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Global Invariants of a Trigonal Fibration

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Global Invariants of a Trigonal Fibration

- Global Invariants
- Gonality of Algebraic Curves
- Recall the Case for $g = 2$
- Non-hyperelliptic Fibration of Genus 3

1. Global Invariants

- Projective Complex Surface. $S \subset \mathbb{P}^n$

- Global Invariants:

$$\chi(\mathcal{O}_S), \quad K_S^2, \quad \chi_{\text{top}}(S)$$

- $12\chi(\mathcal{O}_S) = K_S^2 + \chi_{\text{top}}(S)$

- $f : S \to C$ relatively minimal fibration genus g.

  C smooth curve, $b = g(C)$.

- Relatively Numerical Invariants

$$\begin{aligned} K_f^2 &:= K_S^2 - 8(g-1)(b-1), \\ \chi_f &:= \chi(\mathcal{O}_S) - (g-1)(b-1), \\ e_f &:= \chi_{top}(S) - 4(g-1)(b-1). \end{aligned}$$

- $K_f^2 \geq 0, \chi_f \geq 0, e_f \geq 0$

- $12\chi_f = K_f^2 + e_f$

- $g = 0$ (ruled surface)

$$K_f^2 = \chi_f = e_f = 0$$

- $g = 1$ (elliptic fibration)

$$K_f^2 = 0, \ \chi_f = \frac{1}{12} e_f$$

- If $g \geq 2$, then

  $K_f^2 = 0 \Longleftrightarrow \chi_f = 0 \Longleftrightarrow$ All fibers are isomorphic.

  $e_f = 0$ iff All fibers are smooth.

- $F_1, \cdots, F_s$ are all singular fibers

$$e_f = \sum_i e_{F_i}$$

$$e_{F_i} = 2(g - p_a(F_{i,red})) + \mu(F_{i,\text{red}})$$

- Total Milnor number

$$\mu(F_{i,\text{red}}) = \sum_{p \in F_i} \mu_p(F_i)$$

where $\sum_{p \in F_i}$ run over all singularities of $F_{i,red}$.

($\mu_p(F)$ is Milnor number of F at p.)

- $g = 1$ elliptic fibration (Kodaira)

  there are 11 types of singular fibers.

- $$K_f^2 = 0, \qquad e_f = \sum_i e_{F_i}, \qquad \chi_f = \frac{1}{12} e_f$$

  $$e_{F_i} = 2(g - p_a(F_{i,\mathrm{red}})) + \mu(F_{i,\text{red}})$$

  In this case, all invariants can be computed.

- $g = 2$
  - (Ogg, Namikawa, Ueno) ≈ 246 types of singular fiber
  - (Horrikawa, 1977) Classify singular fibers into 5 types.

    (double cover)

- (G.Xiao, 1985) If $g = 2$, then

$$K_f^2 = \frac{1}{5}s_2 + \frac{7}{5}s_3,$$
$$\chi_f = \frac{1}{10}s_2 + \frac{1}{5}s_3,$$
$$e_f = s_2 + s_3,$$

$$s_2 = \sum_F s_2(F), \;\; s_3 = \sum_F s_3(F)$$

where $s_2(F), s_3(F) \geq 0$.

- (G.Xiao) $g \geq 3 \;\; \&$ hyperelliptic similar formulas.

- **Corollary**

$$2 \leq \lambda_f := \frac{K_f^2}{\chi_f} \leq 7$$

- (G.Xiao)
  If $g = 3$ & hyperelliptic, then

$$K_f^2 = \frac{2}{7}s_2 + \frac{17}{7}s_3 + \frac{10}{7}s_4 + \frac{20}{7}s_5,$$
$$\chi_f = \frac{3}{28}s_2 + \frac{2}{7}s_3 + \frac{2}{7}s_4 + \frac{1}{14}s_5,$$
$$e_f = (s_2 - 2s_5) + s_3 + 2s_4,$$

- $g = 3$ & Non-hyperelliptic

(Horikawa, M.Reid, Z.J.Chen)

$$\lambda_f := \frac{K_f^2}{\chi_f} \geq 3$$

- Horikawa number

$$K_f^2 - 3\chi_f = \sum_F H_F,$$

$$H_F := lengthcoker(S^2 f_*\omega_{S/C} \hookrightarrow f_*(\omega_{S/C}^{\otimes 2}))_p, \ p = f(F)$$

- $H_F \neq 0 \iff F$ singular, or smooth hyperelliptic.

