Some Results on Fibrations and Foliations

JUN LU

Department of Mathematics
East China Normal University

2015. 11

1. Definition of Foliation

- X: algebaic surface,
  T_X: tangent bundle of X.
  $\mathcal{L}^{-1} \subseteq T_X$: maximal sub-line bundle.
- Foliation $\mathcal{F}$ is a section

$$s \in H^0(X, T_X \otimes \mathcal{L}).$$

- Open covering $X = \cup_\alpha U_\alpha$,

$$s|_{U_\alpha} = A(x_\alpha, y_\alpha)\frac{\partial}{\partial x_\alpha} + B(x_\alpha, y_\alpha)\frac{\partial}{\partial y_\alpha}, \quad (x_\alpha, y_\alpha) \in U_\alpha.$$

- $s|_{U_\alpha} = g_{\alpha\beta} s|_{U_\beta}$, $\mathcal{L} = \{g_{\alpha\beta}\}$.

1. Definition of Foliation

- X: algebaic surface,
  T_X: tangent bundle of X.
  $\mathcal{L}^{-1} \subseteq T_X$: maximal sub-line bundle.
- Foliation $\mathcal{F}$ is a section

$$s \in H^0(X, T_X \otimes \mathcal{L}).$$

- Open covering $X = \cup_\alpha U_\alpha$,

$$s|_{U_\alpha} = A(x_\alpha, y_\alpha)\frac{\partial}{\partial x_\alpha} + B(x_\alpha, y_\alpha)\frac{\partial}{\partial y_\alpha}, \quad (x_\alpha, y_\alpha) \in U_\alpha.$$

- $s|_{U_\alpha} = g_{\alpha\beta} s|_{U_\beta}$, $\mathcal{L} = \{g_{\alpha\beta}\}$.

1. Definition of Foliation

- X: algebaic surface,
  T_X: tangent bundle of X.
  $\mathcal{L}^{-1} \subseteq T_X$: maximal sub-line bundle.
- Foliation $\mathcal{F}$ is a section

$$s \in H^0(X, T_X \otimes \mathcal{L}).$$

- Open covering $X = \cup_\alpha U_\alpha$,

$$s|_{U_\alpha} = A(x_\alpha, y_\alpha)\frac{\partial}{\partial x_\alpha} + B(x_\alpha, y_\alpha)\frac{\partial}{\partial y_\alpha}, \quad (x_\alpha, y_\alpha) \in U_\alpha.$$

- $s|_{U_\alpha} = g_{\alpha\beta} s|_{U_\beta}$, $\mathcal{L} = \{g_{\alpha\beta}\}$.

1. Definition of Foliation

- X: algebaic surface,
  T_X: tangent bundle of X.
  $\mathcal{L}^{-1} \subseteq T_X$: maximal sub-line bundle.
- Foliation $\mathcal{F}$ is a section

$$s \in H^0(X, T_X \otimes \mathcal{L}).$$

- Open covering $X = \cup_\alpha U_\alpha$,

$$s|_{U_\alpha} = A(x_\alpha, y_\alpha)\frac{\partial}{\partial x_\alpha} + B(x_\alpha, y_\alpha)\frac{\partial}{\partial y_\alpha}, \quad (x_\alpha, y_\alpha) \in U_\alpha.$$

- $s|_{U_\alpha} = g_{\alpha\beta} s|_{U_\beta}$, $\mathcal{L} = \{g_{\alpha\beta}\}$.

1. Definition of Foliation

- X: algebaic surface,
  T_X: tangent bundle of X.
  $\mathcal{L}^{-1} \subseteq T_X$: maximal sub-line bundle.
- Foliation $\mathcal{F}$ is a section

$$s \in H^0(X, T_X \otimes \mathcal{L}).$$

- Open covering $X = \cup_\alpha U_\alpha$,

$$s|_{U_\alpha} = A(x_\alpha, y_\alpha)\frac{\partial}{\partial x_\alpha} + B(x_\alpha, y_\alpha)\frac{\partial}{\partial y_\alpha}, \quad (x_\alpha, y_\alpha) \in U_\alpha.$$

- $s|_{U_\alpha} = g_{\alpha\beta} s|_{U_\beta}$, $\mathcal{L} = \{g_{\alpha\beta}\}$.

