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INEQUALITIES BETWEEN THE CHERN NUMBERS OF A SINGULAR FIBER IN A FAMILY OF ALGEBRAIC CURVES

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ABSTRACT. In a family of curves, the Chern numbers of a singular fiber are the local contributions to the Chern numbers of the total space. We will give some inequalities between the Chern numbers of a singular fiber as well as their lower and upper bounds. We introduce the dual fiber of a singular fiber, and prove a duality theorem. As an application, we will classify singular fibers with large or small Chern numbers.

1. INTRODUCTION AND MAIN RESULTS

Chern numbers of a singular fiber in a family of curves are the local contributions of the fiber to the global Chern numbers of the total space. The first purpose of this paper is to find the best inequalities between the Chern numbers of a singular fiber. Our second purpose is to try to give a new approach to the classification of singular fibers of genus g. We know that when g is large, there are too many singular fibers of genus g to classify completely (see [6], [8], [9], [17]). Some authors are trying to find a new method to classify singular fibers of small genus (see, for example, [2]). In order to get the local-global relations between the invariants, one possible way is to classify singular fibers according to their contributions to the global invariants. To explain this approach, we will classify singular fibers with large or small Chern numbers and give some applications. See the survey [3] for the background of the study on the local-global properties for families of curves.

A family of curves of genus g over C is a fibration $f : X \to C$ whose general fibers F are smooth curves of genus g, where X is a complex smooth projective surface. The family is called *semistable* if all of the singular fibers are reduced nodal curves. If $X = F \times C$ and f is just the second projection to C, then we call f a *trivial* family. If all of the smooth fibers of f are isomorphic to each other, equivalently, f becomes trivial under a finite base change $\tilde{C} \to C$, then f is called *isotrivial.* We always assume that f is relatively minimal, i.e., there is no (-1)-curve in any singular fiber.

When $g = 1$, Kodaira [7] found the global invariants from the singular fibers. The first Chern number $c_1^2(X)$ is always zero, the second Chern number $c_2(X)$ is

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equal to $12\chi(\mathcal{O}_X)$ by Noether's formula, and

$$c_2(X) = j + 6\nu(\mathrm{I}^*) + 2\nu(\mathrm{II}) + 10\nu(\mathrm{II}^*) + 3\nu(\mathrm{III}) + 9\nu(\mathrm{III}^*) + 4\nu(\mathrm{IV}) + 8\nu(\mathrm{IV}^*), \tag{1.1}$$

where $\nu(\mathrm{T})$ denotes the number of singular fibers of type T, and j is the number of poles of the J-function of the family. Note that the J-function over C induces a holomorphic map of degree j from C to the moduli space $\overline{\mathcal{M}}_1$ of elliptic curves. So j depends only on the generic fibers.

By introducing the Chern numbers $c_1^2(F)$, $c_2(F)$ and χ_F for a singular fiber F, the second author ([13], [14], [15]) generalized Kodaira's formula (1.1) to the higher genus case,

$$\begin{cases} c_1^2(X) = \kappa(f) + 8(g-1)(g(C)-1) + \sum_{i=1}^s c_1^2(F_i), \\ c_2(X) = \delta(f) + 4(g-1)(g(C)-1) + \sum_{i=1}^s c_2(F_i), \\ \chi(\mathcal{O}_X) = \lambda(f) + (g-1)(g(C)-1) + \sum_{i=1}^s \chi_{F_i}, \end{cases} \tag{1.2}$$

where $F_1, \cdots, F_s$ are all singular fibers of f, and $\kappa(f)$, $\delta(f)$ and $\lambda(f)$ are the *modular invariants* of the family. f also induces a holomorphic map from C to the moduli space of semistable curves of genus g:

$$J : C \longrightarrow \overline{\mathcal{M}}_g .$$

Then $\kappa(f) = \deg J^*\kappa$, $\delta(f) = \deg J^*\delta$ and $\lambda(f) = \deg J^*\lambda$, where λ, δ and κ are respectively the Hodge divisor class, the boundary divisor class and $\kappa = 12\lambda - \delta$. In the case of elliptic fibrations, $\kappa(f) = 0$ and $\delta(f) = j$.