- (Viehweg 1977) a complete set of invariants for the curves is described.

  This set agrees in the case $g = 2$ with the invariants of Y. Namikawa and K. Ueno

- Atomic fiber (M.Reid, G.Xiao)
  - Let F be a fiber
    F can be deformed to some more simple singular fibers.
  - Atomic fiber: most simple singular fiber
  - Conjecture: There are only 4 types of atomic fibers.
    (denoted by type (0) - (3))

- Conjecture(M.Reid, G.Xiao)

Relatively invariants

$$K_f^2 = \frac{1}{3}a_0 + \frac{4}{3}a_1 + 3a_2 + \frac{14}{3}a_3,$$
$$\chi_f = \frac{1}{9}a_0 + \frac{1}{9}a_1 + \frac{1}{3}a_2 + \frac{5}{9}a_3$$
$$e_f = a_0 + a_2 + a_3,$$

where a_i is the number of atomic fibers of type (i).

- Main Result

Theorem If $f : S \to C$ is semistable, then

$$
\begin{aligned}
K_f^2 &= \frac{1}{3}s_1 + 3s_2 + \frac{7}{3}s_3 + 2s_4 + \frac{13}{3}s_5 + \frac{4}{3}s_6, \\
\chi_f &= \frac{1}{9}s_1 + \frac{1}{3}s_2 + \frac{4}{9}s_3 + \frac{1}{3}s_4 + \frac{4}{9}s_5 + \frac{1}{9}s_6 \\
e_f &= s_1 + s_2 + 3s_3 + 2s_4 + s_5,
\end{aligned}
$$

where $s_i \geq 0$.

- **The Methods:**

Horikawa-Xiao's:	"Double Cover"
Reid's:	"Atomic Fiber"
Our Method:	"Triple Cover"

Degree of Finite Cover $=$ Gonality

2. Gonality of Algebraic Curves

- "Gonality of C" $=$ "minimal degree d"

$$\pi : C \xrightarrow{d:1} \mathbb{P}^1,$$

- **Classification:** $d = 1, 2, 3, \cdots, \left[\frac{g+3}{2}\right]$

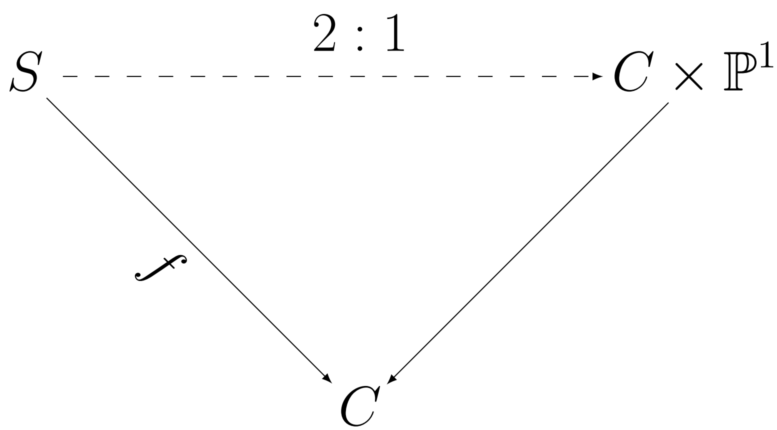
(1) $d = 1$: $C \cong \mathbb{P}^1$ & $g = 0$;
(2) $d = 2$: C Hyperelliptic (e.g. $g = 1, 2$);
(3) $d = 3$: C Trigonal (e.g. $g = 3, 4$);
(4) $d \geq 4$: $\cdots\cdots$