1. Definition of Foliations

- $T_{\mathcal{F}} := \mathcal{L}^{-1}$ tangent bundle of $\mathcal{F}$,
  $K_{\mathcal{F}} := \mathcal{L}$ canonical bundle of $\mathcal{F}$.
- Exact sequence

$$0 \to T_{\mathcal{F}} \xrightarrow{\cdot s} T_X \to \mathcal{I}_{Z(s)} \otimes N_{\mathcal{F}} \to 0,$$

  $N_{\mathcal{F}}$ line bundle,
  $\mathcal{I}_{Z(s)}$ ideal sheaf of $Z(s)$ (zero set of s).
- Canonical bundle

$$\omega_X := \wedge^2 \Omega_X = K_{\mathcal{F}} \otimes N_{\mathcal{F}}^{-1}$$

  Ω_X cotangent bundle of X,
  $N_{\mathcal{F}}^{-1}$ conormal bundle of $\mathcal{F}$.

1. Definition of Foliations

- $T_{\mathcal{F}} := \mathcal{L}^{-1}$ tangent bundle of $\mathcal{F}$,
  $K_{\mathcal{F}} := \mathcal{L}$ canonical bundle of $\mathcal{F}$.
- Exact sequence

$$0 \to T_{\mathcal{F}} \xrightarrow{\cdot s} T_X \to \mathcal{I}_{Z(s)} \otimes N_{\mathcal{F}} \to 0,$$

  $N_{\mathcal{F}}$ line bundle,
  $\mathcal{I}_{Z(s)}$ ideal sheaf of $Z(s)$ (zero set of s).
- Canonical bundle

$$\omega_X := \wedge^2 \Omega_X = K_{\mathcal{F}} \otimes N_{\mathcal{F}}^{-1}$$

  Ω_X cotangent bundle of X,
  $N_{\mathcal{F}}^{-1}$ conormal bundle of $\mathcal{F}$.

1. Definition of Foliations

- $T_{\mathcal{F}} := \mathcal{L}^{-1}$ tangent bundle of $\mathcal{F}$,
  $K_{\mathcal{F}} := \mathcal{L}$ canonical bundle of $\mathcal{F}$.
- Exact sequence

$$0 \to T_{\mathcal{F}} \xrightarrow{\cdot s} T_X \to \mathcal{I}_{Z(s)} \otimes N_{\mathcal{F}} \to 0,$$

  $N_{\mathcal{F}}$ line bundle,
  $\mathcal{I}_{Z(s)}$ ideal sheaf of $Z(s)$ (zero set of s).
- Canonical bundle

$$\omega_X := \wedge^2 \Omega_X = K_{\mathcal{F}} \otimes N_{\mathcal{F}}^{-1}$$

  Ω_X cotangent bundle of X,
  $N_{\mathcal{F}}^{-1}$ conormal bundle of $\mathcal{F}$.

1. Definition of Foliations

- $T_{\mathcal{F}} := \mathcal{L}^{-1}$ tangent bundle of $\mathcal{F}$,
  $K_{\mathcal{F}} := \mathcal{L}$ canonical bundle of $\mathcal{F}$.
- Exact sequence

$$0 \to T_{\mathcal{F}} \xrightarrow{\cdot s} T_X \to \mathcal{I}_{Z(s)} \otimes N_{\mathcal{F}} \to 0,$$

  $N_{\mathcal{F}}$ line bundle,
  $\mathcal{I}_{Z(s)}$ ideal sheaf of $Z(s)$ (zero set of s).
- Canonical bundle

$$\omega_X := \wedge^2 \Omega_X = K_{\mathcal{F}} \otimes N_{\mathcal{F}}^{-1}$$

  Ω_X cotangent bundle of X,
  $N_{\mathcal{F}}^{-1}$ conormal bundle of $\mathcal{F}$.

1. Definition of Foliation

- Equivalently,

$$0 \to N_{\mathcal{F}}^{-1} \to \Omega_X \to \mathcal{I}_{Z(s)} \otimes K_{\mathcal{F}} \to 0.$$

- The second definition of Foliation $\mathcal{F}$:

$$\omega \in H^0(X, \Omega_X \otimes N_{\mathcal{F}}).$$

- $$\omega|_{U_\alpha} = B(x_\alpha, y_\alpha)dx_\alpha - A(x_\alpha, y_\alpha)dy_\alpha, \quad (x_\alpha, y_\alpha) \in U_\alpha.$$

- $\omega|_{U_\alpha} = f_{\alpha\beta}\omega|_{U_\beta}$, $\mathcal{N}_{\mathcal{F}} = \{f_{\alpha\beta}\}$.

