THE RODRIGUEZ-VILLEGAS TYPE CONGRUENCES FOR TRUNCATED \( q \)-HYPERGEOMETRIC FUNCTIONS

VICTOR J. W. GUO, HAO PAN, AND YONG ZHANG

Abstract. In this paper, we establish a Rodriguez-Villegas type congruence for truncated \( q \)-hypergeometric functions. Using this result, we can confirm several conjectures of Guo and Zeng, such as

\[
\sum_{k=0}^{p-1} \left( \frac{q}{q^3}; q^3 \right)_k k \equiv \left( \frac{-3}{p} \right)^{1-p^2} \frac{q^{1-p^2}}{q^{1-p^2}} (1 + q + \cdots + q^{p-1})^2, \\
\]

where \( p \geq 5 \) is a prime, \((a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})\), and \((\frac{a}{p})\) denotes the Legendre symbol modulo \( p \).

1. Introduction

Define the truncated hypergeometric function

\[
_{2}F_{1}\left[ \begin{array}{c} x_{1}, x_{2} \\ y_{1} \end{array} \right| z \right]_{n} = \sum_{k=0}^{n-1} \frac{(x_{1})_{k}(x_{2})_{k}}{(y_{1})_{k}} \frac{z^k}{k!},
\]

where \((x)_{k} = x(x+1) \cdots (x+k-1)\) if \( k \geq 1 \) and \((x)_{0} = 1\). Motivated by the Calabi-Yau manifold, Rodriguez-Villegas [8] conjectured some congruences on truncated hypergeometric functions modulo \( p^2 \) and \( p^3 \). Nowadays, most of those conjectures have been confirmed. For example, with the help of Gross-Koblitz’s

2010 Mathematics Subject Classification. Primary 11B65, Secondary 05A10, 05A30.

Key words and phrases. congruences, truncated \( q \)-hypergeometric function, cyclotomic polynomial, \( q \)-Chu-Vandermonde identity.

The first author was supported by the National Natural Science Foundation of China (Grant No. 11371144). The second and the third authors were supported by the National Natural Science Foundation of China (Grant No. 11271185). The third author was also supported by the National Natural Science Foundation of China (Grant No. 11401301).
formula, Mortenson \[5, 6\] first proved that, for any prime \(p \geq 5\),

\[
\begin{align*}
\binom{1/2, 1/2}{1}_p &= \left(\frac{-1}{p}\right) \pmod{p^2}, \quad (1.1) \\
\binom{1/3, 2/3}{1}_p &= \left(\frac{-3}{p}\right) \pmod{p^2}, \quad (1.2) \\
\binom{1/4, 3/4}{1}_p &= \left(\frac{-2}{p}\right) \pmod{p^2}, \quad (1.3) \\
\binom{1/6, 5/6}{1}_p &= \left(\frac{-1}{p}\right) \pmod{p^2}, \quad (1.4)
\end{align*}
\]

where \(\left(\frac{\cdot}{p}\right)\) is the Legendre symbol modulo \(p\). In \[11\], Z.-W. Sun gave an elementary proof for (1.1)–(1.4). Subsequently, Z.-H. Sun \[10\] generalized the above congruences to the following unified form:

\[
\binom{\alpha, 1-\alpha}{1}_p \equiv (-1)^{\langle -\alpha \rangle_p} \pmod{p^2}. \quad (1.5)
\]

Here \(\alpha\) is a rational number whose denominator is prime to \(p\), and \(\langle x \rangle_p\) denotes the integer in \(\{0, 1, \ldots, p - 1\}\) such that \(x \equiv \langle x \rangle_p \pmod{p}\). Note that we can also define \(\langle x \rangle_n\) similarly on condition that the denominator of \(x\) is prime to \(n\).

