Application of slope inequality for semi-stable families over char 0 (joint work with Tan-Viehweg-Zuo) over char p (joint work with Sheng-Zuo)

Lu Jun

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- **semi-stable family of curves in char** 0

Thereom A (Viehweg and Zuo 2004)
Arakelov inequality
Let $f : X \to Y$ semistable, non-isotrivial, minimal family of n-folds in char 0
Y smooth curve, genus b
$\mathcal{F} \subseteq f_*\omega_{X/Y}^{\otimes\nu}$ subsheaf, $\nu \geq 1$
s number of singular fibers

$$\mu(\mathcal{F}) \leq \frac{n\nu}{2}(2b-2+s), \tag{1}$$

where
$\omega_{X/Y} = \omega_X \otimes (f^*\omega_Y)^\vee$ relatively canonical sheaf
$\mu(\mathcal{F}) = \deg \mathcal{F}/rk(\mathcal{F})$ slope of $\mathcal{F}$

Question: Is it a strict inequality ?

$f : X \to Y$ semistable, non-isotrivial family of curves of genus g.

Proposition 1. Strict Arakelov inequality

$$\mu(f_*\omega_{X/Y}^{\otimes\nu}) < \frac{\nu}{2}(2b - 2 + s), \qquad (2)$$

Thereom B (Voita 1988)
canonical class inequality

$$\omega_{X/Y}^2 \le (2g-2)(2b-2+s) \qquad (3)$$

If $s > 0$, the inequality is strict.

(3) $\iff$ Miyaoka-Yau inequality with logarithmic Chern numbers

$$c_1^2(\Omega_X^1(\log\Delta)) \le c_2(\Omega_X^1(\log\Delta)), \qquad (4)$$

where $\Delta = \sum_{i=1}^{s} F_i$, F_i singular fiber.

If $s > 0$

(4) "=" holds $\Longrightarrow$ all F_i can be contracted

$\Longrightarrow$ F_i is negative, contradiction.

So (3) (Resp. (2)) is strict. □

Remark Arakelov ineq. $\Longrightarrow$ Canonical class ineq.

$$\lim_{v\to\infty} \frac{\mu(f_*\omega_{X/Y}^{\otimes\nu})}{\nu} = \frac{\omega_{X/Y}^2}{4g-4}$$

$$\frac{\mu(f_*\omega_{X/Y}^{\otimes\nu})}{\nu} \le \frac{1}{2}(2b-2+s)$$

$\Longrightarrow$ Canonical class ineq.

Lemma 2

If $\mu(f_*\omega_{X/Y}^{\otimes\nu}) = \frac{\nu}{2}(2b-2+s)$, F general fiber, then

$D = \nu\omega_{X/Y} - \mu(f_*\omega_{X/Y}^{\otimes\nu})F$ is nef, thus $D^2 \geq 0$, i.e.,

$$\nu\omega_{X/Y}^2 \geq (4g-4)\mu(f_*\omega_{X/Y}^{\otimes\nu}). \qquad (5)$$

Proof of Prop. 1.

From (5) and canonical class ineq.,

$$\mu(f_*\omega_{X/Y}^{\otimes\nu}) < \frac{\nu}{2}(2b-2+s),\ s>0$$

contradiction.

If $s = 0$

$$\begin{aligned} 12\deg f_*\omega_{X/Y} &= \omega_{X/Y}^2 \\ \deg(f_*\omega_{X/Y}^{\otimes\nu}) &= \begin{pmatrix} v \\ 2 \end{pmatrix}\omega_{X/Y}^2 + \deg f_*\omega_{X/Y} \\ rk(f_*\omega_{X/Y}^{\otimes\nu}) &= (2\nu-1)(g-1), \nu > 1 \end{aligned}$$

$\Longrightarrow$ strict ineq. is also valid. □

- **semi-stable family of curves in char p**

$f : X \to Y$ semi-stable, non-isotrivial,
$g \geq 2$ genus of general fiber.
$\mathcal{F} \subseteq f_* \omega_{X/Y}^{\otimes \nu}$ subsheaf, $\nu \geq 1$

Szpiro's results
(1) $\omega_{X/Y}$ nef and big.
(2) (Szpiro's inequality) If the class of Kodaira-Spencer isn't null,

$$\omega_{X/Y}^2 < 4(g-1)g(2b-2+s)$$

Remark: class of Kodaira-Spencer isn't null $\Longleftrightarrow$ non-isotrivial and f is not the pull back of Frobenius base change of any fibration $f' : X' \to Y$.

Proposition 3 (joint work with Sheng Mao and Kang Zuo)
(i)

$$\mu(\mathcal{F}) < 2\nu g(2b-2+s), \qquad (6)$$

(ii) $\mathcal{G}$ subsheaf of $R^1 f_* \mathcal{O}_X$

$$\mu(\mathcal{G}) < 2g(g-1)(2b-2+s). \qquad (7)$$

Proof.(i) Recall Harder-Narasimhan filtration

$$\mathcal{F} = \mathcal{E}_m \supseteq \mathcal{E}_{m-1} \supseteq \cdots \supseteq \mathcal{E}_1 \supseteq \mathcal{E}_0 = 0.$$

s.t. $\mathcal{E}_i/\mathcal{E}_{i-1}$ semi-stable
$\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) < \mu(\mathcal{E}_{i-1}/\mathcal{E}_{i-2})$
$\mu(\mathcal{E}_1) \geq \mu(\mathcal{F})$

Step 1. Find smooth curve Γ
s.t. $\pi = f|_\Gamma$ is separable finite morphism.
Thus $\pi^*\mathcal{E}_1$ is also semistable.