1. Definition of Foliation

- Equivalently,

$$0 \to N_{\mathcal{F}}^{-1} \to \Omega_X \to \mathcal{I}_{Z(s)} \otimes K_{\mathcal{F}} \to 0.$$

- The second definition of Foliation $\mathcal{F}$:

$$\omega \in H^0(X, \Omega_X \otimes N_{\mathcal{F}}).$$

-

$$\omega|_{U_\alpha} = B(x_\alpha, y_\alpha)dx_\alpha - A(x_\alpha, y_\alpha)dy_\alpha, \quad (x_\alpha, y_\alpha) \in U_\alpha.$$

- $\omega|_{U_\alpha} = f_{\alpha\beta}\omega|_{U_\beta}$, $\mathcal{N}_{\mathcal{F}} = \{f_{\alpha\beta}\}$.

1. Definition of Foliation

- Equivalently,

$$0 \to N_{\mathcal{F}}^{-1} \to \Omega_X \to \mathcal{I}_{Z(s)} \otimes K_{\mathcal{F}} \to 0.$$

- The second definition of Foliation $\mathcal{F}$:

$$\omega \in H^0(X, \Omega_X \otimes N_{\mathcal{F}}).$$

-

$$\omega|_{U_\alpha} = B(x_\alpha, y_\alpha)dx_\alpha - A(x_\alpha, y_\alpha)dy_\alpha, \quad (x_\alpha, y_\alpha) \in U_\alpha.$$

- $\omega|_{U_\alpha} = f_{\alpha\beta}\omega|_{U_\beta}$, $\mathcal{N}_{\mathcal{F}} = \{f_{\alpha\beta}\}$.

1. Definition of Foliation

- Equivalently,

$$0 \to N_{\mathcal{F}}^{-1} \to \Omega_X \to \mathcal{I}_{Z(s)} \otimes K_{\mathcal{F}} \to 0.$$

- The second definition of Foliation $\mathcal{F}$:

$$\omega \in H^0(X, \Omega_X \otimes N_{\mathcal{F}}).$$

-

$$\omega|_{U_\alpha} = B(x_\alpha, y_\alpha)dx_\alpha - A(x_\alpha, y_\alpha)dy_\alpha, \quad (x_\alpha, y_\alpha) \in U_\alpha.$$

- $\omega|_{U_\alpha} = f_{\alpha\beta}\omega|_{U_\beta}$, $\mathcal{N}_{\mathcal{F}} = \{f_{\alpha\beta}\}$.

1. Definition of Foliation

- Equivalently,

$$0 \to N_{\mathcal{F}}^{-1} \to \Omega_X \to \mathcal{I}_{Z(s)} \otimes K_{\mathcal{F}} \to 0.$$

- The second definition of Foliation $\mathcal{F}$:

$$\omega \in H^0(X, \Omega_X \otimes N_{\mathcal{F}}).$$

-

$$\omega|_{U_\alpha} = B(x_\alpha, y_\alpha)dx_\alpha - A(x_\alpha, y_\alpha)dy_\alpha, \quad (x_\alpha, y_\alpha) \in U_\alpha.$$

- $\omega|_{U_\alpha} = f_{\alpha\beta}\omega|_{U_\beta}$, $\mathcal{N}_{\mathcal{F}} = \{f_{\alpha\beta}\}$.

2. Example (1): fibration

- Fibration $f: X \to C$,
  C smooth curve, f holomorphic and surjective.
- Fiber $F_t = f^{-1}(t)$, $t \in C$.
  Local equation
  $$F_t: \quad f(x,y) = t.$$
  Smooth (Singular) fiber $F_t \overset{def}{\Longleftrightarrow} F_t$ smooth (singular) curve.
- Foliation $\mathcal{F}$ generated by f:
  $$\omega = \frac{1}{\mu(f)}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) \quad \text{(local eq.)},$$
  $\mu(f) = \gcd(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$.

