac{4m - 36}{10m}, \frac{3m - 27}{10m} \right), & \text{if } A \equiv 0 \pmod{10}, \\
\left( \frac{6m - 18}{10m}, \frac{2m - 36}{10m}, \frac{9m - 27}{10m} \right), & \text{if } A \equiv 2 \pmod{10}, \\
\left( \frac{4m - 18}{10m}, \frac{8m - 36}{10m}, \frac{m - 27}{10m} \right), & \text{if } A \equiv 6 \pmod{10}, \\
\left( \frac{8m - 18}{10m}, \frac{6m - 36}{10m}, \frac{7m - 27}{10m} \right), & \text{if } A \equiv 8 \pmod{10}.
\end{cases}
\]
Hence,
\[
\left\{ \frac{15n}{m} \right\} + \left\{ \frac{2n}{m} \right\} = \left\{ \frac{10n}{m} \right\} + \left\{ \frac{4n}{m} \right\} + \left\{ \frac{3n}{m} \right\} - 1,
\]
which means that (5.2) holds. \(\square\)

**Proof of Theorem 1.3.** Since the proofs of the congruences (1.4)–(1.9) are similar in view of Lemmas 5.1 and 5.2, we only give proofs of (1.5) and (1.9). Noticing that \(\gcd(2n + 1, 2n + 7) = 1\) or \(3\), to prove (1.5), it suffices to show that
\[
(2n + 7) \mid 105 \frac{\binom{6n}{3n} \binom{3n}{n}}{\binom{2n}{n}}.
\]  
(5.3)

Let
\[
X_n := \frac{\binom{6n}{3n} \binom{3n}{n}}{(2n + 7) \binom{2n}{n}} = \frac{(2n + 6)! \cdot (6n)! \cdot (n)!}{(2n + 7)! \cdot (3n)! \cdot (2n)!^2}.
\]

By (2.1), for any odd prime \(p\), we have
\[
\text{ord}_p X_n = \sum_{i=1}^{\infty} \left( \left\lfloor \frac{2n + 6}{p^i} \right\rfloor + \left\lfloor \frac{6n}{p^i} \right\rfloor - \left\lfloor \frac{2n + 7}{p^i} \right\rfloor - \left\lfloor \frac{3n}{p^i} \right\rfloor - 2 \left\lfloor \frac{2n}{p^i} \right\rfloor \right).
\]

Note that (5.1) is also true for \(m = 3\) and \(n \equiv 1 \pmod{3}\), and
\[
\left\lfloor \frac{2n + 6}{p^i} \right\rfloor - \left\lfloor \frac{2n + 7}{p^i} \right\rfloor = \begin{cases} -1, & \text{if } p^i | 2n + 7, \\ 0, & \text{otherwise.} \end{cases}
\]

By Lemmas 2.1 and 5.1, we obtain
\[
\begin{cases} \text{ord}_p X_n \geq 0, & \text{if } p \geq 11, \\ \text{ord}_p X_n \geq -1, & \text{if } p = 3, 5, 7. \end{cases}
\]

This proves (5.3).

Similarly, since \(\gcd(10n + 9, 10n + 1) = \gcd(10n + 9, 5) = 1\), the congruence (1.9) is equivalent to
\[
(10n + 9) \mid 43263 \frac{\binom{15n}{5n} \binom{5n}{n}}{\binom{2n}{n}}.
\]  
(5.4)

Let
\[
Y_n := \frac{\binom{15n}{5n} \binom{5n}{n}}{(10n + 9) \binom{3n}{n}} = \frac{(10n + 8)! \cdot (15n)! \cdot (2n)!}{(10n + 9)! \cdot (10n)! \cdot (4n)! \cdot (3n)!}.
\]
Then, for any odd prime $p$, $\text{ord}_p Y_n$ is given by
\[
\sum_{i=1}^{\infty} \left( \left\lfloor \frac{10n + 8}{p^i} \right\rfloor + \left\lfloor \frac{15n}{p^i} \right\rfloor + \left\lfloor \frac{2n}{p^i} \right\rfloor - \left\lfloor \frac{10n + 9}{p^i} \right\rfloor - \left\lfloor \frac{10n}{p^i} \right\rfloor - \left\lfloor \frac{4n}{p^i} \right\rfloor - \left\lfloor \frac{3n}{p^i} \right\rfloor \right).
\]
Note that (5.2) also holds for $m = 7, 13, 17$ and any positive integer $n$ such that $m|10n+9$. Similarly as before, we have
\[
\begin{cases}
\text{ord}_p Y_n \geq 0, & \text{if } p = 5, 7, 13, 17, \text{ or } p \geq 29, \\
\text{ord}_p Y_n \geq -1, & \text{if } p = 11, 19, 23, \\
\text{ord}_p Y_n \geq -2, & \text{if } p = 3.
\end{cases}
\]
Observing that $43263 = 3^2 \cdot 11 \cdot 19 \cdot 23$, we complete the proof of (5.4).

