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An Arakelov inequality in characteristic p and upper bound of p-rank zero locus ☆

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ABSTRACT

In this paper we show an Arakelov inequality for semi-stable families of algebraic curves of genus $g \geqslant 1$ over characteristic p with nontrivial Kodaira–Spencer maps. We apply this inequality to obtain an upper bound of the number of algebraic curves of p-rank zero in a semi-stable family over characteristic p with nontrivial Kodaira–Spencer map in terms of the genus of a general closed fiber, the genus of the base curve and the number of singular fibres. The parallel results for smooth families of Abelian varieties over k with W_2-lifting assumption are also obtained.

1. Introduction

Let $f : X \to C$ be a non-isotrivial semi-stable family of algebraic curves of genus $g \geqslant 1$ over smooth projective curve C over the complex numbers. Let $S \subset C$ be the singular locus over which the fibration f degenerates. The classical Arakelov inequality (cf. [4]) states that the following inequality holds

$$\deg f_*\omega_{X/C} \leqslant \frac{g}{2} \deg \Omega^1_C(S).$$

This is one of key ingredients in the proof by Arakelov on the Shararevich conjecture that the isomorphism classes of genus g curves over a given functional field with fixed degeneracy are finite. There

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are new developments since then on the Arakelov inequalities and their applications to certain geometric problems on moduli spaces of polarized algebraic manifolds. (For more information, refer to the recent survey articles [21,25] and references therein.) In a series of papers [9,22,24], the following generalized form of Arakelov inequality was obtained:

Theorem 1.1. *(See [21, Theorem 1.1]; [25, Theorem 4.4], $n = 1$ case.) Let $f : X \to C$ be a semi-stable family of curves as above. Let $\mathcal{E}$ be a coherent subsheaf of $f_*\omega^{\nu}_{X/C}$, $\nu \geqslant 1$. Then the following inequality holds*:

$$\frac{\deg \mathcal{E}}{\operatorname{rank} \mathcal{E}} \leqslant \frac{\nu}{2} \deg \Omega^1_C(S).$$

In this paper we give a characteristic p analogue of the Arakelov inequality in the above generalized form. Let k be the algebraic closure of the finite field $\mathbb{F}_p$ with p an odd prime. Now we let $f : X \to C$ be a semi-stable family of algebraic curves of genus $g \geqslant 1$ with nontrivial Kodaira–Spencer map over the projective curve C, which is defined over k. The nontriviality of Kodaira–Spencer map means that it is nonzero at one closed point in the smooth locus of the base curve and this assumption is equivalent to saying that the family is non-isotrivial and it is not equal to the semi-stable reduction of the base change of another family $\tilde{f} : \tilde{X} \to C$ under the Frobenius map $F_C : C \to C$.

Theorem 1.2. *(See Theorem 2.1.) Let $f : X \to C$ be a semi-stable family of algebraic curves of genus $g \geqslant 1$ over k with nontrivial Kodaira–Spencer map. Let $\mathcal{E}$ be a coherent subsheaf of $f_*\omega^{\nu}_{X/C}$, $\nu \geqslant 1$. Then the following strict inequality holds*:

$$\frac{\deg \mathcal{E}}{\operatorname{rank} \mathcal{E}} < 2\nu g \deg \Omega^1_C(S).$$

One cannot deduce the above Arakelov inequality in characteristic p directly from the Arakelov inequality in characteristic zero as given in Theorem 1.1. This is because there exists non-liftable family of algebraic curves in characteristic p, and even if the family f in characteristic p comes from the reduction at a prime over p of a semi-stable family in characteristic zero, there exists possibly non-liftable coherent subsheaf in $f_*\omega^{\nu}_{X/C}$. Furthermore one does not expect the same Arakelov inequality holds for families of Abelian varieties over k. Actually Moret-Bailly [10] constructed a semi-stable family of genus 2 curves over $\mathbb{P}^1$ defined over k, whose associated Jacobian fibration is a smooth family of supersingular Abelian surfaces. It was shown furthermore that $f_*\omega_{X/C} = \mathcal{O}_{\mathbb{P}^1}(p) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ (see [10, 3.2]). So the Arakelov inequality as given in the above theorem does not hold for the family of Abelian surfaces. It is known that this family is non-liftable to characteristic zero. However, when a smooth family of Abelian varieties is assumed to be W_2-liftable, one does has an Arakelov inequality. Precisely, let $f : X \to C$ be a smooth family of Abelian varieties of dimension $g \geqslant 2$ over k. We assume that f has nontrivial Kodaira–Spencer map and is W_2-liftable (see Assumption 5.1). We obtain the following Arakelov inequality:

Theorem 1.3. *(See Theorem 5.3.) Let $f : X \to C$ be a smooth family of Abelian varieties which is W_2-liftable and has nontrivial Kodaira–Spencer map. Assume furthermore that $p \geqslant 4g^2 - 6g + 4$. Then for any coherent subsheaf $\mathcal{F}$ of $E^{1,0}$ one has inequality*

$$\frac{\deg \mathcal{F}}{\operatorname{rank} \mathcal{F}} \leqslant \frac{1}{2} \deg \Omega^1_C.$$

Besides giving a characteristic p analogue of the classical Arakelov inequality, we are also motivated by its potential applications to certain geometric problems of the moduli spaces of curves defined over k. The articles by F. Oort [15,16] and G. van der Geer [20] are referred to for a general introduction to the moduli spaces of curves and Abelian varieties in characteristic p. Particularly we shall use the same notations as them and shall not repeat the definitions if they have already been defined there. We start our discussions with a very classical example, the Legendre family of elliptic curves over k. It is given by the affine equation

$$y^2 = x(x-1)(x-t)$$

with parameter t. As well known, the family has three singular fibers and $\frac{p-1}{2}$ (smooth) supersingular fibers. One needs to do semi-stable reduction to the Legendre family in order to fit into our considerations. A possible base change is given by the double cover of the base curve $\mathbb{P}^1$ branching at 0 and ∞. The obtained semi-stable model of the Legendre family has then four singular fibers and $p-1$ supersingular fibers. The base curve is actually modulo p reduction of the modular curve with level structure $\Gamma(2) \cap \Gamma_1(4) \subset SL(2,\mathbb{Z})$. This is one of six examples of semi-stable families of elliptic curves over $\mathbb{P}^1$ due to A. Beauville [1]. From these information plus that of automorphisms of elliptic curves one can deduce the classical Deuring mass formula on the number of supersingular elliptic curves in the coarse moduli space over k. It is remarkable that T. Ekedahl and G. van der Geer have generalized the formula to the Ekedahl–Oort strata in the coarse moduli space of principally polarized Abelian varieties over k (see the article by G. van der Geer [20] and references therein). One can ask as next step for formulas or equalities for Ekedahl–Oort strata in the Torelli locus. Unfortunately, such questions remain difficult in general (see the article by F. Oort [16]). In this paper we want to use the above Arakelov inequality to present certain inequality about p-rank zero locus. To motivate it, it can be shown that for example $p-1$ is the maximal number of supersingular elliptic curves in a semi-stable family of elliptic curves over $\mathbb{P}^1$ with four singular fibers over k, whose Kodaira–Spencer map is nontrivial (see discussions after Proposition 4.1). For a semi-stable family of higher genus curves, the situation is different. The example of Moret-Bailly mentioned above shows that there are possibly infinitely many closed fibers in a semi-stable family of $g \geqslant 2$ curves with nontrivial Kodaira–Spencer map, whose p-ranks are zero. However the following theorem shows that, if one assumes that the generic fiber of the family is not of p-rank zero, then the number of p-rank zero closed fibers in the family cannot be arbitrarily large and is actually bounded by the basic invariants of the family itself and the characteristic of the ground field k.