- Classification of Fibrations

$$f : S \to C, \qquad g(F) = g$$

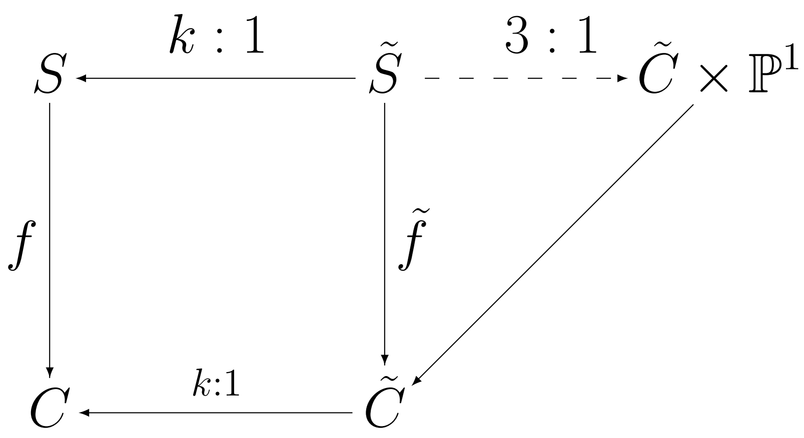
- f is elliptic $\iff F \cong \mathbb{C}/\Lambda$
- f is hyperelliptic $\iff d(F) = 2$
- f is trigonal $\iff d(F) = 3$

- $f : S \to C$ is hyperelliptic $\implies$



- $f : S \to C$ is trigonal $\Longrightarrow$

After some base change, we have



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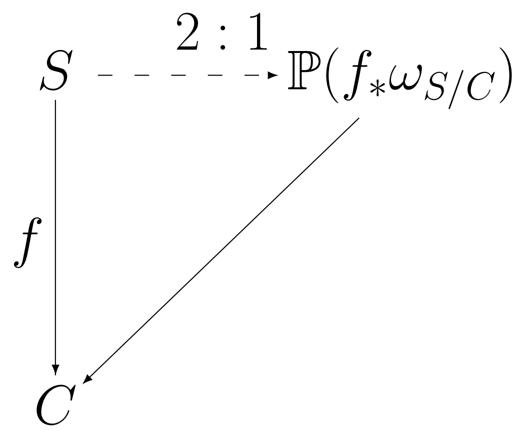
3. Recall the Case for $g = 2$

- Let $f : S \to C$ relatively minimal genus 2.

Step 1



- Then we get a double cover $\pi : S_0 \to P_0$

$$\begin{array}{ccc} S & \overset{1:1}{\dashleftarrow} & S_0 \\ {\scriptstyle 2:1}\,\downarrow & & \downarrow\,{\scriptstyle \pi} \\ \mathbb{P}(f_*\omega_{S/C}) & \underset{1:1}{\dashleftarrow} & P_0 \end{array}$$

- The double cover satisfying

  (1) $\phi : P_0 \to C$ relatively minimal ruled surface.

  (2) B branch locus of π, then (F_0 is a fiber)
  $B \sim -3K_\phi + nF_0$ (for some n).

  $\Longrightarrow BF_0 = 6$.

  (3) B_h horizonal part of B

  (i.e., B_h does not contain any fiber of ϕ and
  $B_v = B - B_h$ is the sum of some fibres.)

  For any singularity p of B_h

  $mult_p(B_h) \leq 3$

- Such ruled surface associated with B is called normalized model.