H ample divisor.
$\because K_{X/Y}$ nef and big
$\therefore mK_{X/Y} + H$ ample, $m \geq 0$ (Nakai's criterion)
From Bertini's theorem
$\exists \Gamma \in |n(mK_{X/Y} + H)|$, Γ smooth curve
s.t. Γ and general fiber F are normal crossing,
i.e. $\forall p \in F \cap \Gamma$, $(F\Gamma)_p = 1$.
$\therefore \pi := f|_\Gamma$ separable finite.

step 2. Natural map $\alpha : f^*\mathcal{F} \to \omega_{X/Y}^{\otimes\nu}$ from

$$f^*\mathcal{F} \hookrightarrow f^* f_* \omega_{X/Y}^{\otimes\nu} \to K_{X/Y}^{\otimes\nu}.$$

$\mathrm{Im}\alpha = \mathcal{I}_\Delta(\nu K_{X/Y} - Z)$
Δ zero-dim subscheme,
$\mathcal{I}_\Delta$ ideal sheaf over Δ,
Z effective divisor, called fix part of $\mathcal{F}$.

Take $D := \nu K_{X/Y} - Z - \mu(\mathcal{E}_1)F$,

$\alpha|_\Gamma : \pi^*\mathcal{E}_1 \to \mathcal{O}_\Gamma(K_{X/Y} - Z)$ non-trivial map between semi-stable sheave.
$\Longrightarrow \mu(\pi^*\mathcal{E}_1) \leq \mu(\mathcal{O}_\Gamma(\omega_{X/Y} - Z))$
$\Longrightarrow D\Gamma \geq 0$,
i.e., $D(mK_{X/Y} + H) \geq 0$
$DK_{X/Y} \geq 0$, if $m \to \infty$.

Step 3. $DK_{X/Y} \geq 0$
$\iff \nu\omega^2_{X/Y} \geq (2g-2)\mu(\mathcal{E}_1)$
$\because \mu(\mathcal{E}_1) \geq \mu(\mathcal{F})$ (construction of Harder's filtration.)
$\therefore \nu\omega^2_{X/Y} \geq (2g-2)\mu(\mathcal{F})$.

Szpiro's ineq. $\omega^2_{X/Y} < 4(g-1)g(2b-2+s)$,
$\Longrightarrow \mu(\mathcal{F}) < 2\nu g(2b-2+s)$

(ii) $R^1 f_* \mathcal{O}_X = (f_*\omega_{X/Y})^\vee$ (Serre duality)
$\deg R^1 f_* \mathcal{O}_X = -\deg f_*\omega_{X/Y} \leq 0$
$(R^1 f_* \mathcal{O}_{X/Y}/\mathcal{G})^\vee$ is subsheaf of $f_*\omega_{X/Y}$
Thus

$$\begin{aligned}\deg \mathcal{G} &= -\deg f_*\omega_{X/Y} + \deg (R^1 f_*\mathcal{O}_X/\mathcal{G})^\vee \\ &\leq \deg (R^1 f_*\mathcal{O}_X/\mathcal{G})^\vee \\ &< 2(g - rk(\mathcal{G}))g(2b-2+s).\end{aligned}$$

Thus $\mu(\mathcal{G}) < 2g(g-1)(2b-2+s)$. $\square$

Remark In char 0, the proof of Lemma 2 is similar to Step 2.

- **semi-stable family of high-dimension manifolds in char** 0

$f : X \to Y$ semi-stable, non-isotrivial, minimal family of n-dimensional manifolds.
Y projective m-fold, $Y_0 = Y \backslash S$,
S singular locus.
$\mathcal{K} \subseteq (f_* \omega_{X/Y}^{\otimes \nu})^{\vee\vee}$ subsheaf.
$\mu(\mathcal{K}_\nu) := c_1(\mathcal{K}) c_1(\omega_Y(S))^{m-1}$.
$\omega_Y(S)$ semistable & ample with respect to Y_0.

Proposition 4. (Viehweg and Zuo)
$\exists$ constant $\rho \leq 1$, s.t.

$$\mu(\mathcal{K}_\nu) \leq \nu \cdot n \cdot \rho \cdot \mu(\Omega_Y^1(\log S)) \qquad (8)$$

Volume inequality

$\mathcal{L}$ linear bundle, volume of $\mathcal{L}$:

$$v(\mathcal{L}) = \limsup \frac{\dim(X)! \cdot \dim(H^0(X, \mathcal{L}^\nu))}{\nu^{\dim(X)}}.$$

(Viehweg 1982) If $\mathcal{L}$ is nef, then

$$\dim(H^i(X, \mathcal{L}^\nu)) \leq a_i \cdot \nu^{\dim(X)-i}$$

So $v(\mathcal{L}) = c_1(\mathcal{L})^{\dim X}$ (Riemann-Roch Theorem)

(Kawamata 1982) If Y is smooth curve of genus b.

$$v(\omega_Y) \cdot v(\omega_F) \leq \frac{1}{n+1} \cdot v(\omega_X).$$

Proposition 5. If Y is smooth curve of genus b, $s = \#S$.

$$v(\omega_{X/Y}) \leq \frac{(n+1)n}{2} \cdot v(\omega_F) \cdot \deg \Omega^1_Y(\log S).$$

If $b \geq 1$ then

$$v(\omega_X) \leq v(\omega_F) \cdot \frac{(n+1)}{2}\Big((n+2)v(\omega_Y) + ns\Big).$$

Proposition 6. Let $\dim Y = m$ and l_0 be the smallest integer s.t. $|l_0\omega_Y(S)|$ defines birational map, then
$\exists$ constant c depending only on n, m and l_0, s.t.

$$v(\omega_{X/Y}) \leq c \cdot v(\omega_F) \cdot v(\omega_Y(S)).$$