On the other hand, in recent years, some congruences for \(q\)-binomial sums have been discussed (cf. \[2, 3, 4, 7, 9, 12\]). It is natural to consider the truncated \(q\)-hypergeometric function as follows:

\[
\Phi_{r+1}
\begin{bmatrix}
\begin{array}{c}
\frac{x_1, \ldots, x_{r+1}}{y_1, \ldots, y_r}
\end{array}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
q, z
\end{array}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{cc}
q, 1
\end{array}
\end{bmatrix}
\end{bmatrix}_n
= \sum_{k=0}^{n-1} \frac{(x_1; q)_k \cdots (x_{r+1}; q)_k}{(y_1; q)_k \cdots (y_r; q)_k (q; q)_k} z^k,
\]

where

\[
(x; q)_k = \begin{cases} 
(1-x)(1-xq)\cdots(1-xq^{k-1}), & \text{if } k \geq 1, \\
1, & \text{if } k = 0.
\end{cases}
\]

Guo and Zeng \[3\] obtained a \(q\)-analogue of (1.1) as follows:

\[
\phi_1
\begin{bmatrix}
\begin{array}{c}
q, q^2
\end{array}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
q^2, 1
\end{array}
\end{bmatrix}
\end{bmatrix}_p \equiv \left(\frac{-1}{p}\right) q^{1-p^2} \pmod{[p]^2}, \quad p \geq 3, \quad (1.6)
\]

where \([p] = 1 + q + \cdots + q^{p-1}\) and the above congruence is considered over the polynomial ring \(\mathbb{Z}[q]\). Furthermore, they \[3, \text{Conjecture 1.2}\] also conjectured that,
THE RODRIGUEZ-VILLEGAS TYPE CONGRUENCES FOR TRUNCATED $q$-HYPERGEOMETRIC FUNCTIONS

for any prime $p \geq 5$,

$$2^{\phi_1} \left[ q, q^2 \left| q^3, 1 \right. \right]_p \equiv \left( \frac{-3}{p} \right) q^{1-p^2 \over p} \pmod {[p]^2}, \quad (1.7)$$

$$2^{\phi_1} \left[ q, q^3 \left| q^4, 1 \right. \right]_p \equiv \left( \frac{-2}{p} \right) q^{3(1-p^2 \over 8} \pmod {[p]^2}, \quad (1.8)$$

$$2^{\phi_1} \left[ q, q^5 \left| q^6, 1 \right. \right]_p \equiv \left( \frac{-1}{p} \right) q^{5(1-p^2 \over 12} \pmod {[p]^2}. \quad (1.9)$$

In this paper, we shall prove the congruences (1.7)–(1.9). More precisely, we shall give a $q$-analogue of (1.5) as follows.

**Theorem 1.1.** Let $n, d \geq 2$ with $\gcd(n, d) = 1$ and let $r$ be an integer. If $n$ is odd, then

$$2^{\phi_1} \left[ q^r, q^{d-r} \left| q^d, 1 \right. \right]_n \equiv (-1)^a q^{(ad+r)(a-\frac{n+1}{2})-d(a+1)} \pmod {\Phi_n(q)^2},$$

where $\Phi_n(q)$ denotes the $n$-th cyclotomic polynomial in $q$ and $a = \langle -r/d \rangle_n$. And if $n$ is even, then

$$2^{\phi_1} \left[ q^r, q^{d-r} \left| q^d, 1 \right. \right]_n \equiv (-1)^a q^{a+\frac{ad+r}{n}} q^{(ad+r)(a-\frac{n+1}{2})-d(a+1)} \pmod {\Phi_n(q)^2}.$$

For example, letting $r = 1$, $d = 3$ and letting $n = p$ be a prime greater than 3, we have

$$a = \langle -1/3 \rangle_p = \begin{cases} (p-1)/3, & \text{if } p \equiv 1 \pmod 3, \\ (2p-1)/3, & \text{if } p \equiv 2 \pmod 3, \end{cases}$$

and so

$$2^{\phi_1} \left[ q, q^2 \left| q^3, 1 \right. \right]_p \equiv \left( \frac{-3}{p} \right) q^{ad+r)(a-\frac{n+1}{2})-d(a+1)} = \left( \frac{-3}{p} \right) q^{1-p^2 \over p} \pmod {[p]^2},$$

which is the congruence (1.7).