- Step 2 Canonical Resolution

$$\begin{array}{ccccc} S & \longleftarrow & \tilde{S} & \longrightarrow & S_0 \\ f \big\downarrow & & \tilde{\pi} \big\downarrow & & \pi \big\downarrow \\ C & \longleftarrow & \tilde{P} & \xrightarrow[can.res.]{} & P_0 \end{array}$$

  – From the computation of global invariants of double cover, the contributions of the singularities of S_0 to invariants are due to those of the singularities of branch locus B.

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- Relative Ramified Index

  D effective divisor in ruled surface $\phi : P_0 \to C$.

  relative ramified index of D:

$$r_D := D^2 + DK_\phi$$

- (Relative adjunction formula)

  If D does not contain any fiber, then r_D is just the ramified index of projection $\phi : D \to C$.

- Relatively Invariants of $f : S \to C$

  $\pi : S_0 \to P_0$ corresponding double cover

  B branch locus

$$K_f^2 = \frac{(g-1)}{(2g+1)} r_B - r_1$$
$$\chi_f = \frac{g}{(8g+4)} r_B - r_2$$
$$e_f = r_B - r_3$$

where r_1, r_2, r_3 are contributions of singularity to relatively invariants, which can be computed from the corresponding canonical resolution.

- The computation of relatively invariants is due to two parts:

  (1) relatively ramified of branch locus B;

  (2) contributions of singular points of B to the relatively invariants.

- Find a good classification of singular points of branch locus.

  Compute contribution of each type of singular point to the relatively invariants.

- Step 3 **Classify the singularities of branch locus** B

  Any singularity p of branch locus B can be replaced by the following two types of singularities, which doesn't influence its contribution to relatively invariants.

- Type(A) $(3 \to 3)$ singularity
  p satisfies that
  $mult_p(B) = mult_{p'}(\tilde{B}) = 3$ and $(\tilde{B}E)_{p'} = 3$.
  (p' infinitely close singular point of p)

  Type(B) negligible singular point
  p satisfies that
  $mult_p(B) = 2$.

  ($\sigma : (P_1, E) \to (P_0, p)$ is a blowing-up at p. E is an exceptional curve. $\tilde{B}$ is strict transform of B

- Step4   Singular Index (G.Xiao)

$$K_f^2 = \frac{1}{5}s_2 + \frac{7}{5}s_3,$$
$$\chi_f = \frac{1}{10}s_2 + \frac{1}{5}s_3,$$
$$e_f = s_2 + s_3,$$

- s_3: the number of $(3 \to 3)$ singularity

  s_2: relates to the number of negligible singularities and relatively ramified index of branch locus.

- $g \geq 3$ & hyperelliptic.

  a similar result

4. Non-hyperelliptic Fibration of Genus 3

- $f : S \to C$ trigonal fibration, genus 3.

- Step 1 After some base change, we can assume

$$
\begin{array}{ccc}
S & \overset{3:1}{\dashrightarrow} & \mathbb{P}(f_*\omega_{S/C}(-\Gamma)) \\
{\scriptstyle f}\downarrow & \swarrow & \\
C & &
\end{array}
$$

- Then we get a triple cover $\pi : S_0 \longrightarrow P_0$

$$\begin{array}{ccc} S & \overset{1:1}{\dashleftarrow} & S_0 \\ {\scriptstyle 3:1}\,\big\downarrow & & \big\downarrow\,{\scriptstyle \pi} \\ \mathbb{P}(f_*\omega_{S/C}(-\Gamma)) & \underset{1:1}{\dashleftarrow} & P_0 \end{array}$$

- The triple cover satisfying

(1) $\phi : P_0 \to C$ relatively minimal ruled surface.

(2) R branch locus of π, then (F_0 is a fiber)

$R \sim -5K_\phi + nF_0$ (for some n).

$\Longrightarrow RF_0 = 10$.