Theorem 1.4. *(See Theorem 4.3.) Let $f: X \to C$ be a semi-stable family of algebraic curves of genus $g \geqslant 1$ over smooth projective algebraic curves C over k with nontrivial Kodaira–Spencer map. If the p-rank of the generic fiber of f is nonzero, then the number of p-rank zero closed fibers of f is strictly bounded from above by*

$$2p^g g^3(2b-2+s),$$

where b is the genus of base curve C and s is the number of singular fibers of f.

Remark 1.5. The upper bound in the above theorem does not follow from the result of Ekedahl and van der Geer on the cycle class of p-rank zero locus V_0. By Theorem 9.2 of [20], the cycle class $[V_0]$ of V_0 is equal to $(p-1)(p^2-1)\cdots(p^g-1)\lambda_g$ where λ_g is the top Chern class of the first Hodge bundle over $\tilde{\mathcal{A}}_g$. For a curve C in $\tilde{\mathcal{A}}_g$, the cardinality of p-rank zero locus in C however cannot be expressed by the intersection of Chern classes of C and V_0. This is because the cup product of elements in $H^{2\dim(\tilde{\mathcal{A}}_g)-2}(\tilde{\mathcal{A}}_g,\mathbb{Z})$ and $H^{2g}(\tilde{\mathcal{A}}_g,\mathbb{Z})$ has degree greater than $2\dim\tilde{\mathcal{A}}_g$ when $g \geqslant 2$, and hence is equal to zero.

For the upper bound of p-rank zero locus of smooth family of Abelian varieties with W_2-lifting and nontrivial Kodaira–Spencer map, we have the following:

Theorem 1.6. *(See Theorem 5.6.) Let $f: X \to C$ be a smooth family of Abelian varieties of dimension $g \geqslant 2$ over k which is W_2-liftable and has nontrivial Kodaira–Spencer map. Assume that $p \geqslant 4g^2-6g+4$. If the* *generic fiber of f is not of p-rank zero, then the number of p-rank zero closed fibers $|V_0(f)|$ in f is bounded from above by*

$$\left[p^g g + 2(g-1)^2 \frac{p^g-1}{p-1}\right](b-1),$$

where b is the genus of C.

The contents of the paper are organized as follows. In Section 2 we prove the Arakelov inequality in characteristic p. In Section 3 we discuss the relative Frobenius morphism of $f: X \to C$ and show an inequality for the slopes of coherent subsheaves in $(F_C^*)^n R^1 f_* \mathcal{O}_X$, $n \geqslant 1$. This section serves as preparation for Section 4, but is separated from Section 4 for its own interests. In Section 4 we prove the claimed upper bound of p-rank zero locus. In Section 5 we prove the Arakelov inequality and give upper bound of p-rank zero locus for smooth families of Abelian varieties satisfying W_2-liftability assumption.

Notations and conventions. In the following sections the notation $f: X \to C$ means a semi-stable family of algebraic curves of genus $g \geqslant 1$ or of Abelian varieties of dimension $g \geqslant 2$ over k, where the base curve C is smooth and projective. The set $S \subset C$ is the singular locus of f. We denote by $E^{1,0} = f_* \Omega^1_{X/C}$ and $E^{0,1} = R^1 f_* \mathcal{O}_X$. They are called the first and respectively the second Hodge bundle of the family f. When f is a family of curves, we write $\Omega^1_{X/C}$ as $\omega_{X/C}$. The slope of a coherent sheaf $\mathcal{F}$ over C is defined to be $\mu(\mathcal{F}) = \frac{\deg \mathcal{F}}{\operatorname{rank} \mathcal{F}}$. The p-rank of a smooth projective algebraic curve of genus $g \geqslant 1$ is defined to be the p-rank of its Jacobian (see F. Oort [16], or Section 4 in [11] for the definition of p-rank). For an algebraic variety X defined over k, the map $F_X: X \to X$ is denoted to be the absolute Frobenius morphism defined by power p map on the structure sheaf of rings $\mathcal{O}_X$ (see [3, Section 9]). In Section 3 we shall use occasionally the notion of F-crystal and crystalline cohomology. The basic reference for F-crystals is the article by N. Katz [7] and for crystalline cohomology it is the book [2]. For the remaining notions on algebraic varieties in the article refer to [6].

2. Arakelov inequality of semi-stable families of algebraic curves in characteristic *p*

Let $f: X \to C$ be a semi-stable family of algebraic curves of genus $g \geqslant 1$ with nontrivial Kodaira–Spencer map. In this section we shall prove an Arakelov inequality for the family f. Our proof is based on certain techniques in the theory of algebraic surfaces, for which one can consult for example the book by G. Xiao [19], and the results of L. Szpiro in [18]. We are going to prove the following

Theorem 2.1. *Let $f: X \to C$ be a semi-stable family of algebraic curves of genus $\geqslant 1$ over k with nontrivial Kodaira–Spencer map. For a coherent subsheaf $\mathcal{E}$ in $f_* \omega^{\nu}_{X/C}$, $\nu \geqslant 1$ one has the following upper bound on the slope of $\mathcal{E}$:*

$$\mu(\mathcal{E}) < 2\nu g \deg \Omega^1_C(S).$$

Proof. We proceed it by considering $g = 1$ and $g \geqslant 2$ separately. First we study $g = 1$ case. By cup product with first Hodge bundle $E^{1,0}$ and then contraction on coefficients, the Kodaira–Spencer map of f induces a nontrivial morphism

$$\theta: E^{1,0} \to E^{0,1} \otimes \Omega^1_C(S).$$

Since they are invertible sheaves, θ induces an embedding

$$\left(E^{1,0}\right)^{\otimes 2} \to \Omega^1_C(S).$$

This implies that

$$\deg E^{1,0} \leqslant \frac{1}{2} \deg \Omega^1_C(S).$$

Because f is a semi-stable elliptic fibration, one has isomorphism

$$f_*\omega^{\nu}_{X/C} \simeq (f_*\omega_{X/C})^{\nu}.$$

Then for any coherent subsheaf $\mathcal{E}$ of $f_*\omega^{\nu}_{X/C}$, which is invertible in this case, one has the following inequality:

$$\mu(\mathcal{E}) = \deg \mathcal{E} \leqslant \nu \deg E^{1,0} \leqslant \frac{\nu}{2} \deg \Omega^1_C(S) < 2\nu \deg \Omega^1_C(S).$$