2. Proof of Theorem 1.1

Recall that the $q$-binomial coefficients $\left[ \begin{array}{c} n \\ k \end{array} \right]_q$ are defined by

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \begin{cases} \frac{(q^{n-k+1}; q)_k}{(q; q)_k}, & \text{if } k \geq 0, \\ 0, & \text{otherwise}, \end{cases}$$
and the $q$-integer $[n]_q$ is defined as $[n]_q = \frac{1-q^n}{1-q}$. Our proof of Theorem 1.1 only requires the following two forms of the $q$-Chu-Vandermonde identity (see, for example, [1, (3.3.10)]):

\[
\begin{align*}
{n + m \choose k}_q &= \sum_{j=0}^{k} q^{j(n-j)(k-j)} \left[{n \choose j}_q \right] \left[{m \choose k-j}_q \right], \\
\left[{n + m \choose k}_q \right] &= \sum_{j=0}^{k} q^{j(m-k+j)} \left[{n \choose j}_q \left[{m \choose k-j}_q \right] \right].
\end{align*}
\] (2.1)

It is clear that

\[
\frac{(q^r; q^d)_k}{(q^d; q^d)_k} = (-1)^k q^{rk + \frac{d(s-1)}{2}} \left[-\frac{r}{d}\right],
\]

So writing $\alpha = -r/d$, we see that Theorem 1.1 is equivalent to

\[
\sum_{k=0}^{n-1} q^{ak^2} \left[{\alpha \choose k}_q \left[\frac{1 - \alpha}{k}\right] \right] \equiv (-1)^{n-1} q^{d(a-\alpha)(a-\frac{n-1}{2}) - d^2} (\mod \Phi_n(q)^2),
\]

where $\delta = 1$ if $n$ is even and $\delta = 0$ otherwise.

Note that $a = \langle \alpha \rangle_n$. Let $s = (\alpha - a)/n$. Then $sd$ is an integer. By the $q$-Chu-Vandermonde identity (2.1), we have

\[
\left[{a + sn \choose k}_q \right] \equiv \sum_{j=1}^{k} q^{j(sn-j)(k-j)} \left[{sn \choose j}_q \left[\frac{a}{k-j}\right] \right] \left[{a \choose k}_q \right] - \sum_{j=1}^{k} (\sum_{j=1}^{k} (-1)^j q^{-sj(k-j)-d} \left[{sn \choose j}_q \right] \left[{a \choose k-j}_q \right] \right) \equiv (-1)^{n-1} q^{-d} \left[{a \choose k}_q \right] \left[\frac{1}{k-j}\right] \left[\frac{a}{k-j}\right] \left[{a \choose k}_q \right] \left[\frac{1}{k-j}\right] (\mod \Phi_n(q)^2),
\]

where we have used the fact that

\[
\frac{[sn]_{q^d}}{[j]_{q^d}} = \frac{1 - q^{sdn}}{1 - q^{jd}} \equiv 0 (\mod \Phi_n(q)),
\]

\[
\left[sn - 1 \choose j - 1\right]_{q^d} \equiv \left[\frac{1}{j-1}\right]_{q^d} \equiv (-1)^{j-1} q^{-d} \left[{a \choose k-j}_q \right] (\mod \Phi_n(q)),
\]