6 A $q$-analogue of Theorem 1.4

Recall that the $q$-binomial coefficients are defined by
\[
\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases}
\frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-k+1})}{(1 - q)(1 - q^2) \cdots (1 - q^k)}, & \text{if } 0 \leq k \leq n, \\
0, & \text{otherwise}.
\end{cases}
\]

We begin with the announced strengthening of Theorem 1.4.

**Theorem 6.1** Let $a, b, m, n \geq 1$. Then
\[
\frac{1 - q^{\text{gcd}(am, m+n)}}{1 - q^{m+n}} \begin{bmatrix} am + bm - 1 \\ am \end{bmatrix}_q \begin{bmatrix} an + bn \\ an \end{bmatrix}_q
\]
(6.1)
is a polynomial in $q$ with non-negative integer coefficients.

**Corollary 6.2** Let $a, b, m, n \geq 1$. Then
\[
\frac{1 - q^{am}}{1 - q^{m+n}} \begin{bmatrix} am + bm - 1 \\ am \end{bmatrix}_q \begin{bmatrix} an + bn \\ an \end{bmatrix}_q
\]
(6.2)
is a polynomial in $q$ with non-negative integer coefficients.

It is easily seen that Theorem 1.4 can be obtained upon letting $q \to 1$ in Corollary 6.2. Moreover, when $a = b = m = 1$, the numbers (6.2) reduce to the $q$-Catalan numbers
\[
C_n(q) = \frac{1 - q}{1 - q^{2n+1}} \begin{bmatrix} 2n \\ n \end{bmatrix}_q.
\]
It is well known that the \(q\)-Catalan numbers \(C_n(q)\) are polynomials with non-negative integer coefficients (see [2, 3, 5, 7]). There are many different \(q\)-analogues of the Catalan numbers (see Fürlinger and Hofbauer [7]). For the so-called \(q,t\)-Catalan numbers, see [8, 13, 12].

Recall that a polynomial \(P(q) = \sum_{i=0}^{d} p_i q^i\) in \(q\) of degree \(d\) is called reciprocal if \(p_i = p_{d-i}\) for all \(i\), and that it is called unimodal if there is an integer \(r\) with \(0 \leq r \leq d\) and \(0 \leq p_0 \leq \cdots \leq p_r \geq \cdots \geq p_d \geq 0\). An elementary but crucial property of reciprocal and unimodal polynomials is the following.

**Lemma 6.3** If \(A(q)\) and \(B(q)\) are reciprocal and unimodal polynomials, then so is their product \(A(q)B(q)\).

Lemma 6.3 is well known and its proof can be found, e.g., in [1] or [16, Proposition 1]. Similarly to the proof of [11, Theorem 3.1], the following lemma plays an important role in the proof of Theorem 6.1. It is a slight generalization of [14, Proposition 10.1.(iii)], which extracts the essentials out of Andrews [4, Proof of Theorem 2].

**Lemma 6.4** Let \(P(q)\) be a reciprocal and unimodal polynomial and \(m\) and \(n\) positive integers with \(m \leq n\). Furthermore, assume that \(A(q) = \frac{1-q^m}{1-q^n} P(q)\) is a polynomial in \(q\). Then \(A(q)\) has non-negative coefficients.

**Proof.** See [11, Lemma 5.1]. \(\Box\)

**Proof of Theorem 6.1.** It is well known that the \(q\)-binomial coefficients are reciprocal and unimodal polynomials in \(q\) (cf. [17, Ex. 7.75.d]), and by Lemma 6.3, so is the product of two \(q\)-binomial coefficients. In view of Lemma 6.4, for proving Theorem 6.1 it is enough to show that the expression (6.1) is a polynomial in \(q\). We shall accomplish this by a count of cyclotomic polynomials.

Recall the well-known fact that

\[
q^n - 1 = \prod_{d \mid n} \Phi_d(q),
\]

where \(\Phi_d(q)\) denotes the \(d\)-th cyclotomic polynomial in \(q\). Consequently,

\[
\frac{1 - q^{\gcd(am,m+n)}}{1 - q^{m+n}} \left[\frac{am + bm - 1}{am}\right]_q \left[\frac{an + bn}{an}\right]_q = \prod_{d=2}^{\min\{am+bm-1,an+bn\}} \Phi_d(q)^{e_d},
\]

with

\[
e_d = \chi(d \mid \gcd(am,m+n)) - \chi(d \mid m+n) + \left[\frac{am + bm - 1}{d}\right] + \left[\frac{an + bn}{d}\right]
- \left[\frac{am}{d}\right] - \left[\frac{bm - 1}{d}\right] - \left[\frac{an}{d}\right] - \left[\frac{bn}{d}\right],
\]

(6.3)
where $\chi(S) = 1$ if $S$ is true and $\chi(S) = 0$ otherwise. This is clearly non-negative, unless $d \mid m + n$ and $d \nmid \gcd(am,m+n)$.