Now we consider $g \geqslant 2$ case. Let

$$f_*\omega^{\nu}_{X/C} = \mathcal{E}_m \supseteq \mathcal{E}_{m-1} \supseteq \cdots \supseteq \mathcal{E}_1 \supseteq \mathcal{E}_0 = 0$$

be the Harder–Narasimhan filtration of $f_*\omega^{\nu}_{X/C}$. Obviously, it suffices to show our inequality for $\mathcal{E}_1$, which has maximal slope among all coherent subsheaves in $f_*\omega^{\nu}_{X/C}$. One notes that the image sheaf of the natural morphism $\alpha : f^*\mathcal{E}_1 \to \omega^{\otimes\nu}_{X/C}$ can be expressed by $\mathcal{I}_Z(\nu K_{X/C} - D)$, where $\mathcal{I}_Z$ is the ideal sheaf of a zero-dimensional subscheme Z, $K_{X/C}$ a relative canonical divisor and D is an effective divisor. Now let H be an ample divisor of X. Since $\omega_{X/C}$ is big and nef by L. Szpiro (see Theorem 1 and Proposition 3 in [18]), $m\nu K_{X/C} + H$ is ample for $m \geqslant 0$ by Nakai's criterion (see Theorem 1 in [8]). By Bertini's theorem, one can find a smooth curve

$$\Gamma \in \big|n(m\nu K_{X/C} + H)\big|$$

for n large enough such that Γ intersects transversally with a fixed smooth closed fiber F of f, and Γ is not contained in the support the kernel of α. We put $\pi = f|_{\Gamma}$. By construction, π is a separable finite morphism and the restriction of α to Γ

$$\alpha|_{\Gamma} : \pi^*\mathcal{E}_1 \to \mathcal{O}_{\Gamma}(\nu K_{X/C} - D)$$

is nontrivial. Because $\mathcal{E}_1$ is semi-stable and π is separable, $\pi^*\mathcal{E}_1$ is still semi-stable by Lemma 3.1 of [17]. Hence we have

$$\mu\big(\pi^*(\mathcal{E}_1)\big) \leqslant \deg \mathcal{O}_{\Gamma}(\nu K_{X/C} - D) = (\nu K_{X/C} - D)\Gamma.$$

We put $N = \nu K_{X/C} - D - \mu(\mathcal{E}_1)F$. Then the intersection number

$$\begin{aligned} N\Gamma &= (\nu K_{X/C} - D)\Gamma - \mu(\mathcal{E}_1)F\Gamma \\ &= (\nu K_{X/C} - D)\Gamma - \mu\big(\pi^*(\mathcal{E}_1)\big) \\ &\geqslant 0. \end{aligned}$$

It implies that $N(m\nu K_{X/C} + H) \geqslant 0$. Since m can be arbitrarily large, we must have $NK_{X/C} \geqslant 0$. Therefore, we see that

$$\begin{aligned}\nu K_{X/C}^2 &= \big(N + D + \mu(\mathcal{E}_1)F\big)K_{X/C}\\ &\geqslant \mu(\mathcal{E}_1)FK_{X/C}\\ &= (2g-2)\mu(\mathcal{E}_1).\end{aligned}$$

Now we recall the Szpiro's inequality (see Proposition 4.2 in [18])

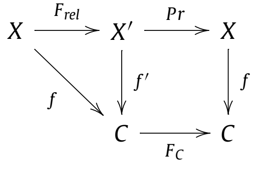
$$K_{X/C}^2 < (4g-4)g\deg\Omega_C^1(S).$$

By combining the last two inequalities, we obtain therefore

$$\mu(\mathcal{E}_1) < 2\nu g\deg\Omega_C^1(S). \qquad \square$$

3. Relative Frobenius morphism and upper bound for the slopes of coherent subsheaves in the pull-back of second Hodge bundle under iterated Frobenius morphism

For the family $f : X \to C$ over k one has the following commutative diagram induced by the absolute Frobenius morphisms of the base C and the total space X:



where $f' : X' \to C$ is the base change of f under F_C and $F_{rel} : X \to X'$ is the so-called relative Frobenius morphism. The C-morphism F_{rel} is the main interest of the first half of this section. The following simple lemma is well known. For the convenience of the reader we include it here with a proof.

Lemma 3.1. *The relative Frobenius morphism F_{rel} induces a natural morphism of $\mathcal{O}_C$-modules*

$$F_{rel}^* : F_C^* E^{0,1} \to E^{0,1}.$$

Proof. The relative Frobenius morphism F_{rel} gives the $\mathcal{O}_{X'}$-morphism of sheaves

$$\mathcal{O}_{X'} \to F_{rel*}\mathcal{O}_X.$$

It induces the morphism on the direct images

$$R^1 f'_*\mathcal{O}_{X'} \to R^1 f'_*(F_{rel*}\mathcal{O}_X).$$

The spectral sequence of composed maps gives the natural morphism

$$R^1 f'_*(F_{rel*}\mathcal{O}_X) \to R^1(f' \circ F_{rel})_*\mathcal{O}_X.$$

Composition of the above two morphisms yields the morphism

$$R^1 f'_*\mathcal{O}_{X'} \to R^1 f_*\mathcal{O}_X.$$

Finally, by the flat base change theorem (cf. [6, Proposition 9.3]) the $R^1 f'_*\mathcal{O}_{X'}$ is isomorphic to $F_C^* R^1 f_*\mathcal{O}_X$. The lemma follows. $\square$

In the following text we denote for brevity the n-th iterated Frobenius morphism $(F_C^*)^n$ by F_n^*. Following Lemma 3.1 one can then consider a sequence of morphisms

$$\Phi_n : F_n^* E^{0,1} \to E^{0,1}, \quad n \geqslant 1,$$

which is the composition of the following morphisms:

$$F_n^* E^{0,1} \xrightarrow{F_{n-1}^* F_{rel}^*} F_{n-1}^* E^{0,1} \xrightarrow{F_{n-2}^* F_{rel}^*} \cdots \to F_1^* E^{0,1} \xrightarrow{F_{rel}^*} E^{0,1}.$$

The following result describes some relations between the morphism Φ_n and the p-rank of the fibers.

Proposition 3.2. *Let $f : X \to C$ be a semi-stable family of algebraic curves or Abelian varieties. Let t be a closed point of C and the closed fiber X_t of f over t be smooth. Then the following statements hold*:

(i) *The morphism Φ_n is isomorphism at t for one n (or for all n) if and only if X_t is ordinary.*
(ii) *The morphism Φ_g is zero at t if and only if X_t is of p-rank zero.*
(iii) *The morphism Φ_1 is zero at t only if X_t is supersingular.*

These statements should be more or less well known. For completeness we present a proof here, although we are aware of the possibility that it is worse than the original one. We remark that the if-part of (iii) above is not true.