Let $\widetilde{f} : \widetilde{X} \to \widetilde{C}$ be a semistable reduction of F under any base change $\pi : \widetilde{C} \to C$ ramified over $p = f(F)$ and some non-critical points of f. The Chern numbers of F are defined as follows:

$$c_1^2(F) = K_f^2 - \frac{1}{d}K_{\widetilde{f}}^2, \quad c_2(F) = e_f - \frac{1}{d}e_{\widetilde{f}}, \quad \chi_F = \chi_f - \frac{1}{d}\chi_{\widetilde{f}}, \tag{1.3}$$

where d is the degree of π, and $K_f^2 = c_1^2(X) - 8(g-1)(g(C)-1)$, $e_f = c_2(X) -4(g-1)(g(C)-1)$ and $\chi_f = \chi(\mathcal{O}_X) - (g-1)(g(C)-1)$ are the relative invariants of f. These Chern numbers are independent of the choice of the semistable reductions π. If $g = 1$, then $c_1^2(F) = 0$ and $c_2(F)$ is exactly the coefficient in (1.1) according to the type of the fiber F. See §3.3 for the computation formulas for the Chern numbers of F. We briefly summarize the known properties of the Chern numbers. Assume that $g = g(F) \geq 2$ and F contains no (-1)-curves.

(1) *Positivity*: $c_1^2(F)$, $c_2(F)$ and χ_F are non-negative rational numbers, and one of the three numbers vanishes if and only if F is semistable.
(2) *Noether's equality*: $c_1^2(F) + c_2(F) = 12\chi_F$.
(3) *Blow-up formulas*: $c_1^2(F') = c_1^2(F) - 1, c_2(F') = c_2(F) + 1, \chi_{F'} = \chi_F$, where $F' = \sigma^*F$ is the pullback of F under the blowing up $\sigma : X' \to X$ at a point p on F.
(4) *Canonical class inequality*: $c_1^2(F) \leq 4g - 4$.
(5) *Miyaoka-Yau type inequality*: $c_1^2(F) \leq 2c_2(F)$ or, equivalently, $c_1^2(F) \leq 8\chi_F$, with equality iff F_{red} is a nodal curve and $F = nF_{\mathrm{red}}$ for some positive integer n.

The positivity is essentially due to Beauville [4], Xiao [19] and the second author [11]. Noether's equality and the blow-up formulas are direct consequences of the definition of Chern numbers. The last two inequalities can be found in [13].

Let $\bar{F}$ be the normal crossing fiber obtained by blowing up the singularities of F ($\bar{F}$ is called the normal crossing model of F). Write $\bar{F} = n_1C_1 + \cdots + n_kC_k$, where the C_i's are the irreducible components. Denote by M_F the least common multiplicity of $n_1, \cdots, n_k$. Let n be a positive integer satisfying $n \equiv -1 \pmod{M_F}$. Denote by F^* the fiber obtained from F by a local base change π defined by $w = z^n$. We call F^* the *dual fiber* of F (see §2). This is a natural generalization of Kodaira's dual fibers for elliptic fibrations. Our first result is the duality theorem for χ_F.

Theorem 1.1 (Duality theorem for χ)**.** *Let $\bar{F}$ and $\bar{F}^*$ be the normal crossing models of F and F^* respectively. Let $N_{\bar{F}} = g - p_a(\bar{F}_{\rm red})$. Then $0 \le N_{\bar{F}} \le g$.*

1) *$N_{\bar{F}} = N_{\bar{F}^*}$, i.e., $p_a(\bar{F}_{\rm red}) = p_a(\bar{F}^*_{\rm red})$.*
2) *$\chi_F + \chi_{F^*} = N_{\bar{F}}$.*
3) *$\frac{1}{6}N_{\bar{F}} \le \chi_F \le \frac{5}{6}N_{\bar{F}}$. $\chi_F = \frac{1}{6}N_{\bar{F}}$ (resp. $\frac{5}{6}N_{\bar{F}}$) if and only if F (resp. F^*) is a reduced curve whose singularities are at worst ordinary cusps or nodes.*

In general, F^{**} is not necessarily equal to F, but we have the equality $\chi_{F^{**}} = \chi_F$.