for $1 \leq j \leq n-1$. Similarly, there holds

\[
\left[-1 - a - sn \choose k\right]_{q^d} \equiv \left[-1 - a \choose k\right]_{q^d} - \sum_{j=1}^{k} (\sum_{j=1}^{k} (-1)^j q^{-sj(k-j)-d} \left[{sn]_{q^d}}{[j]_{q^d}} \left[-1 - a \choose k-j\right]_{q^d} \right) \equiv (-1)^{n-1} q^{-d} \left[-1 - a \choose k\right]_{q^d} (\mod \Phi_n(q)^2).
\]
Therefore,
\[
\sum_{k=0}^{n-1} q^{dk^2} \binom{a + sn}{k} q^t \left[ -1 - a - sn \right]_{q^t} - \sum_{k=0}^{n-1} q^{dk^2} \binom{a}{k} q^t \left[ -1 - a \right]_{q^t} = - \sum_{k=1}^{n-1} q^{dk^2} \binom{a + sn}{k} q^t \left[ -1 - a \right]_{q^t} - \sum_{k=1}^{n-1} q^{dk^2} \binom{-1 - a}{k} q^t \sum_{j=1}^{k} (-1)^j q^{-dj(k-j)-d(z)} \frac{sn_{q^t}}{[j]_{q^t}} \left[ \frac{1}{k-j} \right]_{q^t}
\]
\[
= - \sum_{k=1}^{n-1} q^{dk^2} \binom{-1 - a}{k} q^t \sum_{j=1}^{k} (-1)^j q^{-dj(k-j)-d(z)} \frac{sn_{q^t}}{[j]_{q^t}} \left[ \frac{1}{k-j} \right]_{q^t} \equiv (\text{mod } \Phi_n(q)^2).
\]

By the q-Chu-Vandermonde identity (2.2), we have
\[
\sum_{k=1}^{n-1} q^{dk^2} \binom{a}{k} q^t \sum_{j=1}^{k} (-1)^j q^{-dj(k-j)-d(z)} \frac{sn_{q^t}}{[j]_{q^t}} \left[ \frac{1}{k-j} \right]_{q^t} = \sum_{j=1}^{a} (-1)^j q^{d^2 - d(z)} \frac{a}{[j]_{q^t}} \sum_{k=j}^{a} q^{dk(k-j)} \left[ \frac{a}{a-k} \right]_{q^t} \left[ \frac{-1 - a}{k-j} \right]_{q^t}
\]
\[
= \sum_{j=1}^{a} (-1)^j q^{d^2 - d(z)} \frac{a}{[j]_{q^t}} \left[ \frac{-1}{a-j} \right]_{q^t}
= (-1)^a \sum_{j=1}^{a} q^{\frac{d(n+1)(n-2)}{2}}
\]
and
\[
\sum_{k=1}^{n-1} q^{dk^2} \binom{-1 - a}{k} q^t \sum_{j=1}^{k} (-1)^j q^{-dj(k-j)-d(z)} \frac{sn_{q^t}}{[j]_{q^t}} \left[ \frac{1}{k-j} \right]_{q^t} = \sum_{k=1}^{n-1} q^{dk^2} \binom{n - 1 - a}{k} q^t \sum_{j=1}^{k} (-1)^j q^{-dj(k-j)-d(z)} \frac{sn_{q^t}}{[j]_{q^t}} \left[ \frac{-1 - (n - 1 - a)}{k-j} \right]_{q^t}
\]
\[
= (-1)^{n-1-a} \sum_{j=1}^{n-1-a} q^{\frac{d(n-a)(n-1-a-2)}{2}}
= (-1)^{n-1-a} \sum_{j=a+1}^{n-1} q^{\frac{d(n-a)(2j-n-1-a)}{2}} \left( \text{mod } \Phi_n(q) \right).
\]
If $n$ is even, then
\[ q^n = -1 + \frac{1 - q^n}{1 - q^n} \equiv -1 \pmod{\Phi_n(q)}. \]

Since $d$ is odd in this case, we have
\[ q^{-\frac{d(n-a)(2j-n-1-a)}{2}} = q^{\frac{dn(2j-1-a)-dn(2j-1)}{2}} \equiv -q^{\frac{dn(2j-1-a)}{2}} \pmod{\Phi_n(q)}. \]

While if $n$ is odd, then
\[ q^{-\frac{d(n-a)(2j-n-1-a)}{2}} = q^{\frac{dn(2j-1-a)-dn(2j-1-a)}{2}} \equiv q^{\frac{dn(2j-1-a)}{2}} \pmod{\Phi_n(q)}. \]