Firstly we recall some basic known facts about the crystalline cohomolgy of Abelian variety. Let A be a principally polarized Abelian variety of dimension g over k and let $H^1_{crys}(A/W(k))$ be the first crystalline cohomology. It is torsion free $W(k)$-module of rank $2g$. More importantly, it has a semi-linear endomorphism F on $H^1_{crys}(A/W(k))$ which makes the pair $(H^1_{crys}(A/W(k)), F)$ the structure of F-crystal. By a well-known comparison theorem, the reduction modulo p of the F-crystal is isomorphic to the Frobenius morphism on the de-Rham cohomology $H^1_{dR}(A/k)$. By using the Dieudonne theory, the p-rank of A is equal to the multiplicity of the slope zero part of the F-crystal. The basic (semi-)linear algebra of F-crystals is established in [7]. We shall also use a result due to N. Nygaard [12].

Proof of Proposition 3.2. These are point-wise statements. Let t be a k-point of C and X_t be the closed fiber over t. When X_t is an algebraic curve, it is equivalent to prove the corresponding statements for its Jacobian $Jac(X_t)$. Actually, by a fundamental result of Weil $Jac(X_t)$ has the same field of definition as X_t, and moreover one has a k-morphism $i : X_t \to Jac(X_t)$. Since i induces an isomorphism of k-vector spaces $i^* : H^1(Jac(X_t), \mathcal{O}_{Jac(X_t)}) \to H^1(X_t, \mathcal{O}_{X_t})$, the action $F^*_{X_t}$ on $H^1(X_t, \mathcal{O}_{X_t})$ induced by the absolute Frobenius is identified with the action $F^*_{Jac(X_t)}$ on $H^1(Jac(X_t), \mathcal{O}_{Jac(X_t)})$ via the isomorphism.

Now we let $A = Jac(X_t)$ be the Jacobian of X_t when f is a family of curves, and $A = X_t$ when f is a family of Abelian varieties. It is equivalent to prove the corresponding statements for

$$\left(F_A^*\right)^n : H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A).$$

We look at the F-crystal of A. By the Hodge–Newton decomposition (cf. [7, Theorem 1.6.1]), there is a unique decomposition of F-crystals

$$\left(H^1_{crys}\left(A/W(k)\right), F\right) = (M_0 \oplus M_{>0}, F)$$

where $(M_0, F|_{M_0})$ is the unit-root subcrystal of $(H^1_{crys}(A/W(k)), F)$ and $M_{>0}$ is the complement of M_0 whose Newton slopes are all positive. So the proofs of (i) and (ii) are reduced to the following:

Claim 3.3. *Let A be a g-dimensional Abelian variety defined over k. The Newton slopes of A are all positive if and only if the morphism*

$$\left(F_A^*\right)^g : H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A)$$

is zero map.

Proof. It is known that the k-vector space $H^1(A, \mathcal{O}_A)$ is a natural quotient of $H^1_{cris}(A, W(k))$ modulo p, and the map F modulo p induces a natural morphism on $H^1(A, \mathcal{O}_A)$, which is identical to F_A^*. By N. Katz [7, 1.3.3], all Newton slopes are positive if and only if $F^{2g}(H^1_{cris}(A, W(k))) \subset pH^1_{cris}(A, W(k))$, which implies in particular

$$\left(F_A^*\right)^{2g} : H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A)$$

is zero map. But since $\dim_k H^1(A, \mathcal{O}_A) = g$, the g times iterate $(F_A^*)^g$ is already zero. Conversely, if $(F_A^*)^g$ is a zero morphism, then

$$\left(F_A^*\right)^{g+1} : H^1_{dR}(A/k) \to H^1_{dR}(A/k)$$

is zero. Since $2g \geqslant g + 1$, it implies that

$$F^{2g}\left(H^1_{cris}\left(A, W(k)\right)\right) \subset pH^1_{cris}\left(A, W(k)\right).$$

Then still by N. Katz's remark, the Newton slopes of $H^1_{cris}(A, W(k))$ are all positive. The claims is proved. □

To the part (iii) we invoke a result of N. Nygaard.

Theorem 3.4. *(See Nygaard [12, Theorem 1.2].) Let A and $(H^1_{crys}(A/W(k)), F)$ be as above. Then A is supersingular if and only if*

(a) *F^{g^2-g+2} is divisible by $p^{\frac{g^2+1}{2}-(g-1)}$ if g is odd;*
(b) *F^{g^2-2g+3} is divisible by $p^{\frac{g^2+1}{2}-\frac{3}{2}(g-1)}$ if g is even.*

So the assumption of (iii) says that the map $F_A^* : H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A)$ is zero. It implies that $F^2 : H^1_{crys}(A/W(k)) \to H^1_{crys}(A/W(k))$ is divisible by p. An elementary calculation shows that the divisibility condition in above theorem is satisfied and (iii) follows therefore from Theorem 3.4. □

In the remainder of this section we shall prove the following theorem about various upper bounds of the slopes of coherent subsheaves contained in $F_n^* E^{0,1}$, $n \geqslant 1$.

Theorem 3.5. *Let $f : X \to C$ be a semi-stable family of algebraic curves. Let $\mathcal{F}$ be a coherent subsheaf of $F_n^* E^{0,1}$ for certain $n \geqslant 1$. Then we have the following upper bound of the slope of $\mathcal{F}$:*

(i) *If f has an ordinary closed fiber, then $\mu(\mathcal{F}) \leqslant 0$.*
(ii) *If $n = 0$, then $\mu(\mathcal{F}) < 2g(g-1)\deg \Omega^1_C(S)$.*
(iii) *In any case one has*

$$\mu(\mathcal{F}) < 2g(g-1)(2b-2+s)p^n + 2(g-1)(b-1)\frac{p^n-1}{p-1},$$

where b is the genus of C and s the number of singular fibers of f.

Proof of (i) and (ii). We prove the first statement by contradiction. By assumption that f has an ordinary closed fiber, the generic fiber of f is then ordinary. By Proposition 3.2(i), the morphism Φ_1 is generically isomorphism. Now if there was a coherent subsheaf $\mathcal{F}$ of $F_n^* E^{0,1}$ with positive slope, one deduces a contradiction as follows. One can assume from the first that $\mathcal{F}$ is of maximal slope in $F_n^* E^{0,1}$. We consider the morphism

$$F_n^* \Phi_1 : F_{n+1}^* E^{0,1} \to F_n^* E^{0,1}$$

and we claim that the subsheaf $F_n^* \Phi_1(F_1^* \mathcal{F})$ of $F_n^* E^{0,1}$ has larger slope than $\mathcal{F}$. Actually the rank of $F_n^* \Phi_1(F_1^* \mathcal{F})$ is equal to that of $\mathcal{F}$ because Φ_1 is generically isomorphism and so is $F_n^* \Phi_1$. Secondly one has

$$\deg F_n^* \Phi_1\left(F_1^* \mathcal{F}\right) \geqslant \deg F_1^* \mathcal{F} = p \deg \mathcal{F}.$$

Since $\deg \mathcal{F} > 0$, it is clear that $\deg F_n^* \Phi_1(F_1^* \mathcal{F}) > \deg \mathcal{F}$. A contradiction.