2. Example (1): fibration

- Fibration $f : X \to C$,
  C smooth curve, f holomorphic and surjective.
- Fiber $F_t = f^{-1}(t)$, $t \in C$.
  Local equation
  $$F_t : \quad f(x,y) = t.$$
  Smooth (Singular) fiber $F_t \overset{def}{\Longleftrightarrow} F_t$ smooth (singular) curve.
- Foliation $\mathcal{F}$ generated by f:
  $$\omega = \frac{1}{\mu(f)}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) \quad \text{(local eq.)},$$
  $\mu(f) = \gcd(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$.

2. Example (1): fibration

- Fibration $f : X \to C$,
  C smooth curve, f holomorphic and surjective.
- Fiber $F_t = f^{-1}(t)$, $t \in C$.
  Local equation
  $$F_t : \quad f(x, y) = t.$$
  Smooth (Singular) fiber $F_t \overset{def}{\Longleftrightarrow} F_t$ smooth (singular) curve.
- Foliation $\mathcal{F}$ generated by f:
  $$\omega = \frac{1}{\mu(f)} \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right) \quad \text{(local eq.)},$$
  $\mu(f) = \gcd(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$.

2. Example (1): fibration

- Fibration $f: X \to C$,
  C smooth curve, f holomorphic and surjective.
- Fiber $F_t = f^{-1}(t)$, $t \in C$.
  Local equation
  $$F_t: \quad f(x,y) = t.$$
  Smooth (Singular) fiber $F_t \overset{def}{\Longleftrightarrow} F_t$ smooth (singular) curve.
- Foliation $\mathcal{F}$ generated by f:
  $$\omega = \frac{1}{\mu(f)}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) \quad \text{(local eq.)},$$
  $\mu(f) = \gcd(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$.

2. Example (1): fibration

- Canonical bundle of $\mathcal{F}$:

$$K_{\mathcal{F}} = \omega_{X/C}(-D(f)),$$

where

- $\omega_{X/C} := \omega_X \otimes f^*\Omega_C^{-1}$ (relative canonical bundle)
- $D(f) := \sum_{t \in C} (F_t - F_{t,\mathrm{red}})$ (zero divisor of df).
- Conormal bundle of $\mathcal{F}$:

$$N_{\mathcal{F}}^{-1} = f^*\Omega_C(D(f)).$$

2. Example (1): fibration

- Canonical bundle of $\mathcal{F}$:

$$K_{\mathcal{F}} = \omega_{X/C}(-D(f)),$$

where

- $\omega_{X/C} := \omega_X \otimes f^*\Omega_C^{-1}$ (relative canonical bundle)
- $D(f) := \sum_{t\in C}(F_t - F_{t,\mathrm{red}})$ (zero divisor of df).
- Conormal bundle of $\mathcal{F}$:

$$N_{\mathcal{F}}^{-1} = f^*\Omega_C(D(f)).$$

2. Example (1): fibration

- Canonical bundle of $\mathcal{F}$:

$$K_{\mathcal{F}} = \omega_{X/C}(-D(f)),$$

where

- $\omega_{X/C} := \omega_X \otimes f^*\Omega_C^{-1}$ (relative canonical bundle)
- $D(f) := \sum_{t \in C} (F_t - F_{t,\mathrm{red}})$ (zero divisor of df).
- Conormal bundle of $\mathcal{F}$:

$$N_{\mathcal{F}}^{-1} = f^*\Omega_C(D(f)).$$

2. Example (1): fibration

- Canonical bundle of $\mathcal{F}$:

$$K_{\mathcal{F}} = \omega_{X/C}(-D(f)),$$

where

- $\omega_{X/C} := \omega_X \otimes f^*\Omega_C^{-1}$ (relative canonical bundle)
- $D(f) := \sum\limits_{t \in C} (F_t - F_{t,\mathrm{red}})$ (zero divisor of df).
- Conormal bundle of $\mathcal{F}$:

$$N_{\mathcal{F}}^{-1} = f^*\Omega_C(D(f)).$$

2. Example (1): fibration

- Canonical bundle of $\mathcal{F}$:

$$K_{\mathcal{F}} = \omega_{X/C}(-D(f)),$$

where

- $\omega_{X/C} := \omega_X \otimes f^*\Omega_C^{-1}$ (relative canonical bundle)
- $D(f) := \sum_{t\in C}(F_t - F_{t,\mathrm{red}})$ (zero divisor of df).
- Conormal bundle of $\mathcal{F}$:

$$N_{\mathcal{F}}^{-1} = f^*\Omega_C(D(f)).$$

2. Example (2): Foliations on $\mathbb{P}^2$

- Foliation $\mathcal{F}$ on $\mathbb{C}^2$:

$$\omega = f(x,y)dx + g(x,y)dy$$

- Extension of $\mathcal{F}$ on $\mathbb{P}^2$ (Darboux): $[X,Y,Z] \in \mathbb{P}^2$,

$$\omega = F(X,Y,Z)dX + G(X,Y,Z)dY + H(X,Y,Z)dZ,$$

  $\deg F = \deg G = \deg H = d.$

- Canonical bundle of $\mathcal{F}$: $K_{\mathcal{F}} = \mathcal{O}_X(d-2)$.
  Conormal bundle of $\mathcal{F}$: $N_{\mathcal{F}}^{-1} = \mathcal{O}_X(d-1)$.

2. Example (2): Foliations on $\mathbb{P}^2$

- Foliation $\mathcal{F}$ on $\mathbb{C}^2$:

$$\omega = f(x,y)dx + g(x,y)dy$$

- Extension of $\mathcal{F}$ on $\mathbb{P}^2$ (Darboux): $[X,Y,Z] \in \mathbb{P}^2$,

$$\omega = F(X,Y,Z)dX + G(X,Y,Z)dY + H(X,Y,Z)dZ,$$

  $\deg F = \deg G = \deg H = d.$

- Canonical bundle of $\mathcal{F}$: $K_{\mathcal{F}} = \mathcal{O}_X(d-2)$.
  Conormal bundle of $\mathcal{F}$: $N_{\mathcal{F}}^{-1} = \mathcal{O}_X(d-1)$.

2. Example (2): Foliations on $\mathbb{P}^2$

- Foliation $\mathcal{F}$ on $\mathbb{C}^2$:

$$\omega = f(x,y)dx + g(x,y)dy$$

- Extension of $\mathcal{F}$ on $\mathbb{P}^2$ (Darboux): $[X,Y,Z] \in \mathbb{P}^2$,

$$\omega = F(X,Y,Z)dX + G(X,Y,Z)dY + H(X,Y,Z)dZ,$$

  $\deg F = \deg G = \deg H = d.$

- Canonical bundle of $\mathcal{F}$: $K_{\mathcal{F}} = \mathcal{O}_X(d-2)$.
  Conormal bundle of $\mathcal{F}$: $N_{\mathcal{F}}^{-1} = \mathcal{O}_X(d-1)$.

2. Example (2): Foliations on $\mathbb{P}^2$

- Foliation $\mathcal{F}$ on $\mathbb{C}^2$:

$$\omega = f(x,y)dx + g(x,y)dy$$

- Extension of $\mathcal{F}$ on $\mathbb{P}^2$ (Darboux): $[X,Y,Z] \in \mathbb{P}^2$,

$$\omega = F(X,Y,Z)dX + G(X,Y,Z)dY + H(X,Y,Z)dZ,$$

  $\deg F = \deg G = \deg H = d$.

- Canonical bundle of $\mathcal{F}$: $K_{\mathcal{F}} = \mathcal{O}_X(d-2)$.
  Conormal bundle of $\mathcal{F}$: $N_{\mathcal{F}}^{-1} = \mathcal{O}_X(d-1)$.

3. $\mathcal{F}$-invariant curve

- $\mathcal{F}$ foliation: $\{(U_\alpha, \omega_\alpha)\}$ (or $\{(U_\alpha, s_\alpha)\}$),

$$s_\alpha = A_\alpha \frac{\partial}{\partial x_\alpha} + B_\alpha \frac{\partial}{\partial y_\alpha}$$

  or

$$\omega_\alpha = B_\alpha dx_\alpha - A_\alpha dy_\alpha.$$

- $C \subseteq X$ curve defined by $f_\alpha = 0$ on U_α.
  C is $\mathcal{F}$-invariant $\overset{def}{\Longleftrightarrow}$
  $\forall p \in C$, vector $s(p)$ is tangent to C at p.

3. $\mathcal{F}$-invariant curve

- $\mathcal{F}$ foliation: $\{(U_\alpha, \omega_\alpha)\}$ (or $\{(U_\alpha, s_\alpha)\}$),

$$s_\alpha = A_\alpha \frac{\partial}{\partial x_\alpha} + B_\alpha \frac{\partial}{\partial y_\alpha}$$

or

$$\omega_\alpha = B_\alpha dx_\alpha - A_\alpha dy_\alpha.$$

- $C \subseteq X$ curve defined by $f_\alpha = 0$ on U_α.