Therefore, for any positive integer $n$, we always have
\[
(-1)^{n-1-a} \sum_{j=a+1}^{n-1} \frac{q^{-\frac{d(n-a)(2j-n-1-a)}{2}}}{[n-j]q^d} \equiv (-1)^{a-1} \sum_{j=a+1}^{n-1} \frac{q^{-\frac{d(a+1)(a-2j)}{2}}}{[j]q^d} \pmod{\Phi_n(q)},
\]
since
\[ \frac{1}{[n-j]q^d} - \frac{q^{jd}}{[n]q^d} = \frac{-q^{jd}}{[j]q^d} \equiv q^{jd} \pmod{\Phi_n(q)}. \]

It follows from (2.5) that
\[
\sum_{k=1}^{n-1} q^{dk^2} \left[ -1-a \right]_{q^d} \sum_{j=1}^{k} \frac{(-1)^j q^{-dj(k-j)-d(\frac{j}{2})} \left[ a \right]}{[j]q^d} \left[ k-j \right]_{q^d} \\
\equiv (-1)^{a-1} \sum_{j=a+1}^{n-1} \frac{q^{-\frac{d(a+1)(a-2j)}{2}}}{[j]q^d} \pmod{\Phi_n(q)}.
\]

Substituting (2.4) and (2.6) into (2.3), we get
\[
\sum_{k=0}^{n-1} q^{dk^2} \left[ \frac{a+sn}{k} \right]_{q^d} \left[ -1-a-sn \right]_{q^d} \\
\equiv \sum_{k=0}^{n-1} q^{dk^2} \left[ \frac{a}{k} \right]_{q^d} \left[ -1-a \right]_{q^d} + [sn]_{q^d} \cdot (-1)^a \sum_{j=1}^{n-1} \frac{q^{-\frac{d(a+1)(a-2j)}{2}}}{[j]q^d} \pmod{\Phi_n^2(q)},
\]

by noticing the relation
\[ [-sn]_{q^d} = -q^{-sdn} [sn]_{q^d} \equiv -[sn]_{q^d} \pmod{\Phi_n(q)}. \]
It is clear that
\[
\sum_{j=1}^{n-1} \frac{1}{[j]_q} = \frac{1}{2} \sum_{j=1}^{n-1} \left( \frac{1}{[j]_q} + \frac{1}{[n-j]_q} \right)
\equiv 1 \sum_{j=1}^{n-1} \frac{1-q^{jd}}{[j]_q} = \frac{n-1}{2} (1-q^d) \pmod{\Phi_n(q)}.
\]

Hence,
\[
\sum_{j=1}^{n-1} \frac{1-q^{jd}(a+1)}{[j]_q} = \sum_{j=1}^{n-1} \frac{1}{[j]_q} - (1-q^d) \sum_{j=1}^{n-1} \sum_{k=0}^{a} q^{dkj}
\equiv \sum_{j=1}^{n-1} \frac{1}{[j]_q} - (n-1)(1-q^d) - (1-q^d) \sum_{k=1}^{a} \frac{q^{dk} - q^{dkn}}{1-q^{dk}}
\equiv \frac{n-1}{2} (1-q^d) - (n-1)(1-q^d) + a(1-q^d) \pmod{\Phi_n(q)}. \tag{2.9}
\]

Combining (2.7) and (2.9), we obtain
\[
\sum_{k=0}^{n-1} q^{dk^2} [a+sn] \begin{bmatrix} a+sn \end{bmatrix}_{q^d} = \sum_{k=0}^{n-1} q^{dk^2} [a] \begin{bmatrix} -1-a \end{bmatrix}_{q^d} + \frac{2a+1-n}{2} (-1)^a q^{-d\left(\frac{a+1}{2}\right)} (1-q^{sn}) \pmod{\Phi_n(q)^2}.
\]