Now we let $\mathcal{F}$ be a rank r coherent subsheaf of $E^{0,1}$. By the relative Serre duality, $E^{0,1}$ and $E^{1,0}$ are dual to each other. So the dual of the quotient sheaf $E^{0,1}/\mathcal{F}$ is a coherent subsheaf of $E^{1,0}$. Furthermore, because the family f has nontrivial Kodaira–Spencer map, one has $\deg E^{1,0} > 0$ by Proposition 3 of [18]. Applying the Arakelov inequality (Theorem 2.1) in $\nu = 1$ case, we have

$$\begin{aligned}
\deg(\mathcal{F}) &= -\deg E^{1,0} + \deg\left(E^{0,1}/\mathcal{F}\right)^* \\
&< \deg\left(E^{0,1}/\mathcal{F}\right)^* \\
&< 2g(g-r) \deg \Omega_C^1(S).
\end{aligned}$$

Therefore

$$\mu(\mathcal{F}) < 2g\left(\frac{g-r}{r}\right) \deg \Omega_C^1(S) \leqslant 2g(g-1) \deg \Omega_C^1(S).$$

The second statement is proved. □

Before proving the upper bound in the general case, we shall discuss the instability of vector bundles under Frobenius pull-back. Our discussion follows that in [17]. For a coherent sheaf $\mathcal{E}$ the notation $\mu_{max}(\mathcal{E})$ (resp. $\mu_{min}(\mathcal{E})$) means the maximal (resp. minimal) slope of coherent subsheaves in $\mathcal{E}$. In [17] X. Sun has shown the following celebrated inequality:

Theorem 3.6. *(See Sun [17, Theorem 3.1].) Let C be a smooth projective curve of genus $b \geqslant 1$ over k. Let $\mathcal{E}$ be a semi-stable vector bundle of rank r over C. One has the following inequality*:

$$\mu_{max}\left(F_C^* \mathcal{E}\right) - \mu_{min}\left(F_C^* \mathcal{E}\right) \leqslant 2(r-1)(b-1).$$

Based on the above result, we shall show the following:

Theorem 3.7. *Let C be curve as above. The morphism $F_n : C \to C$, $n \geqslant 1$ denotes the n-th iterated Frobenius morphism. Let $\mathcal{E}$ be a vector bundle of rank r over C. The following inequality holds*:

$$\mu_{max}\left(F_n^* \mathcal{E}\right) - \mu_{min}\left(F_n^* \mathcal{E}\right) \leqslant p^n\left(\mu_{max}(\mathcal{E}) - \mu_{min}(\mathcal{E})\right) + 4(r-1)(b-1)\frac{p^n - 1}{p - 1}.$$

The generalized inequality follows from the following:

Proposition 3.8. *Let C and $\mathcal{E}$ be as above. The following inequality holds*:

$$\mu_{max}\big(F_n^*\mathcal{E}\big) \leqslant p^n \mu_{max}(\mathcal{E}) + 2(r-1)(b-1)\frac{p^n-1}{p-1}.$$

We shall use only Proposition 3.8 rather than Theorem 3.7 in our paper. It is stated for its independent interest as certain generalization of Sun's inequality. We deduce first Theorem 3.7 from the above proposition.

Proof of Theorem 3.7. Let

$$\mathcal{E} = \mathcal{E}_m \supseteq \mathcal{E}_{m-1} \supseteq \cdots \supseteq \mathcal{E}_1 \supseteq \mathcal{E}_0 = 0$$

be the Harder–Narasimhan filtration of $\mathcal{E}$. It is characterized by two properties, namely the semi-stability of each grading $\frac{\mathcal{E}_{i+1}}{\mathcal{E}_i}$ and the strict increase of slopes of gradings

$$\mu(\mathcal{E}_1) > \mu\left(\frac{\mathcal{E}_2}{\mathcal{E}_1}\right) > \cdots > \mu\left(\frac{\mathcal{E}_m}{\mathcal{E}_{m-1}}\right).$$

Let $\mathcal{E}^\vee$ be the dual vector bundle of $\mathcal{E}$. It is clear that the following filtration

$$\mathcal{E}^\vee = \left(\frac{\mathcal{E}_m}{\mathcal{E}_0}\right)^\vee \supseteq \left(\frac{\mathcal{E}_m}{\mathcal{E}_1}\right)^\vee \supseteq \cdots \supseteq \left(\frac{\mathcal{E}_m}{\mathcal{E}_{m-1}}\right)^\vee \supseteq \left(\frac{\mathcal{E}_m}{\mathcal{E}_m}\right)^\vee = 0$$

is the Harder–Narasimhan filtration of $\mathcal{E}^\vee$ by the characterization. In particular it follows that

$$-\mu_{min}(\mathcal{E}) = -\mu\left(\frac{\mathcal{E}_m}{\mathcal{E}_{m-1}}\right) = \mu\left(\left(\frac{\mathcal{E}_m}{\mathcal{E}_{m-1}}\right)^\vee\right) = \mu_{max}\big(\mathcal{E}^\vee\big).$$

Thus we apply Proposition 3.8 to the vector bundles $\mathcal{E}$ and its dual $\mathcal{E}^\vee$. Hence

$$\begin{aligned}\mu_{max}\big(F_n^*\mathcal{E}\big) - \mu_{min}\big(F_n^*\mathcal{E}\big) &= \mu_{max}\big(F_n^*\mathcal{E}\big) + \mu_{max}\big(F_n^*\mathcal{E}^\vee\big) \\ &\leqslant p^n\mu_{max}(\mathcal{E}) + p^n\mu_{max}\big(\mathcal{E}^\vee\big) + 4(r-1)(b-1)\frac{p^n-1}{p-1} \\ &= p^n\big(\mu_{max}(\mathcal{E}) - \mu_{min}(\mathcal{E})\big) + 4(r-1)(b-1)\frac{p^n-1}{p-1}. \qquad \square\end{aligned}$$

So it is left to show Proposition 3.8.

Proof of Proposition 3.8. We prove it by induction on n.