Theorem 1.2. *Assume that $g \ge 2$. We have the following optimal inequalities.*

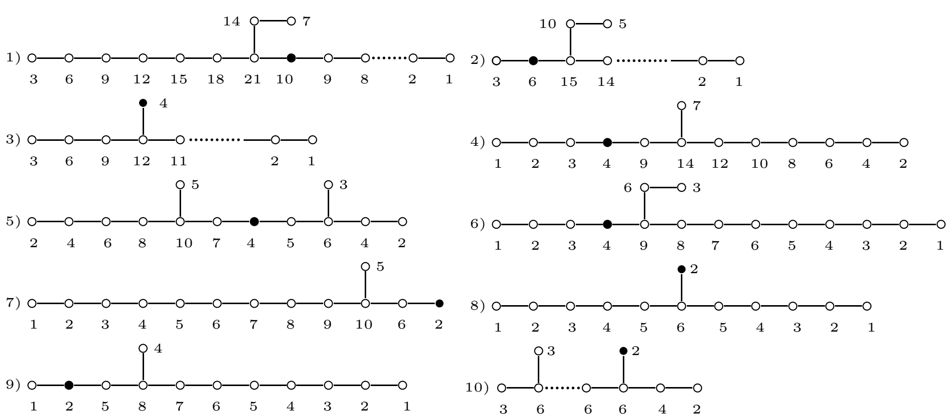
1) *If F is not semistable, then $c_2(F) \ge \frac{11}{6}$ and $\chi_F \ge \frac{1}{6}$. One of the equalities holds if and only if F is a reduced curve with one ordinary cusp and some nodes.*
2) *$c_1^2(F) \le 4g - \frac{24}{5}$. More precisely, if $g \ge 7$ or $g = 5$, then $c_1^2(F) \le 4g - \frac{11}{2}$.*

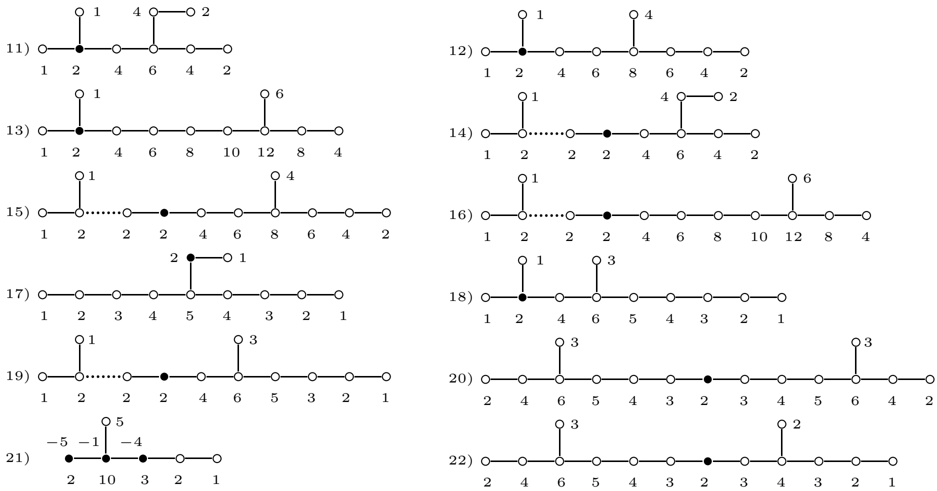
$$c_1^2(F) \le \begin{cases} \frac{16}{5}, & g = 2, \\ 7, & g = 3, \\ \frac{54}{5}, & g = 4, \\ \frac{130}{7}, & g = 6. \end{cases}$$

3) (Arakelov type inequality) *$\chi_F \le \frac{5g}{6}$, with equality iff F^* is a reduced curve with nodes and ordinary cusps as its singularities, and its normal crossing model is a tree of smooth rational curves.*

For any $g \ge 2$, there is a numerical fiber F with $c_1^2(F) = 4g - \frac{11}{2}$ (see Example 5.1).

Theorem 1.3. *Let F be a minimal singular fiber of genus $g \ge 2$ satisfying $c_1^2(F) > 4g - \frac{11}{2}$. Then $g \le 6$ and F is one of the following 22 fibers. $\circ$ is a (-2)-curve, and $\bullet$ is a (-3)-curve.*





See §5.5 for the Chern numbers of these 22 fibers.

Theorem 1.4. *Assume that $g \geq 2$. If $2c_2(F) - c_1^2(F) < 6$, then either $F = nC$ for some smooth curve C or F_{red} admits at most one singular point p other than nodes. One of the following cases occurs.*

I) $F = nF_{\text{red}}$.