By the $q$-Chu-Vandermonde identity (2.2),
\[
\sum_{k=0}^{n-1} q^{dk^2} [a] \begin{bmatrix} -1-a \end{bmatrix}_{q^d} = [1]_{q^d} = (-1)^a q^{-d\left(\frac{a+1}{2}\right)}.
\]

Therefore,
\[
\sum_{k=0}^{n-1} q^{dk^2} [a+sn] \begin{bmatrix} a+sn \end{bmatrix}_{q^d} \equiv (-1)^a q^{-d\left(\frac{a+1}{2}\right)} \left( 1 + \frac{2a+1-n}{2} (1-q^{sn}) \right) \pmod{\Phi_n(q)^2}.
\]

If $n$ is odd, then
\[
q^{d(a-n)} q^{d\left(\frac{n+1}{2}-a\right)} = (1 - (1-q^{sn}))^{\frac{n+1}{2}-a}
\equiv 1 + \frac{2a+1-n}{2} (1-q^{sn}) \pmod{\Phi_n(q)^2}.
\]
While if $n$ is even, then $q^{n/2} \equiv -1 \pmod{\Phi_n(q)}$ and

$$q^{d(\alpha - a)(\frac{n-1}{2} - a)} = ((q^{\frac{n}{2}})^{\frac{(\alpha - a)d}{n}} - (-1)^{\frac{(\alpha - a)d}{n}}) + (-1)^{\frac{(\alpha - a)d}{n}}n - 2a$$

$$\equiv (-1)^{sd} + (n - 1 - 2a)(-1)^{sd(n-2-2a)}(q^{\frac{n}{2}sd} - (-1)^{sd})$$

$$= (-1)^{sd} + (n - 1 - 2a)\frac{q^{sdn} - 1}{q^{\frac{n}{2}sd} + (-1)^{sd}}$$

$$\equiv (-1)^{sd} + (n - 1 - 2a)\frac{q^{sdn} - 1}{(-1)^{sd} + (-1)^{sd}} \pmod{\Phi_n(q^2)},$$

i.e.,

$$(−1)^{sd}q^{(\alpha - a)d(\frac{n-1}{2} - a)} \equiv 1 + \frac{2a + 1 - n}{2} (1 - q^{sdn}) \pmod{\Phi_n(q^2)}.$$  

This completes the proof.

3. Further remarks

Guo and Zeng [3, Corollaries 2.4 and 2.6] also gave two invariants of (1.6) as follows:

$$2\phi_1\left[q, q^{d-r} \mid q^2, q^d\right] = \left(\frac{-1}{p}\right) q^{\frac{n-r}{2}} \pmod{[p]^2},$$  \hspace{1cm} (3.1)

$$3\phi_2\left[q, q^{-1} - q^2 \mid q^2, q^d\right] = \left(\frac{-1}{p}\right) \pmod{[p]^2}. \hspace{1cm} (3.2)$$

In this section we shall give some generalizations of (3.1) and (3.2).

**Corollary 3.1.** Let $n, d \geq 2$ with $\gcd(n, d) = 1$. Let $r$ be an integer and $a = \langle -r/d \rangle_n$. Then

$$2\phi_1\left[q^r, q^{d-r} - q^d \mid q^d, q^d\right]_n \equiv \begin{cases} (-1)^a q^{d(a+1)-(ad+r)(\alpha - a)} \pmod{\Phi_n(q^2)}, & n \text{ odd,} \\ (-1)^{ad+r} n \cdot q^{d(a+1)-(ad+r)(\alpha - a)} \pmod{\Phi_n(q^2)}, & n \text{ even.} \end{cases} \hspace{1cm} (3.3)$$

**Proof.** Just notice that

$$2\phi_1\left[q^r, q^{d-r} - q^d \mid q^d, q^d\right]_n = 2\phi_1\left[q^{-r}, q^{r-d} - q^{-d} \mid q^{-d}, 1\right]_n,$$

and $\Phi_n(q^{-1}) = \Phi_n(q)q^{-\varphi(n)}$ for $n \geq 2$, where $\varphi(n)$ is Euler’s totient function. \hspace{1cm} \square