***n* = 1 case.** It is to show

$$\mu_{max}\big(F_C^*\mathcal{E}\big) \leqslant p\mu_{max}(\mathcal{E}) + (r-1)(2b-2).$$

We let $\mathcal{F} \subset F_C^*\mathcal{E}$ be the subsheaf with maximal slope. As above, we let

$$\mathcal{E} = \mathcal{E}_m \supseteq \mathcal{E}_{m-1} \supseteq \cdots \supseteq \mathcal{E}_1 \supseteq \mathcal{E}_0 = 0$$

be the Harder–Narasimhan filtration of $\mathcal{E}$. It pulls back to a filtration of $F_C^*\mathcal{E}$:

$$F_C^*\mathcal{E} = F_C^*\mathcal{E}_m \supseteq F_C^*\mathcal{E}_{m-1} \supseteq \cdots \supseteq F_C^*\mathcal{E}_1 \supseteq 0.$$

We consider the natural map $\alpha_1 : \mathcal{F} \to \frac{F_C^*\mathcal{E}_m}{F_C^*\mathcal{E}_{m-1}}$ which is the composition of the morphisms

$$\mathcal{F} \hookrightarrow F_C^*\mathcal{E}_m \to \frac{F_C^*\mathcal{E}_m}{F_C^*\mathcal{E}_{m-1}}.$$

The kernel of α_1 is denoted by $\mathcal{F}_1$. So we have a subsheaf $\frac{\mathcal{F}}{\mathcal{F}_1}$ of $\frac{F_C^*\mathcal{E}_m}{F_C^*\mathcal{E}_{m-1}}$. Since the grading $\frac{\mathcal{E}_m}{\mathcal{E}_{m-1}}$ is semi-stable, one has after Theorem 3.6:

$$\mu_{max}\left(F_C^*\left(\frac{\mathcal{E}_m}{\mathcal{E}_{m-1}}\right)\right) - \mu_{min}\left(F_C^*\left(\frac{\mathcal{E}_m}{\mathcal{E}_{m-1}}\right)\right) \leqslant 2(r-1)(b-1).$$

Since furthermore

$$\mu\left(\frac{\mathcal{F}}{\mathcal{F}_1}\right) \leqslant \mu_{max}\left(\frac{F_C^*\mathcal{E}_m}{F_C^*\mathcal{E}_{m-1}}\right) = \mu_{max}\left(F_C^*\left(\frac{\mathcal{E}_m}{\mathcal{E}_{m-1}}\right)\right)$$

and

$$\mu_{min}\left(F_C^*\left(\frac{\mathcal{E}_m}{\mathcal{E}_{m-1}}\right)\right) \leqslant \mu\left(F_C^*\left(\frac{\mathcal{E}_m}{\mathcal{E}_{m-1}}\right)\right) = p\mu\left(\frac{\mathcal{E}_m}{\mathcal{E}_{m-1}}\right)$$

hold, we get

$$\begin{aligned}\mu\left(\frac{\mathcal{F}}{\mathcal{F}_1}\right) &\leqslant 2(r-1)(b-1) + p\mu\left(\frac{\mathcal{E}_m}{\mathcal{E}_{m-1}}\right)\\ &\leqslant 2(r-1)(b-1) + p\mu_{max}(\mathcal{E}).\end{aligned}$$

One notes that $\mathcal{F}_1$ is a coherent subsheaf of $F_C^*\mathcal{E}_{m-1}$. In case that $\mathcal{F}_1$ is not zero sheaf one considers further the morphism $\alpha_2 : \mathcal{F}_1 \to \frac{F_C^*\mathcal{E}_{m-1}}{F_C^*\mathcal{E}_{m-2}}$ with kernel $\mathcal{F}_2$. By keeping on doing this, one obtains a filtration

$$\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_l \supset \mathcal{F}_{l+1} = 0$$

such that each grading $\frac{\mathcal{F}_i}{\mathcal{F}_{i+1}}$ is subsheaf of $F_C^*(\frac{\mathcal{E}_{m-i}}{\mathcal{E}_{m-i-1}})$. One applies then Theorem 3.6 for each grading $\frac{\mathcal{E}_{m-i}}{\mathcal{E}_{m-i-1}}$ and notes that the inequality

$$\mu\left(\frac{\mathcal{E}_i}{\mathcal{E}_{i-1}}\right) \leqslant \mu_{max}(\mathcal{E})$$

holds for all $1 \leqslant i \leqslant m$. By the same argument as above, one has

$$\mu\left(\frac{\mathcal{F}_i}{\mathcal{F}_{i+1}}\right) \leqslant 2(r-1)(b-1) + p\mu_{max}(\mathcal{E}), \quad 0 \leqslant i \leqslant l.$$

Therefore it follows that

$$\begin{aligned}\deg\mathcal{F}&=\sum_{i=0}^{l}\deg\frac{\mathcal{F}_i}{\mathcal{F}_{i+1}}\\&\leqslant\sum_{i=0}^{l}\bigl[2(r-1)(b-1)+p\mu_{max}(\mathcal{E})\bigr]\operatorname{rank}\frac{\mathcal{F}_i}{\mathcal{F}_{i+1}}\\&=\bigl[2(r-1)(b-1)+p\mu_{max}(\mathcal{E})\bigr]\operatorname{rank}(\mathcal{F}).\end{aligned}$$

Induction step. We show the truth for $n-1$ implies that for n. It is direct to show this. One suffices to notice that the same argument in the above step applying to the sheaf $F_{n-1}^{*}\mathcal{E}$ yields the following inequality

$$\mu_{\max}\bigl(F_n^{*}\mathcal{E}\bigr)\leqslant 2(r-1)(b-1)+p\mu_{max}\bigl(F_{n-1}^{*}\mathcal{E}\bigr).$$

By the inductive assumption one has the inequality

$$\mu_{max}\bigl(F_{n-1}^{*}\mathcal{E}\bigr)\leqslant p^{n-1}\mu_{max}(\mathcal{E})+2(r-1)(b-1)\frac{p^{n-1}-1}{p-1}.$$

Combining the last two inequalities one gets the claimed inequality

$$\mu_{max}\bigl(F_n^{*}\mathcal{E}\bigr)\leqslant p^{n}\mu_{max}(\mathcal{E})+2(r-1)(b-1)\frac{p^{n}-1}{p-1}.$$

The proof is completed. □

Proof of (iii). It follows from Proposition 3.8 for $E^{0,1}$ and (ii). □

In characteristic zero the system of Hodge bundles $(E^{1,0}\oplus E^{0,1},\theta)$ is known to be Higgs semi-stable of slope zero. Since the Higgs field on the second Hodge bundle is trivial, for any coherent subsheaf $\mathcal{F}\subset E^{0,1}$ $(\mathcal{F},0)$ is a Higgs subsheaf and by the Higgs semi-stability the inequality $\mu(\mathcal{F})\leqslant\mu(E^{1,0}\oplus E^{0,1})=0$ holds, the same as in the case (i) of the above theorem. The violation of the Higgs semi-stability in characteristic p was shown by the example of Moret-Bailly [10], in which there is a positive degree sub line bundle in $E^{0,1}$. However, thanks to the Vologodsky–Ogus Theorem (see Theorem 5.2) the system of Hodge bundles for a smooth family of Abelian variety which is W_2-liftable, is again Higgs semi-stable of slope zero. It is interesting to ask the stability-type question about the Hodge bundles over strata (for example p-rank zero stratum or Ekedahl–Oort strata in general) in the moduli space of algebraic curves and the moduli space of Abelian varieties over k where the relative Frobenius degenerates.