(3) R_h horizonal part of R

For any singularity p of R_h
$mult_p(R_h) \leq 5$

- Step 2 Canonical Resolution

$$
\begin{array}{ccccc}
S & \longleftarrow & \tilde{S} & \longrightarrow & S_0 \\
\Big\downarrow f & & \Big\downarrow \tilde{\pi} & & \Big\downarrow \pi \\
C & \longleftarrow & \tilde{P} & \xrightarrow[can.res.]{} & P_0
\end{array}
$$

  – From the computation of global invariants of triple cover, the contributions of the singularities of triple cover to invariants are due to those of the singularities of branch locus R.

- Step 3
  Classify the singularities of branch locus R

  Most difficult problem : How to classify the singularities of the branch locus ?

- Select a good standard of classification:

  multiplicity of a singular point ?

  (compare with the case for $g = 2$)

  It becomes more complex in triple cover.

- A good classification can simplify the computation of relatively invariants.

- Branch locus of triple cover

  Let $\pi : X \to Y$ triple cover
  X normal surface
  Y smooth surface

- triple cover data $(s, t, \mathcal{L})$

  (1) $\mathcal{L}$ invertible sheaf;

  (2) $s \in H^0(X, \mathcal{L}^2)$

  (3) $0 \neq t \in H^0(X, \mathcal{L}^3)$

  (4)Y is the normalization of the surface defined by
  $z^3 + sz + t = 0$
  in the line bundle of $\mathcal{L}$

- $z^3 + sz + t = 0$ is minimal $\Longleftrightarrow$ there is no prime p,

$$p^2 \mid s, \qquad p^3 \mid t.$$

- Any triple cover can be defined by a minimal equation.
- (S.-L. Tan 2000) definition of (a, b, c)

$$a = \frac{4s^3}{\gcd(s^3, t^2)}, b = \frac{27t^2}{\gcd(s^3, t^2)}, c = \frac{4s^3 + 27t^2}{\gcd(s^3, t^2)}.$$

- Decomposition

$$a = 4a_1a_2^2a_0^3, b = 27b_1b_0^2, c = c_1c_0^2,$$

where a_1, a_2, b_1, c_1 are square-free and $\gcd(a_1, a_2) = \gcd(a_i, b_j) = 1$.

- (S.-L. Tan 2000)

  Relation between (a, b, c) and (s, t)

$$s = a_1 a_2^2 b_1 a_0, t = a_1 a_2^2 b_1^2 b_0.$$

- Branch locus of triple cover

$$A_i = \text{Div}(a_i), B_i = \text{Div}(b_i), C_i = \text{Div}(c_i).$$

- Simple ramified locus: $D_1 = B_1 + C_1$

- Totally ramified locus: $D_2 = A_1 + A_2$

- Branch locus: $R = D_1 + 2D_2$

- (S.-L. Tan 2000)
  $D_1 \equiv 2\eta$, $\eta = 3L - A_1 - B_1 - 2A_2 - C_0$.
- (S.-L. Tan & D.-Q.Zhang 2004)
  π is Galois iff $D_1 = 0$ and $\eta \equiv 0$.
- ϕ is the double cover of (D_1, η),

$$\begin{array}{ccc} X' & \xrightarrow{\phi'} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow[\phi]{z^2 = b_1 c_1} & Y \end{array}$$

- π' is $\mathbb{Z}_3 - cover$

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- Canonical resolution

$$\begin{array}{ccccccccccc}
Y_k & \xrightarrow{\tau_k} & Y_{k-1} & \xrightarrow{\tau_{k-1}} & \cdots & \longrightarrow & Y_2 & \xrightarrow{\tau_2} & Y_1 & \xrightarrow{\tau_1} & Y \\
\tilde{\pi} = \pi_k \downarrow & & \pi_{k-1} \downarrow & & \cdots & & \pi_2 \downarrow & & \pi_1 \downarrow & & \pi \downarrow \\
X_k & \xrightarrow{\sigma_k} & X_{k-1} & \xrightarrow{\sigma_{k-1}} & \cdots & \longrightarrow & X_2 & \xrightarrow{\sigma_2} & X_1 & \xrightarrow{\sigma_1} & X
\end{array}$$

- Branch locus of π_k is smooth, so is Y_k.