  C is $\mathcal{F}$-invariant $\overset{def}{\Longleftrightarrow}$

  $\forall p \in C$, vector $s(p)$ is tangent to C at p.

3. $\mathcal{F}$-invariant curve

- $\mathcal{F}$ foliation: $\{(U_\alpha, \omega_\alpha)\}$ (or $\{(U_\alpha, s_\alpha)\}$),

$$s_\alpha = A_\alpha \frac{\partial}{\partial x_\alpha} + B_\alpha \frac{\partial}{\partial y_\alpha}$$

  or

$$\omega_\alpha = B_\alpha dx_\alpha - A_\alpha dy_\alpha.$$

- $C \subseteq X$ curve defined by $f_\alpha = 0$ on U_α.
  C is $\mathcal{F}$-invariant $\overset{def}{\Longleftrightarrow}$
  $\forall p \in C$, vector $s(p)$ is tangent to C at p.

3. $\mathcal{F}$-invariant curve

C is $\mathcal{F}$-invariant iff

-
$$f_\alpha \mid s(f_\alpha)$$

- iff f_α is the solution of ODE
$$\omega_\alpha = 0.$$

Example

Let $\mathcal{F}$ be a foliation generated by a fibration$f : X \to C$. Then $C \subseteq X$ is $\mathcal{F}$-invariant iff C lies in the fibers of f.

3. $\mathcal{F}$-invariant curve

C is $\mathcal{F}$-invariant iff

- $$f_\alpha \mid s(f_\alpha)$$
- iff f_α is the solution of ODE
$$\omega_\alpha = 0.$$

Example

Let $\mathcal{F}$ be a foliation generated by a fibration$f : X \to C$. Then $C \subseteq X$ is $\mathcal{F}$-invariant iff C lies in the fibers of f.

3. $\mathcal{F}$-invariant curve

C is $\mathcal{F}$-invariant iff

-
$$f_\alpha \mid s(f_\alpha)$$

- iff f_α is the solution of ODE
$$\omega_\alpha = 0.$$

Example

Let $\mathcal{F}$ be a foliation generated by a fibration$f : X \to C$. Then $C \subseteq X$ is $\mathcal{F}$-invariant iff C lies in the fibers of f.

3. $\mathcal{F}$-invariant curve

C is $\mathcal{F}$-invariant iff

- $$f_\alpha \mid s(f_\alpha)$$
- iff f_α is the solution of ODE
$$\omega_\alpha = 0.$$

Example

Let $\mathcal{F}$ be a foliation generated by a fibration$f : X \to C$. Then $C \subseteq X$ is $\mathcal{F}$-invariant iff C lies in the fibers of f.

3. $\mathcal{F}$-invariant curve

$s :=$ the number of irreducible compact $\mathcal{F}$-invariant curves.

- Question 1: When does $s = \infty$?
- $\mathcal{F}$ generated by a fibration $\Longrightarrow s = \infty$.

Theorem (Jouanolou, 1978)

If

$$s \geq h^0(X, K_{\mathcal{F}}) + h^{1,1}(X) - h^{1,0}(X) + 2,$$

then $\mathcal{F}$ is generated by a fibration.

- Question 2: How to determine all $\mathcal{F}$-invariant curves when $s < \infty$?

3. $\mathcal{F}$-invariant curve

$s :=$ the number of irreducible compact $\mathcal{F}$-invariant curves.

- Question 1: When does $s = \infty$?
- $\mathcal{F}$ generated by a fibration $\Longrightarrow s = \infty$.

Theorem (Jouanolou, 1978)

If

$$s \geq h^0(X, K_{\mathcal{F}}) + h^{1,1}(X) - h^{1,0}(X) + 2,$$

then $\mathcal{F}$ is generated by a fibration.

- Question 2: How to determine all $\mathcal{F}$-invariant curves when $s < \infty$?

3. $\mathcal{F}$-invariant curve

$s :=$ the number of irreducible compact $\mathcal{F}$-invariant curves.

- Question 1: When does $s = \infty$?
- $\mathcal{F}$ generated by a fibration $\Longrightarrow s = \infty$.