So, let us assume that $d \mid m + n$ and $d \nmid \gcd(am,m+n)$, which means that $d \nmid am$ and therefore

$$\left\lfloor \frac{am + bm - 1}{d} \right\rfloor + \left\lfloor \frac{an + bn}{d} \right\rfloor = \frac{(a+b)(m+n)}{d} - 1,$$

$$\left\lfloor \frac{am}{d} \right\rfloor + \left\lfloor \frac{an}{d} \right\rfloor = \frac{a(m+n)}{d} - 1,$$

$$\left\lfloor \frac{bm - 1}{d} \right\rfloor + \left\lfloor \frac{bn}{d} \right\rfloor = \frac{b(m+n)}{d} - 1,$$

and so $e_d = 0$ is also non-negative in this case. This completes the proof of polynomiality of (6.1).

**Proof of Corollary 6.2.** This follows immediately from Theorem 6.1 and the fact that $\gcd(am,m+n) \mid am$. \hfill \Box

### 7 Concluding remarks and open problems

On January 2, 2014 T. Amdeberhan and V.H. Moll (personal communication) found the following generalization of Theorem 1.1, which was soon proved by Q.-H. Yang [21] and C. Krattenthaler.

**Conjecture 7.1** Let $a, b$ and $n$ be positive integers with $a > b$. Then

$$(2bn + 1)(2bn + 3) \left( \frac{2bn}{bm} \right) \left| 3(a-b)(3a-b) \left( \frac{2an}{an} \right) \left( \frac{2an}{bn} \right) \right|.$$ 

Let $[m]! = (1 - q) \cdots (1 - q^m)$. By a result of Warnaar and Zudilin [20, Proposition 3], one sees that, for any positive integer $n$, the polynomial

$$\frac{[6n]![n]!}{[3n]![2n]!}$$

has non-negative integer coefficients. Similarly as before, we can prove the following generalization of congruences (1.3)–(1.5).

**Theorem 7.2** Let $n$ be a positive integer. Then all of

$$\frac{(1 - q)[6n][n]!}{(1 - q^{2n+1})[3n]![2n]!^2}, \quad \frac{(1 - q^3)[6n][n]!}{(1 - q^{2n+3})[3n]![2n]!^2}, \quad \frac{(1 - q)(1 - q^3)[6n][n]!}{(1 - q^{2n+1})(1 - q^{2n+3})[3n]![2n]!^2},$$

$$\frac{(1 - q^2)(1 - q^5)(1 - q^7)[6n][n]!}{(1 - q^{2n+3})(1 - q^{2n+5})(1 - q^{2n+7})[3n]![2n]!^2} \quad (n \geq 2),$$

$$\frac{(1 - q^2)^2(1 - q^5)(1 - q^7)[6n][n]!}{(1 - q^{2n+1})(1 - q^{2n+3})(1 - q^{2n+5})(1 - q^{2n+7})[3n]![2n]!^2} \quad (n \geq 2)$$

are polynomials in $q$. 

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We have the following two related conjectures.

**Conjecture 7.3** All the polynomials in Theorem 7.2 have non-negative integer coefficients.

**Conjecture 7.4** Let \( n \geq 2 \). Then the polynomial \( \frac{[6n]![n]!}{[3n]![2n]!^2} \) is unimodal.

It is obvious that the polynomial \( \frac{[6n]![n]!}{[3n]![2n]!^2} \) is reciprocal. If Conjecture 7.4 is true, then, applying Lemma 6.3, we conclude that the first two polynomials in Theorem 7.2 have non-negative integer coefficients.

It was conjectured by Warnaar and Zudilin (see [20, Conjecture 1]) that

\[
\frac{[15n]![2n]!}{[10n]![4n]![3n]!}
\]

has non-negative integer coefficients. Similarly, we have the following generalization of Theorem 1.2.

**Theorem 7.5** Let \( n \) be a positive integer. Then both

\[
\frac{(1 - q)[15n]![2n]!}{(1 - q^{10n+1})[10n]![4n]![3n]!}, \quad \text{and} \quad \frac{(1 - q^3)(1 - q^7)[15n]![2n]!}{(1 - q)(1 - q^{10n+3})[10n]![4n]![3n]!}
\]

are polynomials in \( q \).

We end the paper with the following conjecture, strengthening the above theorem.

**Conjecture 7.6** The two polynomials in Theorem 7.5 have non-negative integer coefficients.

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References