1) *F_{red} is a smooth or nodal curve.*
2) *p is of type A_2.*
3) *p is of type A_3 and any (-2)-curve does not pass through p.*
4) *p is of type A_3 and one (-2)-curve passes through p.*
5) *p is of type D_4.*

II) *$F = nA + 2nB$, where A and B are reduced nodal curves without common components, $AB = 2$, $A^2 = -4$ and $B^2 = -1$. A has at most two connected components A_1 and A_2.*

6) *$A \cap B = \{p, q\}$ and any (-2)-curve is not a connected component of A.*
7) *A has two connected components and one is a (-2)-curve.*
8) *A and B are tangent at a point p.*

The invariants of these fibers F are as follows, where $0 \leq N = g(F) - p_a(F_{\text{red}}) \leq g$.

F	1	2	3	4	5	6	7	8
$2c_2 - c_1^2$	0	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{21}{4}$	5	3	$\frac{9}{2}$	$\frac{11}{2}$
$c_1^2 - 4N$	0	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{4}$	1	-1	$-\frac{3}{2}$	$-\frac{1}{2}$
$c_2 - 2N$	0	$\frac{11}{6}$	$\frac{5}{2}$	$\frac{11}{4}$	3	1	$\frac{3}{2}$	$\frac{5}{2}$
$\chi - \frac{1}{2}N$	0	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{3}$	0	0	$\frac{1}{6}$

Note that $2c_2 - c_1^2 < 6$ is equivalent to $8\chi - c_1^2 < 2$. Hence the fibers satisfying $c_2(F) \leq 3$ or $\chi \leq \frac{1}{4}$ are included in the classification list 1) $\sim$ 8). For a non-semistable fiber, c_1^2, c_2 and χ are positive. Therefore, one can check that $\frac{11}{6}$ (resp.

$\frac{1}{6}$) is the lower bound of c_2 (resp. χ) for non-semistable fibers. All of the fibers from 2) to 8) cannot be the fibers in an isotrivial family of curves because their semistable models are not smooth.

Corollary 1.5. *Let s be the number of singular fibers of $f : X \to C$ and $g \geq 2$.*
1) *If f is non-trivial, then $\chi_f \leq \frac{g}{2}\left(2g(C) - 2 + \frac{8}{3}s\right)$.*
2) *If f is isotrivial, then $K_f^2 \leq \left(4g - \frac{24}{5}\right)s$, and $\chi_f \leq \frac{5gs}{6}$.*

As an application of Theorem 1.4, we have

Corollary 1.6. *Assume $f : X \to C$ is isotrivial. Let s be the number of singular fibers that are not multiples of a smooth curve. Then $K_X^2 \leq 8\chi(\mathcal{O}_X) - 2s$.*

This gives a new proof of Polizzi's theorem that $K_X^2 \neq 8\chi(\mathcal{O}_X) - 1$ when $f : X \to C$ is isotrivial [10]. We will give some other applications of the main results in each section.

2. Dual models F^* of a fiber F

We recall several models of a singular fiber in this section, including the minimal model, normal crossing model, n-th root model, semistable model, and the dual model.

2.1. Normal crossing model.

A curve B on X is a non-zero effective divisor.

Definition 2.1. A *partial resolution* of the singularities of B is a sequence of blowing-ups $\sigma = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_r : \bar{X} \to X$,

$$(\bar{X}, \sigma^* B) = (X_r, B_r) \xrightarrow{\sigma_r} X_{r-1} \xrightarrow{\sigma_{r-1}} \cdots \xrightarrow{\sigma_2} (X_1, B_1) \xrightarrow{\sigma_1} (X_0, B_0) = (X, B),$$

satisfying the following conditions:
(i) $B_{r,\mathrm{red}}$ has at worst ordinary double points as its singularities.
(ii) $B_i = \sigma_i^* B_{i-1}$ is the total transform of B_{i-1}.

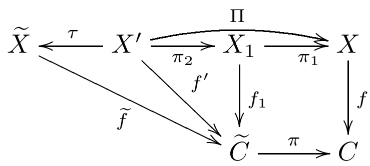
Furthermore, σ is called the *minimal partial resolution* of the singularities of B if
(iii) σ_i is the blowing-up of X_{i-1} at a singular point $(B_{i-1,\mathrm{red}}, p_{i-1})$ which is not an ordinary double point for any $i \leq r$.