4. Upper bound of p-rank zero locus

In this section we shall discuss the problem as described in the introduction, namely the upper bound of p-rank zero locus in a semi-stable family of algebraic curves $f: X\to C$ over k. Of course, in order that our question makes sense we must assume that the generic fiber of f is not of p-rank zero. This is the basic assumption in this section. We denote by $V_0(f)$ the proper subset of C which supports the p-rank zero fibers in f. The notation $|V_0(f)|$ means the cardinality of $V_0(f)$.

We treat first the simplest case, a family of elliptic curves.

Proposition 4.1. *Let $f : X \to C$ be a semi-stable family of elliptic curves with nontrivial Kodaira–Spencer map. Then the number of supersingular elliptic curves in the family f is bounded from above by $\frac{p-1}{2}(2b-2+s)$ where b is the base curve genus and s the number of singular fibers of f.*

When the base curve is $\mathbb{P}^1$, there are at least four singular fibers. It follows from the positivity of direct images of relative differentials over a universal family of Abelian variety in the dimension one case. So the maximal number of supersingular elliptic curves is $p-1$, when the base curve is $\mathbb{P}^1$ and the family degenerates at four points. The six examples of A. Beauville of semi-stable elliptic curves over $\mathbb{P}^1$ have integral model and their good reductions at k have exactly $p-1$ supersingular fibers. It is interesting to ask the converse. Namely, one asks if the good reduction of Beauville's six examples at a prime p can be characterized by the minimal number of singular fibers and maximal number of supersingular fibers.

Proof. We consider the morphism $\Phi_1 : F_1^* E^{0,1} \to E^{0,1}$. By Proposition 3.2(iii), the supersingular locus $V_0(f)$ is the support of the effective divisor of $E^{0,1} \otimes F_1^*(E^{0,1})^{-1}$ defined by Φ_1. As shown in the proof of the Arakelov inequality (Theorem 2.1), one has inequality

$$\deg E^{1,0} \leqslant \frac{1}{2} \deg \Omega^1_C(S).$$

So it follows that

$$\begin{aligned}
\left|V_0(f)\right| &\leqslant \deg\left(E^{0,1} \otimes F_1^*\left(E^{0,1}\right)^{-1}\right) \\
&= \deg\left(E^{0,1}\right) - \deg\left(F_C^*\left(E^{0,1}\right)\right) \\
&= (p-1)\deg\left(E^{1,0}\right) \\
&\leqslant \frac{p-1}{2}(2b-2+s). \qquad \square
\end{aligned}$$

In the above proof the effective divisor of $E^{0,1} \otimes F_1^*(E^{0,1})^{-1} = (E^{1,0})^{p-1}$ is classically known as the Hasse locus. When the family is a modular family of elliptic curves over k, it was known that the Hasse locus is reduced. (See the article [13] for the multiplicity problem of the Hasse locus of CY family.) So the first inequality in the above calculation is indeed an equality for Beauville's six examples. On the other hand, for these families the second inequality is also an equality because over $\mathbb{C}$ it is known that the Arakelov inequality reaches equality for them.

In the next step we want to bound the cardinality of the non-ordinary locus, which we denote by $H(f)$, when the family f has an ordinary fiber. One can regard this as the first generalization of elliptic curve case. Since the p-rank zero locus is contained in the non-ordinary locus, we also obtain an upper bound for $|V_0(f)|$.

Theorem 4.2. *Let $f : X \to C$ be a semi-stable family of algebraic curves over k as above. If there is an ordinary closed fiber in f, then we have the following upper bound for the number of non-ordinary fibers $|H(f)|$:*

$$\left|H(f)\right| \leqslant 2(p-1)g^2(2b-2+s).$$

Proof. By taking the wedge g power of Φ_1, we obtain a morphism of invertible sheaves over C:

$$\det(\Phi_1) : \bigwedge^g F_1^*\left(E^{0,1}\right) \to \bigwedge^g E^{0,1}.$$

It is nontrivial because the family f contains an ordinary fiber and the morphism Φ_1 is therefore generically isomorphism by Proposition 3.2(i). It must be then injective and we have a short exact sequence of coherent sheaves over C:

$$0 \to \bigwedge^g\big(F_1^*\big(E^{0,1}\big)\big) \xrightarrow{\det \Phi_1} \bigwedge^g\big(E^{0,1}\big) \to Q \to 0$$

where Q is a torsion sheaf. By Proposition 3.2(i) again, one has $|H(f)| \leqslant \deg Q$. On the other hand, by the Arakelov inequality $\deg(E^{1,0}) < 2g^2(2b-2+s)$ (cf. Theorem 2.1) it follows that

$$\begin{aligned}\big|H(f)\big| &\leqslant \deg(Q)\\ &= \deg\big(E^{0,1}\big) - \deg\big(F_1^*\big(E^{0,1}\big)\big)\\ &= (p-1)\deg\big(E^{1,0}\big)\\ &< 2(p-1)g^2(2b-2+s). \qquad \square\end{aligned}$$

Now we complete our discussions of this problem by considering the most general case. That is, we are going to provide an upper bound for $|V_0(f)|$ when the family f does not necessarily contain an ordinary fiber.

Theorem 4.3. *Let $f : X \to C$ be a semi-stable family of algebraic curves of genus $g \geqslant 2$ over k whose Kodaira–Spencer map is nonzero. The notations is as above. If the generic fiber of f is not of p-rank zero, then the number of p-rank zero closed fibers $|V_0(f)|$ in f is strictly bounded from above by the numerical function*

$$P(p,g,b,s) = 2p^g g^3(2b-2+s).$$

Proof. The proof is in the same line as above. Instead of considering Φ_1 we need to study Φ_g in the current case. Since by assumption the p-rank of the generic fiber of f is not zero, the morphism Φ_g is nontrivial by Proposition 3.2(ii). Thus one obtains the factorization of Φ_g

$$\big(F_g^*\big)E^{0,1} \to \mathcal{F} \xrightarrow{\phi} \mathcal{E} \to E^{0,1}$$

such that ϕ is an isomorphism at the generic point.

