Some Results on Global Invariants and Local Invariants of fibrations

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1. Fibred surface

- Fibration $f : S \to C \overset{def}{\iff}$ holomorphic & surjective, S smooth surface, C smooth curve of genus b.
- Fiber $F_t := f^{-1}(t)$ connected, genus g
- f relatively minimal $\overset{def}{\iff}$ all singular fibers contains no (-1)-curves.
- S can be viewed as a smooth curve $\mathcal{C}$ of genus g over the function field $K = \mathbb{C}(C)$.
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2. Global invariants

- Chern numbers:

$$c_1^2(S),\ \ c_2(S),\ \ \chi(\mathcal{O}_S)=\frac{1}{12}(c_1^2(S)+c_2(S))$$

- Relative invariants:

$$K_f^2 = c_1^2(S) - 8(g-1)(b-1) = K_{S/C}^2 \geq 0,$$
$$e_f = c_2(S) - 4(g-1)(b-1) = \sum(\chi_{\rm top}(F_i) - (2-2g)) \geq 0,$$
$$\chi_f = \chi(\mathcal{O}_S) - (g-1)(b-1) = \deg f_*\omega_{S/C} \geq 0,$$

- Relative Noether Formula: $K_f^2 + e_f = 12\chi_f$.
- Relative irregularity: $0 \leq q_f = q(S) - b \leq g$

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- Picard number : $\rho(S) = \operatorname{rank} \mathrm{NS}(S)$
- Mordell-Weil rank: $\mathrm{mw}(f) = \operatorname{rank} \mathrm{MW}(f)$

$$\mathrm{MW}(f) := J(K)/\tau A \cong \mathrm{NS}(S)/T$$

- $\tau A \subset J(K)$: $K/\mathbb{C}$-trace, or the fixed part of $J(K)$.

$$\tau A = 0 \iff q_f = 0$$

- $T \subset \mathrm{NS}(S)$: generated by a section O, F and all of the components of singular fibers

$$\mathrm{mw}(f) = \rho(S) - 2 - \sum_{i=1}^{s} (\ell_i - 1).$$

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3. Modular invariants

- "Relative invariant"="modular invariant" + "local invariant"
- Modular invariants: (modular map $J : C \to \overline{\mathcal{M}}_g$)

$$\kappa(f) = \deg J^*\kappa, \quad \delta(f) = \deg J^*\delta, \quad \lambda(f) = \deg J^*\lambda$$

- Noether formula

$$\lambda(f) = \frac{1}{12}(\kappa(f) + e(f))$$

- Semistable family $f : S \to C$,

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3. Modular Invariants

- $g = 0$:

$$K_f^2 = e_f = \chi_f = 0, \quad \kappa(f) = \delta(f) = \lambda(f) = 0.$$

- $g = 1$ (Kodaira's Theory):

$$\kappa(f) = 0, \quad \delta(f) = j, \quad \lambda(f) = \frac{1}{12} j$$

- J-map (or J-function):

$$J : C \to \mathbb{C} \cup \{\infty\} \cong \overline{\mathcal{M}_1}$$
$$t \mapsto J(F_t)$$

- Modular invariant: $\delta(f) = \deg J^*(\infty) = \deg J = j$

$$j = \text{number of poles of } J(F_t)$$
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4. Kodaira's Formula

$g = 1$

- **Kodaira's formula I:**

$$K_f^2 = 0,$$
$$e_f = j + 0\,\nu(\mathrm{I}) + 2\,\nu(\mathrm{II}) + 3\,\nu(\mathrm{III}) + 4\,\nu(\mathrm{IV})$$
$$6\,\nu(\mathrm{I}^*) + 10\,\nu(\mathrm{II}^*) + 9\,\nu(\mathrm{III}^*) + 8\,\nu(\mathrm{IV}^*)$$

$\nu(*)$: the number of singular fibers of type $(*)$.

