Some Results on Fibred Algebraic Surfaces

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2013. 7

1. Fibred surface

- Fibration $f : S \to C \overset{def}{\iff}$ holomorphic & surjective, S smooth surface, C smooth curve of genus b.
- Fiber $F_t := f^{-1}(t)$ connected,
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- Global invariants:
  $c_1^2(S), \quad c_2(S), \quad \chi(\mathcal{O}_S), \quad q(S) := h^{1,0}(S), \quad h^{1,1}(S), \cdots$
- Noether Formula: $c_1^2(S) + c_2(S) = 12\chi(\mathcal{O}_S)$.
- Relative invariants:

$$\begin{cases} K_f^2 &= c_1^2(S) - 8(g-1)(b-1), \\ e_f &= c_2(S) - 4(g-1)(b-1), \\ \chi_f &= \chi(\mathcal{O}_S) - (g-1)(b-1) = \deg f_*\omega_{S/C}, \\ q_f &= q(S) - b. \end{cases}$$

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- $g = 0$: Ruled surface.

$$K_f^2 = \chi_f = e_f = 0.$$

- $g = 1$: Elliptic fibration.

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- $$e_f = \sum_{F_t} e_{F_t}.$$

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Assume $g > 1$

- Positivity(Fujita, Parshin, Arakelov, Beauville):

$$K_f^2 \geq 0, \quad \chi_f \geq 0, \qquad " = " \iff f \text{ local trivial}$$

- $e_f \geq 0$ ($i.e., 12\chi_f \geq K_f^2$), $\qquad "=" \iff$ all fibers smooth
- Relative Irregularity: $0 \leq q_f \leq g$.
  "$q_f = g$" $\iff f$ trivial.
- Cornalba-Harris-Xiao Inequality:

$$K_f^2 \geq \left(4 - \frac{4}{g}\right)\chi_f.$$

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- Vojta inequality(Canonical class inequality):

$$K_f^2 \leq (2g-2)(2b-2+s)$$

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$$\chi_f \leq \frac{g-q_f}{2}(2b-2+s) \leq \frac{g}{2}(2b-2+s).$$

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4. $h^{1,1}$ Inequality

$F_1, \cdots, F_s$ all singular fibers.

- $\ell(F_i)$ = the number of irreducible components in F_i.
- $g(F_i)$ = sum of geometric genus of irreducible components.
- $NS(S)$ Neron-Severi group,

$$\rho(S) := \dim_{\mathbb{Q}} NS(S) \otimes \mathbb{Q}.$$

- (Beauville, 1981) $q_f \leq g(F_i)$.
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$$h^{1,1}(S) \geq \rho(S) \geq 2 + \sum_{i=1}^{s} (\ell(F_i) - 1).$$

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- $h^{1,1}$ ineuqality for a fibred surface

Theorem (J. Lu, S.-L. Tan, F. Yu, K. Zuo, 2013)

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- What happens if S does not admit any fibration over a curve of genus $b \geq 1$?
- $h^{1,1}(S) \geq 2q(S) - 1$ (if $b \geq 2$)
- (R.Lazarsfeld, M.Popa, 2010)

$$h^{1,1}(S) \geq \begin{cases} 3q(S) - 2, & \text{if } q(S) \text{ is even,} \\ 3q(S) - 1, & \text{if } q(S) \text{ is odd.} \end{cases}$$

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5. χ_f formula (semistale)

- $\tilde{s} =$ the number of singular fibers satisfying $g(F_i) < g$.
- $\tilde{s} \leq s$.
- f semistable $\stackrel{def}{\Longleftrightarrow}$ all singular fibers are reduced nodal curves.
- f semistable $\Longrightarrow$

  $\tilde{s} =$ the number of singular fibers with non-compact jacobian.
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$$2\chi_f = (g - q_f)(2b - 2 + \tilde{s}) - A - B.$$

- $A := \sum_{i=1}^{\tilde{s}} (g(F_i) - q_f) \geq 0$ (Beauville)
- $B = h^{1,1}(S) - 2q_f b - 2 - \sum_{i=1}^{s} (\ell(F_i) - 1) \geq 0$ ($h^{1,1}$ inequality).

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6. Relations between Classical Inequalities

- χ_f formula & $h^{1,1}$ inequality
  $\Longrightarrow$ Viehweg-Zuo inequality
  $\Longrightarrow$ Arakelov-Faltings inequality.
- Viehweg-Zuo inequality (for subsheaf of $f_*\omega_{S/C}^{\otimes\nu}$)
  $\Longrightarrow$ Vojta inequality
  $\Longleftrightarrow$ Miyaoka-Yau inequality (for open surface)
- Vojta inequality & Cornalba-Harris-Xiao inequality
  $\Longrightarrow$ weaker Arakelov inequality

$$\chi_f < \frac{g}{2}(2b-2+s).$$

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- f semistable

Corollary

$\chi_f = \frac{g-q_f}{2}(2b-2+\tilde{s})$ *iff*
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8. χ_f formula (non-semistale)

- χ_f formula (non-semistable)

Theorem (J. Lu, S.-L. Tan, F. Yu, K. Zuo, 2013)

$$2\chi_f = (g - q_f)(2b - 2 + \tilde{s}) - A - B + C.$$

- $C := \sum_{i=1}^{\tilde{s}} N_{\overline{F}_i}, \quad 0 \leq N_{\overline{F}_i} = g - p_a(\overline{F}_{i,red}) \leq g.$
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Assume f non-trivial, semistable, $g \geq 1$.

- f Kodaira fibration $\overset{def}{\iff}$ $s = 0$.
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- Example (Xiao) $g = 2$, $\tilde{s} = 4$, $s = 7$, $q = 1$, $h^{1,1} = 5$.
- Example (J. Lu, S.-L. Tan, F. YU, K. Zuo) $g = 2$, $\tilde{s} = 4$, $s = 6$, $q = 1$, $h^{1,1} = 6$.
- Problem: Is there f such that $\tilde{s} = 4$ and $q(S) = 0$?
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A smooth curve C can be defined over algebraic number field $\overline{\mathbb{Q}}$ iff there is a cover $\pi : C \to \mathbb{P}^1$ with three branching points $0, 1, \infty$.

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Let F_1, F_2 be singular fibers.
(1) S is a ruled surface.
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- Viehweg-Zuo: $\frac{\deg \mathcal{F}}{rk\mathcal{F}} \leq \frac{n\nu}{2}(2b - 2 + s)$.
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- global invariants
  $=$ moduli invariants $+$ monodromy invariants
- Compute moduli invariants: semistable reduction (Deligne-Mumford)

$$\kappa(f) = \deg j^*\kappa, \quad \lambda(f) = \deg j^*\lambda, \quad \delta(f) = \deg j^*\delta.$$

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- Positivity (S.-L. Tan, 1996): $c_1^2(F) \geq 0$, $c_2(F) \geq 0$, $\chi_F \geq 0$. One of "=" holds iff F is semistable.
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