It is easy to see that, for $d = 3, 4, 6$, $r = 1$ and $n$ being a prime greater than 3, we may simplify (3.3) as follows:
Corollary 3.2. Let $p \geq 5$ be a prime. Then
\[
2\phi_1 \left[ \begin{array}{c} q, q^2 \\ q^3 \end{array} \right]_{p} \equiv \left( \frac{-3}{p} \right) q^{\frac{p^2-1}{3}} \pmod{[p]^2},
\]
\[
2\phi_1 \left[ \begin{array}{c} q, q^3 \\ q^4 \end{array} \right]_{p} \equiv \left( \frac{-2}{p} \right) q^{\frac{3(p^2-1)}{8}} \pmod{[p]^2},
\]
\[
2\phi_1 \left[ \begin{array}{c} q, q^5 \\ q^6 \end{array} \right]_{p} \equiv \left( \frac{-1}{p} \right) q^{\frac{3(p^2-1)}{12}} \pmod{[p]^2}.
\]

Note that Corollary 3.2 confirms another conjecture of Guo and Zeng [3, Conjecture 3.4]. However, it seems still rather difficult to tackle [3, Conjectures 7.1 and 7.2] and [4, Conjectures 6.2 and 6.3], which are further generalizations of (1.7)–(1.9) (or, equivalently, Corollary 3.2).

Numerical calculation suggests the following generalization of (3.2).

Conjecture 3.3. Suppose that $n \geq 3$ is an odd integer. Let $d \geq 2$ be an integer with $\gcd(n,d) = 1$. Then, for any integer $r$, there holds
\[
3\phi_2 \left[ \begin{array}{c} q^r, q^{d-r}, -1 \\ q^d, q^d \end{array} \right]_{n} \equiv (-1)^{\langle -r/d \rangle n} \pmod{\Phi_n(q)^2}. \tag{3.4}
\]

It seems that there does not exist a congruence similar to (3.4) for an even integer $n$. Note that (3.4) in the case where $n$ is an odd prime still remains open. What we have known is, for any odd prime $p$,
\[
3\phi_2 \left[ \begin{array}{c} q^r, q^{d-r}, -1 \\ q^d, q^d \end{array} \right]_{p} \equiv (-1)^{\langle -r/d \rangle p} \pmod{[p]^2} \tag{3.5}
\]
(see [3, (3.8)]). For another difficult conjecture related to (3.5), we refer the reader to [3, Conjecture 7.3].

We are unable to prove Conjecture 3.3. However, we shall give some partial results related to this conjecture.

Lemma 3.1. Let $n$ and $j$ be non-negative integers with $j \leq n$. Then
\[
\sum_{k=j}^{n} q^{k(k-j+1)} \left[ \begin{array}{c} [n] \\ [k] \end{array} \right]_{q} \frac{[1 - n]}{[k - j]} = (-1)^{n-j} \frac{(-x; q)_{j} (-q/x; q)_{n-j} x^{n-j} q^{j(n-j+1)}}{(-x; q)_{n+1}}. \tag{3.6}
\]

Proof. This is just the Lagrange interpolation formula for $(-q/x; q)_{n-j} x^{n-j}$ at the values $q^{-k} (j \leq k \leq n)$ of $x$. \qed