- Compute the data (a', b', c') of π_1:

$$a + b = c \;\Rightarrow\; \sigma_1^* a + \sigma_1^* b = \sigma_1^* c$$

Eliminate the common factors,

$$a' + b' = c'.$$

- **Global Invariants of triple cover (S.-L.Tan 2000)**

$$K_{\tilde{X}}^2 = 3K_{\tilde{Y}}^2 + \frac{1}{2}D_1^2 + 2D_1K_{\tilde{Y}} + \frac{4}{3}D_2^2 + 4D_2K_{\tilde{Y}} - r_1.$$

$$\chi(\mathcal{O}_{\tilde{X}}) = 3\chi(\mathcal{O}_{\tilde{Y}}) + \frac{1}{8}D_1^2 + \frac{1}{4}D_1K_{\tilde{Y}} + \frac{5}{18}D_2^2 + \frac{1}{2}D_2K_{\tilde{Y}} - r_2$$

where r_1, r_2 are the contributions of the singularities of triple cover to global invariants which can be computed by canonical resolution.

- Consider canonical resolution

$$
\begin{array}{ccc}
X_k & \longrightarrow & X \\
\Big\downarrow \tilde{\pi} & & \Big\downarrow \pi \\
Y_k & \xrightarrow[can.res.]{} & Y
\end{array}
$$

Let p singularity of branch locus R
σ_1 blowing up p
E_1 strict transform of σ_1
$\mathcal{E}_1$ totally transform of E_1 in Y_k

- Define

$$Z_{\pi,p} := \tilde{\pi}^* \mathcal{E}_1$$

- $Z_{\pi,p}$ is a cycle of exceptional curves.

- For simplicity, we denote

$$Z_{\pi,p} := \tilde{\pi}^*\sigma^*(p)$$

- Another definition of $Z_{\pi,p}$:

$$Z_{\pi,p} = \gcd\{div(\tilde{\pi}^*\sigma^* g) \mid g \in m_p\},$$

where m_p is the maximal ideal of local ring $\mathcal{O}_{Y,p}$.

- Fundamental Cycle

  Theorem $Z_{\pi,p}$ has a unique decomposition as follows.

  $$Z_{\pi,p} = Z_1 + Z_2 + Z_3, \;\; Z_i \geq 0,$$

  satisfying that
  (1) $Z_i Z_j = 0$, $(i \neq j)$;
  (2) either $Z_i = 0$, or Z_i is the fundamental cycle of its support;
  (3) if p totally ramified, then $Z_1 \geq Z_2 \geq Z_3$.

- **Corollary** $Z_1^2 \geq -3$,
  equality holds iff $Z_2 = Z_3 = 0$.

- **Definition**

  Z_1 : The fist fundamental cycle;

  Z_2 : The second fundamental cycle;

  Z_3 : The third fundamental cycle.

- Any singularity of branch locus has unique triples (Z_1, Z_2, Z_3).

- Local Invariants of p

$$Z_i^2, \qquad p_a(Z_i) \qquad (i = 1, 2, 3)$$

- The invariants are independent of resolution.

- Assuming p totally ramified, $q = \pi^{-1}(p)$, then

  (1) q rational double point

  $\iff p_a(Z_1) = p_a(Z_2) = 0$ & $Z_3 = 0$

  (2) q rational triple point

  $\iff p_a(Z_1) = 0 \; Z_2 = Z_3 = 0$

  (3) q is smooth

  $\iff p_a(Z_1) = p_a(Z_2) = p_a(Z_3) = 0$.

- $R = D_1 + 2D_2$ branch locus

  $mult_p(R) \geq 2p_a(Z_1) + 2p_a(Z_2) + 2p_a(Z_3)$.

- **Remark**
  In the case for double cover, we also have a similar result.

