Singularities of triple covers on smooth surfaces

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- **hyperelliptic fibration** — an application of double cover

$f: S \to C$ hyperelliptic fibration of genus 2.

Induce a double cover π on rule surface $\varphi_0 : P_0 \to C$.

G.Xiao proved

$$\begin{cases} K_f^2 = & \frac{1}{5}s_2 + & \frac{7}{5}s_3, \\ \chi_f = & \frac{1}{10}s_2 + & \frac{1}{5}s_3, \\ e_f = & s_2 + & s_3. \end{cases} \tag{1}$$

s_2, s_3 : singular index
(due to singular points of branch locus of corresponding double cover)

- **fibration of genus 3**

$f : S \to C$ fibration of genus 3.

M.Reid's conjecture:

$$\begin{aligned} \chi_f = & \ \tfrac{1}{9}(a_0 + a_1 + 3a_2 + 5a_3), \\ K_f^2 = & \ \tfrac{1}{3}(a_0 + 4a_1 + 9a_2 + 14a_3). \end{aligned} \qquad (2)$$

a_i: numerical invariants.
(atomic fibers, singular index?)

- trigonal fibration

$f : S \to C$ fibration of genus $g \geq 3$,

general fiber is a triple cover of $\mathbb{P}^1$

After some base changes, we induce a triple cover π on rule surface $\varphi_0 : P_0 \to C$.

Canonical resolution

$$\begin{array}{ccc} S_0 & \xleftarrow{\tau} & \tilde{S} \\ \downarrow \pi & & \downarrow \tilde{\pi} \\ P_0 & \xleftarrow{\sigma} & \tilde{P} \end{array}$$

$\sigma : \tilde{P} \to P_0$ blowing-ups of P_0.

$\tilde{\pi}$ triple cover with smooth branch locus

$\rho : \tilde{S} \to S$ contracted all (-1)-curves in fibres.
Then we get $f : S \to C$

- compute the relative numerical invariants by triple cover (local problem).

- find singular indexes like Xiao's.

Difficulties in this work:

- know little about singularities of triple cover.

- more complex process of canonical resolution.

- When is an exceptional curve contractible in a fibre?

- What do the singular indexes look like?