Let $\alpha = -r/d$, $a = \langle \alpha \rangle /n$, and $s = (\alpha - a)/n$ as before. It is easy to see that the congruence (3.4) is equivalent to
\[
\sum_{k=0}^{n-1} 2q^{d(k^2+k)} \left[ \begin{array}{c} [a + sn] \\ [k] \end{array} \right]_{q^d} \frac{[1 - a - sn]}{[k]} \equiv (-1)^{a} \pmod{\Phi_n(q)^2}.
\]
Since \( n \) is odd, it is well known that \( \gcd(1 + q^k, 1 - q^n) = 1 \) for all non-negative integers \( k \). Therefore, similarly to (2.3), modulo \( \Phi_n(q)^2 \),

\[
\sum_{k=0}^{n-1} \frac{2q^{d(k^2+k)}}{1+q^k} \binom{a+sn}{k} q^d \left[ -1 - a - sn \right] = \sum_{k=0}^{n-1} \frac{2q^{d(k^2+k)}}{1+q^k} \binom{a}{k} q^d \left[ -1 - a \right]
\]

by the identity (3.6) \((x = 1)\), we have

\[
\sum_{k=1}^{n-1} \frac{2q^{d(k^2+k)}}{1+q^k} \binom{a}{k} q^d \left[ -1 - a \right] \equiv \sum_{k=1}^{n-1} \frac{2q^{d(k^2+k)}}{1+q^k} \sum_{j=1}^{k} (-1)^j q^{-dj(k-j)-\frac{d(j)}{2}} \left[ -sn \right] q^d \left[ -1 - a \right] q^d \left[ k - j \right] q^d
\]

\[
\sum_{k=1}^{n-1} \frac{2q^{d(k^2+k)}}{1+q^k} \binom{a}{k} q^d \left[ -1 - a \right] \equiv \sum_{k=1}^{n-1} \frac{2q^{d(k^2+k)}}{1+q^k} \sum_{j=1}^{k} (-1)^j q^{-dj(k-j)-\frac{d(j)}{2}} \left[ sn \right] q^d \left[ a \right] q^d \left[ k - j \right] q^d.
\]

By the identity (3.6) \((x = 1)\), we have

\[
\sum_{k=0}^{n-1} \frac{2q^{d(k^2+k)}}{1+q^k} \binom{a}{k} q^d \left[ -1 - a \right] = (-1)^a,
\]

and

\[
\sum_{k=1}^{n-1} \frac{2q^{d(k^2+k)}}{1+q^k} \binom{a}{k} q^d \left[ -1 - a \right] \equiv \sum_{k=1}^{n-1} \frac{2q^{d(k^2+k)}}{1+q^k} \sum_{j=1}^{k} (-1)^j q^{-dj(k-j)-\frac{d(j)}{2}} \left[ a \right] q^d \left[ k - j \right] q^d
\]

\[
\sum_{k=1}^{n-1} \frac{2q^{d(k^2+k)}}{1+q^k} \binom{a}{k} q^d \left[ -1 - a \right] \equiv (-1)^{n-1-a} \sum_{j=1}^{n-1-a} \frac{q^{d(\frac{2n-2n+1+j}{2})}}{[j] q^d} \left( -1 ; q^d \right)_j \left( -q^d ; q^d \right)_{n-a-1-j} \quad \text{mod } \Phi_n(q).
\]

For any non-negative integer \( m \), let

\[
F_q(m, d) := \sum_{j=1}^{m} \frac{q^{d(\frac{2m+3-j}{2})}}{[j] q^d} \left( -1 ; q^d \right)_j \left( -q^d ; q^d \right)_{m-j}.
\]

Substituting (3.8)–(3.10) into (3.7), and noticing (2.8), we have

\[
\sum_{k=0}^{n-1} \frac{2q^{d(k^2+k)}}{1+q^k} \binom{a+sn}{k} q^d \left[ -1 - a - sn \right] \equiv (-1)^a + (-1)^a [sn] q^d (F_q(a, d) - F_q(n-1-a, d)) \quad \text{mod } \Phi_n^2(q).
\]
Therefore, Conjecture 3.3 can be deduced from the following conjectural congruence modulo $\Phi_n(q)$.

**Conjecture 3.4.** Let $n \geq 3$ be an odd integer, and let $d$ be a positive integer with $\gcd(n, d) = 1$. Then, for any non-negative integer $a \leq n - 1$,

$$F_q(a, d) \equiv F_q(n - 1 - a, d) \pmod{\Phi_n(q)}.$$ 

**References**