  Theorem If π is a double cover, p is the singular point of branch locus B, then $Z_{\pi,p}$ has a unique decomposition as follows.

$$Z_{\pi,p} := Z_1 + Z_2, \ \ Z_i \geq 0,$$

  satisfying that
  (1) $Z_1 Z_2 = 0$;
  (2) either $Z_2 = 0$, or Z_2 is the fundamental cycle of its support;
  (3) $Z_1 \geq Z_2$;
  (4) $Z_1^2 \geq -2$, equality holds iff $Z_2 = 0$.

- (Double cover)

  $(3 \to 3)$ singularity $\Longleftrightarrow p_a(Z_1) = 1$ and $p_a(Z_2) = 0$.

  negligible singular point $\Longleftrightarrow Z_1^2 = -2$ and $p_a(Z_1) = 0$.

- It suggest that we can classify singular points of branch locus by considering the local invariants Z_i^2 and $p_a(Z_i)$.

- Classify singularities of branch locus of triple cover
  - There are 9 classes of singularities of branch locus of triple cover.
  - Any singularity of branch locus can be replaced by 9 classes of singularities above, which doesn't influence its contribution to global invariants.

- Standard of Classification:

(1) $Z_i^2 =$?

(2) $p_a(Z_i) = 0$ or not?

- **Example**

type(0) good cusp: p is totally ramified &

$$p_a(Z_1) = p_a(Z_2) = p_a(Z_3) = 0$$

(defined by local equation $x^2 + y^{3n} = 0$, $n \geq 1$.)

type(1) $Z_1^2 = -3$ (triple point)

......(omit)

- **Relatively Invariants of $f : S \to C$**

  $\pi : S_0 \longrightarrow P_0$ corresponding triple cover

  $R = D_1 + 2D_2$ branch locus

  $D_1 F = \alpha_1$, $D_2 F = \alpha_2$

  where F is a generic fiber.

$$\begin{aligned} 3(g+1)K_f^2 &= \alpha r_{D_1} + 4(g-1) r_{D_2} - r_1' \\ 36(g+1)\chi_f &= \beta r_{D_1} + 2(5g+1) r_{D_2} - r_2' \\ e_f &= r_{D_1} + 2 r_{D_2} - r_3' \\ D_1 D_2 &= \frac{\alpha_2}{(2\alpha_1 - 2)} r_{D_1} + \frac{\alpha_1}{(2\alpha_2 - 2)} r_{D_2} \end{aligned}$$

where

$$\alpha = \frac{(3g-1)}{2} - \frac{(g-3)}{2\alpha_1 - 2}, \quad \beta = \frac{(9g+5)}{2} - \frac{(g-3)}{2\alpha_1 - 2}$$

r_1', r_2', r_3' are contributions of singularities of triple cover to relatively invariants.

- The computation of relatively invariants is also due to two parts (compare with the case for $g = 2$):

  (1) relatively ramified of branch locus $R = D_1 + 2D_2$;

  (2) contributions of singular points of R to the relatively invariants.

- We only need to compute contribution of each type of singular point to the relatively invariants.

- Step4 $(g=3,$ & non-hyperelliptic)

  Singular Index

  From the computation of global invariants of triple cover, we have (semistable)

$$K_f^2 = \frac{1}{3}s_1 + 3s_2 + \frac{7}{3}s_3 + 2s_4 + \frac{13}{3}s_5 + \frac{4}{3}s_6,$$
$$\chi_f = \frac{1}{9}s_1 + \frac{1}{3}s_2 + \frac{4}{9}s_3 + \frac{1}{3}s_4 + \frac{4}{9}s_5 + \frac{1}{9}s_6$$
$$e_f = s_1 + s_2 + 3s_3 + 2s_4 + s_5,$$

  where $s_i \geq 0$.

- s_i relates to the number of singularities of branch locus.

- $g \geq 4$ & trigonal fibration.

  a similar result.

Thanks