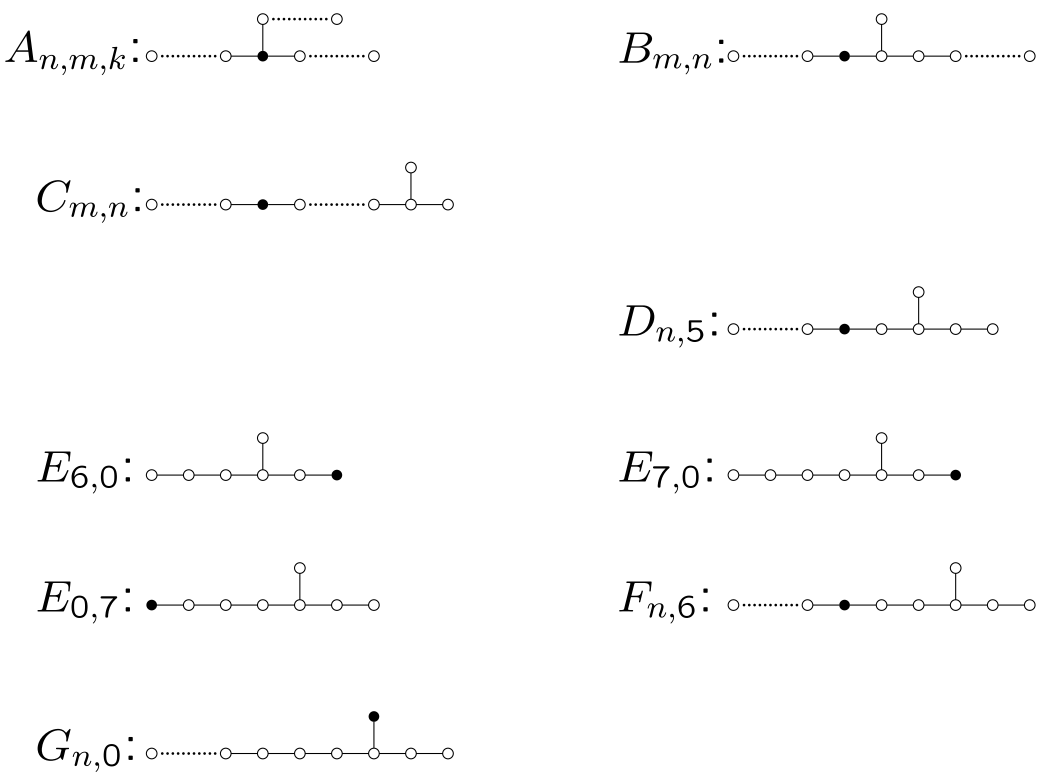
- rational double point— double cover

$$
\begin{aligned}
A_n: &\quad z^2 = \quad x^2 + y^{n+1}, \\
D_n: &\quad z^2 = \quad y(x^2 + y^{n-2}, \\
E_6: &\quad z^2 = \quad x^3 + y^4, \\
E_7: &\quad z^2 = \quad x(x^2 + y^3), \\
E_8: &\quad z^2 = \quad x^3 + y^5.
\end{aligned}
$$

- rational triple point — triple cover

There are 9 classes of rational triple points (M.Artin)

Tyurina (1968) gave explicitly 3 defining equations for each singularity.



Where ∘ is a (-2)-curve, • is a (-3)-curve.

- a rational point (double & triple) is isomorphic to the normalization of the local surface defined by a cubic equation (Z.-J. Chen, R.Du, S.-L. Tan, F.Yu).

- a rational point (double & triple) may have many kinds of such cubic equations with analytically distinct branch loci.

- find all possible branch loci for each rational singularity.

- Introduction of triple cover
  (R. Miranda & S.-L. Tan)

$f : Y \to X$ normal triple cover (X smooth).

- triple cover data $(s, t, \mathcal{L})$:
  $\mathcal{L}$ invertible sheaf.
  $s \in H^0(X, \mathcal{L}^2)$, $0 \neq t \in H^0(X, \mathcal{L}^3)$

- Y is normalization of the surface defined by

$$z^3 + sz + t = 0.$$

- Let

$$a = \frac{4s^3}{\mathrm{gcd}\,(s^3, t^2)}, b = \frac{27t^2}{\mathrm{gcd}\,(s^3, t^2)}, c = a + b.$$

(a, b, c) coprime sections.

- decompositions

$$a = 4a_1 a_2^2 a_0^3, b = 27 b_1 b_0^2, c = c_1 c_0^2,$$

a_1, a_2, b_1, c_1 square-free, $\mathrm{gcd}\,(a_1, a_2) = 1$.

- decompositions of s, t

$$s = a_1 a_2^2 b_1 a_0, t = a_1 a_2^2 b_1^2 b_0.$$

- Set

$$A_i = \mathrm{Div}(a_i), B_i = \mathrm{Div}(b_i), C_i = \mathrm{Div}(c_i).$$

- branch locus $R = D_1 + 2D_2$.

$$D_1 = B_1 + C_1, D_2 = A_1 + A_2.$$

  π totally ramified over D_2,
  π simply ramified over D_1.

- non-normal locus $A_2 + B_1 + C_0$.

- Canonical resolution of singularities of triple cover.

Canonical resolution $\tau : \tilde{Y} \to Y$ is the following communicative diagrams.

$$\begin{array}{ccccccc}
\tilde{Y} = Y_k & \xrightarrow{\tau_k} & \cdots & \xrightarrow{\tau_2} & Y_1 & \xrightarrow{\tau_1} & Y_0 = Y \\
{\scriptstyle \tilde{\pi}=\pi_k}\downarrow & & \cdots & & {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_0} \\
\tilde{X} = X_k & \xrightarrow{\sigma_k} & \cdots & \xrightarrow{\sigma_2} & X_1 & \xrightarrow{\sigma_1} & X_0 = X
\end{array}$$

σ_i : the blowing-up of X_i at singular point p_i.
Y_{i+1} :the normalization of $X_{i+1} \times_{X_i} Y_i$

- The corresponding data $(a^{(i)}, b^{(i)}, c^{(i)})$ of π_i is obtained from

$$(\sigma_i^* a^{(i-1)}, \sigma_i^{(*)} b^{(i-1)}, \sigma_i^* c^{(i-1)})$$

by eliminating the common factors.

