Riccati foliations and Double Riccati foliations

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1. Definition of Foliation

- X: algebaic surface,
  T_X: tangent bundle of X.
  $\mathcal{L}^{-1} \subseteq T_X$: maximal sub-line bundle.
- Foliation $\mathcal{F}$ is a section

$$s \in H^0(X, T_X \otimes \mathcal{L}).$$

- Open covering $X = \cup_\alpha U_\alpha$,

$$s|_{U_\alpha} = A(x_\alpha, y_\alpha)\frac{\partial}{\partial x_\alpha} + B(x_\alpha, y_\alpha)\frac{\partial}{\partial y_\alpha}, \quad (x_\alpha, y_\alpha) \in U_\alpha.$$

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1. Definition of Foliations

- $K_{\mathcal{F}} := \mathcal{L}$ canonical bundle of $\mathcal{F}$.
- Exact sequence

$$0 \to K_{\mathcal{F}}^{-1} \xrightarrow{\cdot s} T_X \to \mathcal{I}_{Z(s)} \otimes N_{\mathcal{F}} \to 0,$$

  $N_{\mathcal{F}}$ line bundle,
  $\mathcal{I}_{Z(s)}$ ideal sheaf of $Z(s)$ (zero set of s).
- Canonical bundle

$$\omega_X := \wedge^2 \Omega_X = K_{\mathcal{F}} \otimes N_{\mathcal{F}}^{-1}$$

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2. Example (1): fibration

- Fibration $f: X \to C$,
  C smooth curve, f holomorphic and surjective.
- Fiber $F_t = f^{-1}(t)$, $t \in C$.
- Algebraic foliation $\mathcal{F}$:

$$\omega = \frac{1}{\mu(f)}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) \quad \text{(local eq.)},$$

$\mu(f) = \gcd(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$.

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- Canonical bundle of $\mathcal{F}$:

$$K_{\mathcal{F}} = \omega_{X/C}(-D(f)),$$

where

- $\omega_{X/C} := \omega_X \otimes f^*\Omega_C^{-1}$ (relative canonical bundle)
- $D(f) := \sum_{t \in C} (F_t - F_{t,\mathrm{red}})$ (zero divisor of df).
- Conormal bundle of $\mathcal{F}$:

$$N_{\mathcal{F}}^{-1} = f^*\Omega_C(D(f)).$$

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- Ruled surface $\varphi : X \to B$.
  Riccati Foliation $\mathcal{F}$ w.r.t. $\varphi \overset{def}{\Longleftrightarrow}$ general fiber F of φ transverse to $\mathcal{F}$.
- $\mathcal{F}$ Riccati foliation $\Longleftrightarrow K_{\mathcal{F}}F = 0$.
- Local equation $(p_i, q \in \mathbb{C}\{x\})$:

$$\omega = \left(p_0(x)y^2 + p_1(x)y + p_2(x)\right) dx - q(x)dy, \quad x \in B,\ y \in F.$$

  For convenience,

$$\omega = \left(g_0(x)y^2 + g_1(x)y + g_2(x)\right) dx - dy, \quad g_i := \frac{p_i}{q}.$$

- Canonical bundle of $\mathcal{F}$: $K_{\mathcal{F}} = rF$,
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- Double cover $\pi : Y \to X$.
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3. $\mathcal{F}$-invariant curve

- $\mathcal{F}$ foliation: $\{(U_\alpha, \omega_\alpha)\}$ (or $\{(U_\alpha, s_\alpha)\}$),

$$s_\alpha = A_\alpha \frac{\partial}{\partial x_\alpha} + B_\alpha \frac{\partial}{\partial y_\alpha}$$

  or

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- $C \subseteq X$ curve defined by $f_\alpha = 0$ on U_α.
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- C is $\mathcal{F}$-invariant $\Longleftrightarrow$ f_α is the solution of ODE

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- **Example** $\mathcal{F}$ generated by a fibration $f : X \to B$.
  $C \subseteq X$ is $\mathcal{F}$-invariant iff C lies in the fibers of f.

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- $s = \infty \Longleftrightarrow \mathcal{F}$ is algebraic.

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- **Question 1** (Poincaré 1891): *Is it possible to decide if $\mathcal{F}$ on a rational surface is algebraic?*

Theorem (Jouanolou, 1978)

If

$$s \geq h^0(X, K_{\mathcal{F}}) + h^{1,1}(X) - h^{1,0}(X) + 2,$$

then $\mathcal{F}$ is algebraic.

- **Question 2** (Painlevé 1974) *Can we recognize the genus g of an algebraic foliation from its defining differential equation?*

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4. Invariants of $\mathcal{F}$

- Kodaira dimension of $\mathcal{F}$:

$$Kod(\mathcal{F}) := \limsup_{n\to+\infty} \frac{\log h^0(nK_{\mathcal{F}})}{\log n}.$$

- Pluri-genus of $\mathcal{F}$: $p_n(\mathcal{F}) := h^0(nK_{\mathcal{F}})$
- Chern number of $\mathcal{F}$ (S.-L. Tan 2015):

$$c_1^2(\mathcal{F}) \geq 0, c_2(\mathcal{F}) \geq 0, \chi(\mathcal{F}) \geq 0,$$