- **Kodaira's formula II:**

$$j = \sum_{n\geq 1} n\left(\nu({}_m\mathrm{I}_n) + \nu(\mathrm{I}_n^*)\right)$$

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5. Local Invariants

$g \geq 2$

- Chern numbers of a singular fiber F (S.-L. Tan, 1996)

$$c_1^2(F) \geq 0, \quad c_2(F) \geq 0, \quad \chi_F = \frac{1}{12}(c_1^2(F) + c_2(F)).$$

- Generalization of Kodaira's Formula I(S.-L. Tan, 1996):

$$\begin{cases} K_f^2 = \kappa(f) + \sum_{i=1}^{s} c_1^2(F_i) \\ e_f = \delta(f) + \sum_{i=1}^{s} c_2(F_i) \\ \chi_f = \lambda(f) + \sum_{i=1}^{s} \chi_{F_i} \end{cases}$$

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6. Formulae of Chern numbers

- $$\begin{cases} c_1^2(F) = 4N_F + F_{\mathrm{red}}^2 + \alpha_F - \beta_F^- \\ c_2(F) = 2N_F + \mu_F - \beta_F^+ \end{cases}$$

- Total Milnor number μ_F:

$$\mu_F := \sum_{p \in F} \mu_p(F_{\mathrm{red}}) \quad (\mathrm{Milnor\ number})\ .$$

- $N_F := g - p_a(F_{red}), \quad 0 \leq N_F \leq g.$
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6. Formulae of local Chern numbers

Normal Crossing Model $\overline{F}$ of F:

- $\sigma = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_r : \ \overline{S} \longrightarrow S$: sequence of blowing-ups
  $\overline{F} = \sigma^* F$: normal crossing divisor.
  - $\sigma_i : S_{i+1} \longrightarrow S_i$: blowing-up at a point p_i.
  - F_i: pullback fiber of F on S_i.
  - $m_i = \mathrm{mult}_{p_i}(F_{i,\mathrm{red}}) \geq 2$
- Definition of α_F:

$$\alpha_F := \sum_{i=1}^{r} (m_i - 2)^2.$$

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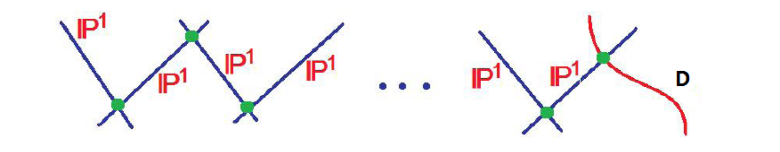
9. Local Invariants

- Let q be a node of $\overline{F}$ locally defined by $x^a y^b = 0$.

$$\beta_q = \frac{\gcd(a,b)^2}{ab}$$

- A HJ branch of $\overline{F}$ is the following chain of rational curves



$D \not\cong \mathbb{P}^1$, or D meets the other components in at least 3 points.

$$\beta_F^- := \sum_{q \in \mathrm{HJ}} \beta_q, \qquad \beta_F^+ := \sum_{q \notin \mathrm{HJ}} \beta_q,$$

Example ($g = 1$): F singular fiber

- $$c_1(F) = 0, \quad \chi_F = \frac{1}{12} c_2(F).$$

F	${}_mI_n$	I^*	II	III	IV
$c_2(F)$	0	6	2	3	4
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Jun Lu, S.-L. Tan, Trans. of AMS, 365 (213)

- Noether formula: $\chi_F = \frac{1}{12}(c_1^2(F) + c_2(F))$.
- Positivity: $c_1^2(F) \geq 0$, $c_2(F) \geq 0$, $\chi_F \geq 0$.
  If $g \geq 2$, then
  $$c_1^2(F) = 0 \iff c_2(F) = 0 \iff \chi_F = 0 \iff F \text{semistable}$$
- Blow-up Formulas: $\sigma : S' \to S$, $F' = \sigma^* F$.
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- Miyaoka-Yau Type Inequality: $c_1^2(F) \leq 2c_2(F)$.

$$c_1^2(F) = 2c_2(F) \iff F = nF_{\mathrm{red}},\ F_{\mathrm{red}} \text{ is a nodal curve.}$$

- If $2c_2(F) - c_1^2(F) < 6 \Longrightarrow F$ can be classified into 8 kinds.

F	1	2	3	4	5	6	7	8
$2c_2 - c_1^2$	0	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{21}{4}$	5	3	$\frac{9}{2}$	$\frac{11}{2}$

- **Corollary**: F non-semistable $\Longrightarrow c_2(F) \geq \frac{11}{6}$, $\chi_F \geq \frac{1}{6}$.
  One of "=" holds $\iff F$ reduced nodal-cuspidal curves with one cusp.