The *minimal model* of F is obtained by contracting all (-1)-curves in F. Denote by $\bar{F}$ the partial resolution of the singularities of the minimal model of F.

Definition 2.2. $\bar{F}$ is called the *normal crossing model* of F. If σ is minimal, then we say that $\bar{F}$ is the *minimal normal crossing model* of F.

A (-1)-curve in $\bar{F}$ is called *redundant* if it meets the other components in at most two points. It is obvious that a redundant (-1)-curve can be contracted without introducing singularities worse than ordinary double points. The minimal normal crossing model of F contains no redundant (-1)-curves, and it can be obtained from any normal crossing model by contracting all redundant (-1)-curves. In fact, the minimal normal crossing model of F is determined uniquely by F.

2.2. **The n-th root model and the semistable model of F.** Let $\pi : \widetilde{C} \to C$ be a base change of degree n. Then we can construct the pullback fibration $\widetilde{f} : \widetilde{X} \to \widetilde{C}$ of $f : X \to C$ as follows:



where $X_1 = X \times_C \widetilde{C}$, π_1 and f_1 are the projections. X' is the minimal resolution of the singularities of the normalization of X_1 and τ is the contraction of those (-1)-curves in the fibers. Then we get the pullback fibration $\widetilde{f}$ of f under the base change π.

Now we consider the above construction locally. Let F be a fiber of f over $p \in C$. Assume that π is totally ramified over p, i.e., $\pi^{-1}(p)$ contains only one point $\widetilde{p}$. In this case, π is defined locally by $z = w^n$ near $p = 0$.

Now denote by $\widetilde{F}$ (resp. F') the fiber of $\widetilde{f}$ (resp. f') over $\widetilde{p} \in \widetilde{C}$. In fact, $F' = \frac{1}{n}\Pi^*(F)$.

Definition 2.3. The fiber $\widetilde{F}$ of $\widetilde{f}$ over $\widetilde{p}$ is called the *n-th root model* of F.

Note that F and any of its normal crossing model $\bar{F}$ have the same n-th root model $\widetilde{F}$ for any n. In fact, if F is normal crossing, then F' is also normal crossing. In particular, $\bar{F}'$ is the normal crossing model of $\widetilde{F}$.

Indeed, we can assume that $F = \bar{F} = \sum_{i=1}^{k} n_i C_i$ is normal crossing, where C_i is irreducible. Let p be a singular point of F_{red}. Without loss of generality, we assume that p is an intersection point of C_i with C_j. Near p, π_1 is defined locally by $z^n = x^{n_i} y^{n_j}$. Then we see that the singularities of the normalization of X_1 are of Hirzebruch-Jung type. Hence, F' is normal crossing. By the computation of the normalization, we see that the multiplicity of the strict transform of C_i in F' is $n_i / \gcd(n, n_i)$.

If n_i divides n for any i, then one can prove that F' and $\widetilde{F}$ are semistable. This is the famous Semistable Reduction Theorem. Denote by $M_F = \operatorname{lcm}\{n_1, \cdots, n_k\}$. Then the n-th root model of F is always semistable if n satisfies $n \equiv 0 \pmod{M_F}$.

Definition 2.4. If $\widetilde{F}$ is semistable, then $\widetilde{F}$ is called the *semistable model* of F or the *semistable reduction* of F.

2.3. **Dual model F^* of F.**

Definition 2.5. If $n \equiv -1 \pmod{M_F}$, then the n-th root model of F is called the *dual model* of F, denoted by F^*.

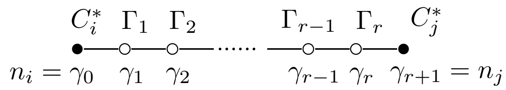
The dual model is first introduced by Kodaira for elliptic fibrations. Our definition is a natural generalization. In general, $(F^*)^*$ doesn't coincide with F unless the semistable model of F is smooth. (If the uniqueness of the dual model is needed, one may choose n to be the minimal positive integer satisfying $n \equiv -1 \pmod{M_F}$.)

Let $\bar{F} = \sum_{i=1}^{k} n_i C_i$ be the minimal normal crossing model of F, where the C_i's are all irreducible components. We have seen that $\bar{F}'$ is the normal crossing model of F^*.