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- (S.-L. Tan 2015) By Xiao's inequality, one can find that any foliation $\mathcal{F}$ satisfying that

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- **Assume:** $\mathcal{F}$ Riccati foliation w.r.t. a Hirzebruch surface $\varphi : X \to B$.
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2. Classification

- Classification of $\mathcal{F}$ with $kod(\mathcal{F}) = -\infty$

Theorem (J. Lu, S.-L. Tan)

Up to a birational map, we have

1. $\omega = dy$;
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$\mathcal{F}$ is algebraic iff it occurs in one of the following cases (up to a birational map):

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- A Riccati fibration $f: X \to C$ of type A_n (D_n, E_k) $\overset{def}{\Longleftrightarrow}$ $\omega = df$ gives a Riccati foliation of type A_n (D_n,E_k)

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Up to a birational map, $\mathcal{F}$ algebraic & $\mathrm{Kod}(\mathcal{F}) = 0$ iff

Type	Riccati foliations	Families	Singular fibers
A_1	$(3x^2+1)ydx - 2(x^3+x+c)dy$	$y^2 = t(x^3+x+c)$	$\mathrm{I}_0^*, \mathrm{I}_0^*$
	$3x^2ydx - 2(x^3+1)dy$	$y^2 = t(x^3+1)$	
A_2	$(2x-1)ydx - 3x(x-1)dy$	$y^3 = tx(x-1)$	$\mathrm{IV}, \mathrm{IV}^*$
A_3	$(2x-1)ydx - 4x(x-1)dy$	$y^4 = tx(x-1)$	$\mathrm{III}, \mathrm{III}^*$
A_5	$(3x-2)ydx - 6x(x-1)dy$	$y^6 = tx^2(x-1)$	$\mathrm{II}, \mathrm{II}^*$
E_6	$(3y^2-2xy-1)dx - 6(x^2-1)dy$	$z^3 = t(x^2-1)$	$\mathrm{IV}, \mathrm{IV}^*, 2\mathrm{I}_0$
D_{n+2}	$\frac{\psi'}{\psi(\psi-1)}(y^2+n(\psi-1)y-\psi)dx - 2ndy$	$\left(\frac{y+\sqrt{\psi}}{y-\sqrt{\psi}}\right)^n = t\left(\frac{\sqrt{\psi}+1}{\sqrt{\psi}-1}\right)$	$\mathrm{I}_0^*, \mathrm{I}_0^*, n\mathrm{I}_0$

where $c \in \mathbb{C}$ satisfies $4+27c^3 \neq 0$,

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Corollary (C. Gong, J. Lu, S.-L. Tan)

$\mathcal{F}$ is an algebraic foliation of type A_n
$\Longleftrightarrow$ it has two $\mathcal{F}$-invariant section of φ
$\Longleftrightarrow$ it is from a fibration $f : X \to \mathbb{P}^1$ with two singular fibers.

- **Remark:** Let s be the number of critical points of a fibration $f : X \to \mathbb{P}^1$. Then $s \geq 2$. Furthermore,

$$s \geq \begin{cases} 4, & \text{semistable (Beauville 1981)} \\ 5, & \text{semistable and } g > 1 \text{(Tan 1995)}. \end{cases}$$

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- $\omega = (g_0 y^2 + g_1 y + g_2)dx - dy$, $g_i \in \mathbb{C}(x)$.
- Discriminant of $\mathcal{F}$:

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- $\Delta(\mathcal{F}) \in H^0(S^2\Omega_{\mathbb{P}^1}(\log T))$ where

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- If all singularities of $\mathcal{F}$ have non-zero eigenvalue, then

$$\Delta(\mathcal{F}) = \sum_{p} \frac{1 - {\lambda_p}^2}{4(x-p)^2} + \sum_{p} \frac{\mu_p}{x-p}$$

where

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- Invariance of Discriminant

Theorem (C. Gong, J. Lu, S.-L. Tan)

$\Delta(\mathcal{F}) = \Delta(\widetilde{\mathcal{F}})$ iff $\widetilde{\mathcal{F}}$ can becomes $\mathcal{F}$ by choosing suitable coordinates and flipping maps.

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Theorem (C. Gong, J. Lu, S.-L. Tan)

$\mathcal{F}$ is algebraic iff

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for some $\psi \in \mathbb{C}(x)$ and a Riccati foliation $\mathcal{F}_0$ with Kodaira dimension $-\infty$.

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1.Formulae for Chern numbers

- Double cover $\pi : X \to Y$ with branch locus R
  Riccati foliation $\mathcal{G}$ on Y w.r.t. φ
  Double Riccati foliation $\mathcal{F} = \pi^*\mathcal{G}$
- π and φ give a hyperelliptic fibration of genus g.
- Let $p \in R$ be a node or the tangent points of R to $\mathcal{G}$.
  local invariants $s_1(p)$ and $s_2(p)$ of the branch locus R w.r.t. $\mathcal{G}$.
- Let F be an $\mathcal{G}$-invariant fiber of φ.
  local invariants $\nu(F)$ of F w.r.t. $\mathcal{G}$.

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Theorem (J. Hong, J. Lu, S.-L. Tan)

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$$\lambda(\mathcal{F}) := c_1^2(\mathcal{F})/\chi(\mathcal{F}).$$

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Thank you!