By taking the wedge product of ϕ one has the following short exact sequence of coherent sheaves over C:

$$0 \to \det \mathcal{F} \xrightarrow{\det \phi} \det \mathcal{E} \to Q \to 0$$

where Q is a torsion sheaf. Now one can estimate $|V_0(f)|$ as in the last theorem:

$$\begin{aligned}\big|V_0(f)\big| &\leqslant \deg Q\\ &= \deg \mathcal{E} - \deg \mathcal{F}\\ &= \deg \mathcal{E} - \deg\big(F_g^* E^{0,1}\big) + \deg(\ker \Phi)\\ &= p^g \deg\big(E^{1,0}\big) + \deg \mathcal{E} + \deg(\ker \Phi)\\ &< 2p^g g^2 \deg \Omega_C^1(S) + 2g(g-1)\deg \Omega_C^1(S) + 2p^g g(g-1)^2 \deg \Omega_C^1(S)\\ &\quad + 2(g-1)^2(b-1)\frac{p^g-1}{p-1}\end{aligned}$$

$$= \left[\frac{p^{g+1}-1}{p-1}(g-1)^2 + 2p^g g\big(g^2-g+1\big) + 2g(g-1)\right]\deg \Omega_C^1(S) - s(g-1)^2\frac{p^g-1}{p-1}$$
$$\leqslant \left[\frac{p^{g+1}-1}{p-1}(g-1)^2 + 2p^g g\big(g^2-g+1\big) + 2g(g-1)\right](2b-2+s)$$
$$< 2p^g g^3(2b-2+s).$$

In the first strict inequality above we use the Arakelov inequality for $E^{1,0}$ together with Theorem 3.5(ii), (iii). The second strict inequality is elementary which makes the expression simpler. The whole proof is completed. □

5. Arakelov inequality and upper bound of *p*-rank zero locus for smooth families of Abelian varieties in characteristic *p*

In this section we discuss some extensions of previous results for families of algebraic curves to smooth families of Abelian varieties. So we let $f : X \to C$ be a smooth family of Abelian varieties of dimension $g \geqslant 2$ over k. Our basic assumption about f is as follows:

Assumption 5.1. We assume that f is W_2-liftable. Namely, there is a smooth family $\tilde{f} : \tilde{X} \to \tilde{C}$ over $W_2(k)$ such that the reduction of $\tilde{f}$ at k is f. We assume furthermore that the Kodaira–Spencer map of f is nonzero.

We recall first several recent remarkable results due to Ogus and Vologodsky [14].

Theorem 5.2. *(See Ogus and Vologodsky [14, Theorem 4.14(3), Proposition 4.19] for smooth family of Abelian varieties.) Let $f : X \to C$ be a smooth family of Abelian varieties over k which is W_2-liftable. Then the first relative de Rham cohomology $R^1 f_*^{DR}(\mathcal{O}_X)$ of f is a Fontaine module over C. The Higgs bundle*

$$(E,\theta) = \big(E^{1,0} \oplus E^{0,1}, \theta^{1,0} \oplus \theta^{0,1}\big),$$

over C, which is the grading of $R^1 f_*^{DR}(\mathcal{O}_X)$ with respect to the Hodge filtration, is Higgs semi-stable of slope zero when $p \geqslant 4g^2 - 6g + 4$.

Theorem 5.3. *Let $f : X \to C$ be a smooth family of Abelian varieties which satisfies Assumption 5.1. Assume furthermore that $p \geqslant 4g^2 - 6g + 4$. Then for any coherent subsheaf $\mathcal{F}$ of $E^{1,0}$ one has inequality of the slope of $\mathcal{F}$:*

$$\mu(\mathcal{F}) \leqslant b - 1,$$

where b is genus of C. Furthermore, for $\mathcal{E}$ a coherent subsheaf $F_n^* E^{0,1}$, $n \geqslant 0$, one has inequality

$$\mu(\mathcal{E}) \leqslant 2(g-1)(b-1)\frac{p^n-1}{p-1}.$$

Corollary 5.4. *Let $f : X \to C$ be a family as in the above theorem. Then the genus of the base curve C is greater than two.*

Proof. From the proof of Theorem 5.3 (see below), we have the inequality $\mu(E^{1,0}) \leqslant b - 1$ as by the assumption $\theta(E^{1,0}) \neq 0$. On the other hand, it is known that $\det E^{1,0}$ is ample (see Theorem 2.3 in [5]) and hence $\deg(E^{1,0}) > 0$. It follows that $b \geqslant 2$. □

Corollary 5.5. *Let $f : X \to C$ be a smooth family of Abelian varieties of dimension $g \geqslant 2$ over k. Assume that $p \geqslant 4g^2 - 6g + 4$. If f has nontrivial Kodaira–Spencer map and C is $\mathbb{P}^1$ or an elliptic curve, then f is non-W_2-liftable.*

From this corollary one obtains the non-W_2-liftability of the Moret-Bailly's family of supersingular Abelian surfaces over $\mathbb{P}^1$ for $p \geqslant 11$.

Proof of Theorem 5.3. The idea of the proof is similar to that in the characteristic zero situation (cf. [23, Proposition 1.2]). So let $\mathcal{G} \otimes \Omega_C^1$ be the image of $\mathcal{F}$ under $\theta^{1,0}$. Then $\mathcal{F} \oplus \mathcal{G}$ forms a Higgs subsheaf of (E, θ). By the Higgs semi-stability in Theorem 5.2, it follows that $\deg \mathcal{F} + \deg \mathcal{G} \leqslant \deg E = 0$. If $\mathcal{G} = 0$, then one has $\deg \mathcal{F} \leqslant 0$. We assume $\mathcal{G} \neq 0$ in the following. Hence

$$\begin{aligned} \deg \mathcal{F} &\leqslant \deg \mathcal{G} + \operatorname{rank} \mathcal{G} \cdot \deg \Omega_C \\ &\leqslant \deg \mathcal{G} + \operatorname{rank} \mathcal{F} \cdot \deg \Omega_C \\ &\leqslant -\deg \mathcal{F} + \operatorname{rank} \mathcal{F} \cdot \deg \Omega_C. \end{aligned}$$

It follows that $\mu(\mathcal{F}) \leqslant \frac{1}{2} \deg \Omega_C = b - 1$. So one has

$$\mu(\mathcal{F}) \leqslant \max\{0, b-1\}.$$

Because it is shown above that $b \geqslant 2$, one has $\mu(\mathcal{F}) \leqslant b - 1$.

Now let $\mathcal{E}$ be a coherent subsheaf of $E^{0,1}$. One notes that the second component of the Higgs field $\theta^{0,1}$ is simply zero map. Then $(\mathcal{E}, 0)$ forms a Higgs subsheaf of (E, θ) and by the Higgs semi-stability Theorem 5.2 $\mu(\mathcal{E}) \leqslant 0$. So the second inequality is proved for $n = 0$ case. For general n case one applies the result for $n = 0$ case and the inequality in Proposition 3.8. □

The following statement is analogous to that in Theorem 4.3.

Theorem 5.6. *Let $f : X \to C$ be a smooth family of Abelian varieties of dimension $g \geqslant 2$ over k with Assumption 5.1. Assume furthermore that $p \geqslant 4g^2 - 6g + 4$. If the generic fiber of f is not of p-rank zero, then the number of p-rank zero closed fibers $|V_0(f)|$ in f is bounded from above by the numerical function*

$$Q(p, g, b) = \left[p^g g + 2(g-1)^2 \frac{p^g - 1}{p - 1} \right] (b - 1).$$

Proof. The proof is the same as that in Theorem 4.3, except that we replace the estimates of degrees by those proved in this section. The result can be easily checked. □

Let $f : X \to C$ be a family of algebraic curves of genus $\geqslant 2$ over k with nontrivial Kodaira–Spencer map with the W_2-liftability assumption as that in Assumption 5.1. Then the above theorem provides a better upper bound $Q(p, g, b)$ than $P(p, g, b, 0)$.

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