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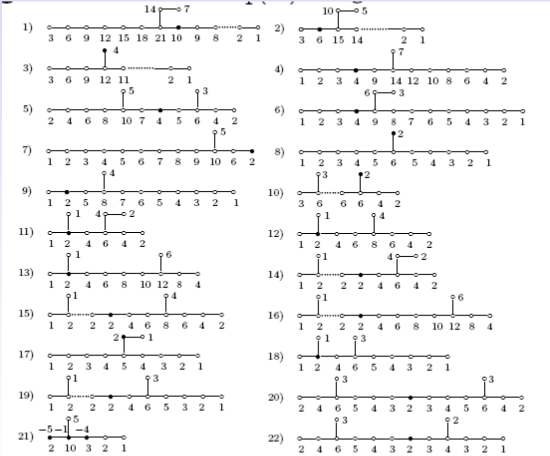
- Canonical Class Inequality: $c_1^2(F) \leq 4g - 4.8$
- $c_1^2(F) > 4g - 5.5 \Longrightarrow g \leq 6$, F can be classified into 22 kinds.



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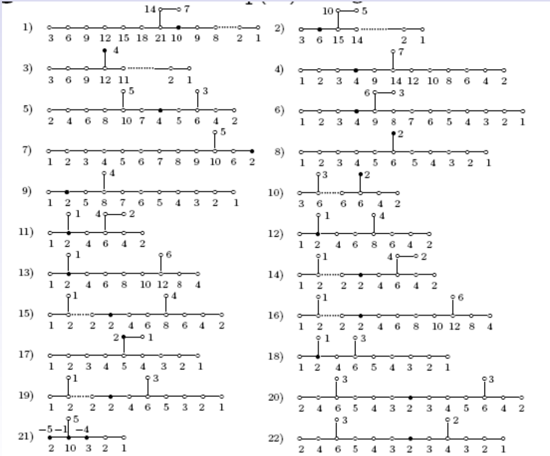

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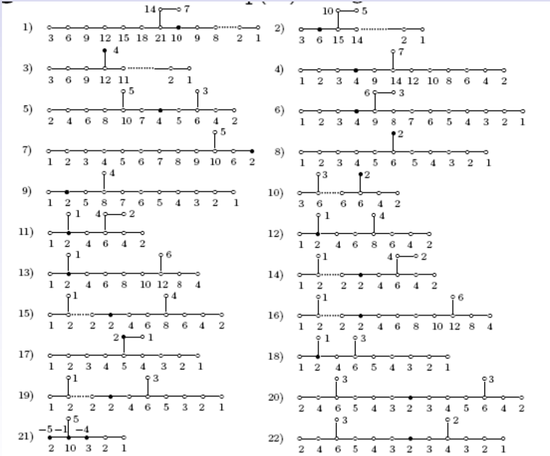

4. Arakelov Inequality and Dual Theorem

- Dedekind Number

$$\chi(p,q) = \frac{1}{12}\left(\frac{q}{p} + \frac{p}{q} + \frac{(p,q)^2}{pq}\right) - \frac{1}{4}.$$

- $\overline{F} = n_1 C_1 + \cdots + n_k C_k$

$$\chi_F = \frac{1}{2} N_{\overline{F}} - \sum_{i<j} \chi(n_i, n_j) C_i C_j.$$

- Denote by

$$M_F := \text{l.c.m.}\{\, n_1, \cdots, n_k \,\}$$

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- Let $\pi : \widetilde{C} \to C$ be a locally base change defined by $w^d = t$. $f(F) = p \in C$ is defined by $t = 0$.

$$\begin{array}{ccc} \widetilde{S} & \longrightarrow & S \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ \widetilde{C} & \xrightarrow{\pi} & C \end{array}$$

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- Duality:

$$\begin{cases} N_{\overline{F^*}} = N_{\overline{F}}, \\ \chi_F + \chi_{F^*} = N_{\overline{F}} \end{cases}, \qquad \frac{1}{6}N_{\overline{F}} \le \chi_F \le \frac{5}{6}N_{\overline{F}}$$

- $\chi_F = \frac{1}{6}N_{\overline{F}} \iff F$ reduced nodal-cuspidal curves.
- $\chi_F = \frac{5}{6}N_{\overline{F}} \iff F^*$ reduced nodal-cuspidal curves.
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