Theorem (Jouanolou, 1978)

If

$$s \geq h^0(X, K_{\mathcal{F}}) + h^{1,1}(X) - h^{1,0}(X) + 2,$$

then $\mathcal{F}$ is generated by a fibration.

- Question 2: How to determine all $\mathcal{F}$-invariant curves when $s < \infty$?

3. $\mathcal{F}$-invariant curve

$s :=$ the number of irreducible compact $\mathcal{F}$-invariant curves.

- Question 1: When does $s = \infty$?
- $\mathcal{F}$ generated by a fibration $\Longrightarrow s = \infty$.

Theorem (Jouanolou, 1978)

If

$$s \geq h^0(X, K_{\mathcal{F}}) + h^{1,1}(X) - h^{1,0}(X) + 2,$$

then $\mathcal{F}$ is generated by a fibration.

- Question 2: How to determine all $\mathcal{F}$-invariant curves when $s < \infty$?

3. $\mathcal{F}$-invariant curve

$s :=$ the number of irreducible compact $\mathcal{F}$-invariant curves.

- Question 1: When does $s = \infty$?
- $\mathcal{F}$ generated by a fibration $\Longrightarrow s = \infty$.

Theorem (Jouanolou, 1978)

If

$$s \geq h^0(X, K_{\mathcal{F}}) + h^{1,1}(X) - h^{1,0}(X) + 2,$$

then $\mathcal{F}$ is generated by a fibration.

- Question 2: How to determine all $\mathcal{F}$-invariant curves when $s < \infty$?

1. Canonical bundle $K_{\mathcal{F}}$

- Pseudo-effective divisor $D \overset{def}{\Longleftrightarrow} DH \geq 0, \forall \text{ample} H.$

Theorem (Miyaoka,1985)

If $K_{\mathcal{F}}$ is not pseudo-effective, then $\mathcal{F}$ is generated by a rational fibration.

-
- (Zariski decomposition) $K_{\mathcal{F}}$ pseudo-effective $\Longrightarrow$

$$K_{\mathcal{F}} = P + N, \quad PN = 0,$$

Positive part P: nef $\mathbb{Q}$-divisor.
Negative part N: effective $\mathbb{Q}$-divisor, $Supp(N)$ negative curve.

1. Canonical bundle $K_{\mathcal{F}}$

- Pseudo-effective divisor $D \overset{def}{\Longleftrightarrow} DH \geq 0, \forall \text{ample} H$.

Theorem (Miyaoka,1985)

If $K_{\mathcal{F}}$ is not pseudo-effective, then $\mathcal{F}$ is generated by a rational fibration.

-
- (Zariski decomposition) $K_{\mathcal{F}}$ pseudo-effective $\Longrightarrow$

$$K_{\mathcal{F}} = P + N, \quad PN = 0,$$

Positive part P: nef $\mathbb{Q}$-divisor.
Negative part N: effective $\mathbb{Q}$-divisor, $Supp(N)$ negative curve.

1. Canonical bundle $K_{\mathcal{F}}$

- Pseudo-effective divisor $D \overset{def}{\Longleftrightarrow} DH \geq 0, \forall \text{ample} H$.

Theorem (Miyaoka,1985)

If $K_{\mathcal{F}}$ is not pseudo-effective, then $\mathcal{F}$ is generated by a rational fibration.

- (Zariski decomposition) $K_{\mathcal{F}}$ pseudo-effective $\Longrightarrow$

$$K_{\mathcal{F}} = P + N, \quad PN = 0,$$

Positive part P: nef $\mathbb{Q}$-divisor.
Negative part N: effective $\mathbb{Q}$-divisor, $Supp(N)$ negative curve.

1. Canonical bundle $K_{\mathcal{F}}$

- Pseudo-effective divisor $D \overset{def}{\Longleftrightarrow} DH \geq 0, \forall \text{ample} H$.

Theorem (Miyaoka,1985)

If $K_{\mathcal{F}}$ is not pseudo-effective, then $\mathcal{F}$ is generated by a rational fibration.

- (Zariski decomposition) $K_{\mathcal{F}}$ pseudo-effective $\Longrightarrow$

$$K_{\mathcal{F}} = P + N, \quad PN = 0,$$

Positive part P: nef $\mathbb{Q}$-divisor.
Negative part N: effective $\mathbb{Q}$-divisor, $Supp(N)$ negative curve.