- Resolution data.

$$d_i = \min\{m_{p_i}(A^{(i)}), m_{p_i}(B^{(i)}), m_{p_i}(C^{(i)})\},$$

$$m_i = [\frac{m_{p_i}(D_1^{(i)})}{2}],$$

$$n_i = \begin{cases} m_{p_i}(D_2^{(i)}), & \text{if } 3 \mid d_i - m_{p_i}(D_2^{(i)}); \\ m_{p_i}(D_2^{(i)}) - 1, & \text{otherwise.} \end{cases}$$

$$w_i = m_i + n_i.$$

$D_1^{(i)}$ ($D_2^{(i)}$) simply (totally) ramified locus of π_i.

- Numerical invariants.

$$
\begin{aligned}
\chi(\mathcal{O}_{\tilde{Y}}) = & 3\chi(\mathcal{O}_{\tilde{X}}) + \frac{1}{8}D_1^2 + \frac{1}{4}D_1 K_{\tilde{X}} \\
& + \frac{5}{18}D_2^2 + \frac{1}{2}D_2 K_{\tilde{X}} \\
& - \sum_{i=0}^{k-1} \frac{m_i(m_i-1)}{2} - \sum_{i=0}^{k-1} \frac{n_i(5n_i-9)}{18}, \\
K_{\tilde{Y}}^2 = & 3K_{\tilde{X}}^2 + \frac{1}{2}D_1^2 + 2D_1 K_{\tilde{X}} \\
& + \frac{4}{3}D_2^2 + 4D_2 K_{\tilde{X}} \\
& - \sum_{i=0}^{k-1} 2(m_i-1)^2 - \sum_{i=0}^{k-1} \frac{4n_i(n_i-3)}{3} - k.
\end{aligned}
$$

- Singularities of triple covers
  - Topology of singularity
  - Decomposition Theorem (fundamental Cycle of exceptional curves)
  - Contraction Theorem (When is an exceptional curve contractible?)
  - Criterion for rational points.

- Topology of singularities of triple covers.

$\pi : Y \to X$ triple cover;
p singular point of branch locus (totally ramified);
$p' = \pi^{-1}(p)$;
E'_p exceptional curves of p'(no (-1)-curve);
$\mu_p(D)$ Milnor number of D at p.

We get

$$\mu_p(D_1) + 2\mu_p(D_2) = \chi_{top}(E'_p) + H_p + \frac{1}{2}(D_1 D_2)_p + 9\tau_p,$$

where

$$\tau_p = \sum_{i=0}^{k-1} \frac{m_i(m_i - 1)}{2} + \sum_{i=0}^{k-1} \frac{n_i(5n_i - 9)}{18}.$$

$H_p = \frac{1}{2} \sum_i (2 - w_i)(w_i - 3) + \varepsilon_p$
(Horikawa Number)

- Betti number of rational point of triple cover

$$b_2(E'_p) = \mu_p(D_1) + 2\mu_p(D_2) - \frac{1}{2}D_1D_2 - 1 - 9\tau_p.$$

- Laufer's formula on Galois triple cover.

$$2\mu_p(D_2) = \chi_{top}(E'_p) - 1 + K^2 + 12p_g + \frac{4}{3}(A_1A_2)_p.$$

K rational canonical divisor of E'_p
p_g geometric genus of p'.

- Decomposition Theorem.

  Recall canonical resolution

$$\begin{array}{ccc} (\tilde{Y}, E_p) & \xrightarrow{\tau} & (Y, p) \\ \tilde{\pi}\downarrow & & \downarrow\pi_0 \\ (\tilde{X}, \mathcal{E}_1) & \xrightarrow{\sigma} & (X, p) \end{array}$$

  p singular point of branch locus (totally ramified);
  $p' = \pi^{-1}(p)$;
  $E_p = (\sigma\pi)^{-1}(p)$;
  $\mathcal{E}_1$ the totally transformation of σ at p.

$$\tilde{\pi}^*\mathcal{E}_1 = Z_0 + Z_1 + Z_2,\ Z_i \geq 0,$$
$$Z_i Z_j = 0, i \neq j.$$

  Moreover, $Z_0 > Z_1 > Z_2$,
  Z_i is either zero or a fundamental cycle on it's support ($i = 0, 1, 2$).

- This decomposition is unique.

- Contraction Theorem
  It tell us when an exceptional curve is contractible in E_p.
  (The details are omitted.)

- Horikawa number of singular point p of branch locus.

$$H_p = \frac{1}{2}\sum_i (2 - w_i)(w_i - 3) + \varepsilon_p$$

  ε_p the number of the exceptional curve contracted by some blow-downs.

The following conditions are equivalent.
(1) the points of $\pi^{-1}(p)$ are rational singularities or smooth;
(2) $H_p = 0$ and $w_i \leq 2$, $\forall i$.

- Criterion for rational point of triple cover.
  - Rational point is mainly due to the branch locus.
  - a rational point of triple cover may have many kinds of such cubic equations with analytically distinct branch loci.

- List of branch locus of rational point of triple cover.
  (in the meaning of topological equivalence.)

type	equations of (D_1, D_2)
A_n	$(1,\ x^2+y^2)$ $(x^2+y^{3m+1},\ y)$ $((x^2+y^{3m+1})(x^2+y^{3l+1}),\ 1\ )$ $(x,\ x+y^{2m})$ $((x+y^{2m})(x^2+y^{3l+4m}),\ 1)$ $(x^3+y^{2n+2},\ 1)$
D_n	$(1,\ x^2+y^3)$ $((x^2+y^3)^2+x^{m+3},\ 1)$ $((x^2+y^3)(x^2+y^{n-1}),\ 1)$
E_6	$(1,\ x^2+y^4)$ $(x^2+y^{3m+2},\ x+y^2)$ $((x^2+y^{3n+2})((x+y^2)^2+y^{3m+2}),\ 1)$
E_7	$(x^4+y^9,\ 1)$
E_8	$(1,\ x^2+y^5)$ $((x^2+y^5)^2+y^{3n+9},\ 1\ )$ $((x^2+y^5)^2+x^3y^{3n},\ 1)$

(List of rational triple cover is omitted.)

- Trigonal fibration $f : S \to C$.

  After some base changes, we induce a triple cover π on rule surface $\varphi_0 : P_0 \to C$ with data $(s, t, \mathcal{L})$.

  Canonical resolution

$$\begin{array}{ccc} S_0 & \xleftarrow{\tau} & \tilde{S} \\ \Big\downarrow \pi & & \Big\downarrow \tilde{\pi} \\ P_0 & \xleftarrow{\sigma} & \tilde{P} \end{array}$$

  $\rho : \tilde{S} \to S$ contracted all (-1)-curves in fibres. Then we get $f : S \to C$

$$\begin{array}{ccc} S_0 & \xleftarrow{\tau} & \tilde{S} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tilde{\pi}} \\ P_0 & \xleftarrow{\sigma} & \tilde{P} \end{array}$$

F_0 fibre in rule surface P_0.
$\tilde{F}_0$ strict transform of F_0 under σ.

- When is a component of $\tilde{\pi}^*\tilde{F}_0$ contractible?

- Case 1. $\tilde{\pi}^*\tilde{F}_0 = 3C$, i.e., F_0 is totally ramified component.
  C contractible $\Leftrightarrow$ $F_0^2 = -3$.

- Case 2. $\tilde{\pi}^*\tilde{F}_0 = 2C + C'$, i.e., F_0 is simply ramified component.
  C contractible $\Leftrightarrow$ $F_0^2 = -2$;
  C' contractible $\Leftrightarrow$ $\exists!$ totally ramified point in F_0.

- Case 3. $\tilde{\pi}^* \tilde{F}_0 = C$, C irreducible.
  C is not contractible.

- Case 4. $\tilde{\pi}^* \tilde{F}_0 = C + C'$, C double cover over $\tilde{F}_0$.
  C is not contractible;
  C' contractible $\Leftrightarrow$ one of the following cases.

  – $\exists!$ totally ramified point in F_0 (including infinitely near point)

  – $\exists$ a good cusp (defined by $x^2 + y^3 = 0$ at $(0,0)$) in F_0;

  – all points in F_0 is simply ramified & $\exists!\ p \in F_0$, s.t. D_1 smooth and $(D_1 F_0)_p = 2$.

- Case 5. $\tilde{\pi}^* \tilde{F}_0 = C + C' + C''$
  C contractible $\Leftrightarrow$ one of the following cases.
  - $\exists!$ totally ramified point in F_0
    (including infinitely near point)
  - all points in F_0 is simply ramified &
    $\exists!\ p \in F_0$, s.t. D_1 smooth and $(D_1 F_0)_p = 2$.

- Our ultimate goal: find singular indexes in trigonal fibration.
  However, it is still hard to us.