Let $n \equiv -1 \pmod{M_F}$. Denote by C_i^* the strict transform of C_i in $\bar{F}'$. Because n_i is prime to n for any i, C_i^* is irreducible. The multiplicity of C_i^* in $\bar{F}'$ is still n_i. By the resolution of Hirzebruch-Jung singularities, we see that $\bar{F}'$ is obtained by inserting a chain of rational curves,

$$\bar{F}' = \sum_{i=1}^{k} n_i C_i^* + \sum_p \Gamma_p^*,$$

where p runs over all double points of $\bar{F}$, $\Gamma_p^* = \sum_{i=1}^{r} \gamma_i \Gamma_i$. Assume that p is an intersection point of two local components C_i and C_j. Then near Γ_p^*, $\bar{F}'$ is as follows, where $\gamma_0 = n_i$ and $\gamma_{r+1} = n_j$:



Lemma 2.6. 1) *For* $i = 1, \cdots, r$, *we have* $\gamma_i \,|\, \gamma_{i-1} + \gamma_{i+1}$.
2) $\gamma_0 \,|\, \gamma_1 + \gamma_{r+1}$ *and* $\gamma_{r+1} \,|\, \gamma_r + \gamma_0$.

Proof. The local base change over p is defined by $z^n = x^{n_i} y^{n_j}$. Note that n is prime to n_i and n_j, the equation is equivalent to $z^n = xy^{n-q}$ for some q satisfying $n_j + qn_i \equiv 0 \pmod{n}$ and $1 \le q < n$ (see [5], Ch. III, §5). By definition, n_i divides $n+1$. One can see that $q_0 = -(n+1)n_j/n_i = -(n+1)\gamma_{r+1}/\gamma_0$ is an integer satisfying $q \equiv q_0 \pmod{n}$. The singular point over p is of Hirzebruch-Jung type.

For convenience, we take $\Gamma_0 = C_i^*$, $\gamma_0 = n_i$, $\Gamma_{r+1} = C_j^*$ and $\gamma_{r+1} = n_j$. Let $e_i = -\Gamma_i^2$. By Zariski's lemma ([5], Ch. III, §8), $\bar{F}' \cdot \Gamma_i = 0$ for $i = 1, \cdots, r$. Thus we have

$$\begin{cases} -\gamma_0 + \gamma_1 e_1 - \gamma_2 = 0, \\ -\gamma_1 + \gamma_2 e_2 - \gamma_3 = 0, \\ \qquad \vdots \\ -\gamma_{r-1} + \gamma_r e_r - \gamma_{r+1} = 0. \end{cases} \tag{2.1}$$

So we have proved 1). For fixed γ_0 and γ_{r+1}, this is a linear system of the r variables $\gamma_1, \cdots, \gamma_r$. We denote by $A = [e_1, \cdots, e_r]$ the coefficient matrix. It is well known that the determinant of A is equal to n, and the determinant of the submatrix $[e_2, \cdots, e_r]$ is equal to q. By the Gramer Rule,

$$\gamma_1 = \frac{\gamma_0 q + \gamma_{r+1}}{n} = \frac{\gamma_0 q_0 + \gamma_{r+1}}{n} + \gamma_0 \frac{q - q_0}{n} = -\gamma_{r+1} + \gamma_0 \frac{q - q_0}{n},$$

so $\gamma_0 \,|\, \gamma_1 + \gamma_{r+1}$. Symmetrically, $\gamma_{r+1} \,|\, \gamma_r + \gamma_0$. □

Lemma 2.7. *The reduced normal crossing models of F and F^* have the same arithmetic genus, i.e.,* $p_a(\bar{F}_{\mathrm{red}}) = p_a(\bar{F}'_{\mathrm{red}})$.

Proof. This follows from the fact that the arithmetic genus of $\bar{F}$ is equal to the sum of the geometric genus of each component plus the number of cycles in the dual graph of $\bar{F}$. Note that the geometric genera of C_i and C_i^* are the same. So the arithmetic genus is not changed by inserting a Hirzebruch-Jung chain of rational curves. □

3. Local invariants of a fiber

In order to obtain the computation formulas for the Chern numbers of a singular fiber, we need to introduce several local invariants for a singular point of a curve, not necessarily reduced. See [13].

3.1. **Invariants α and β for a curve singularity.** In Definition 2.1, we denote by m_{i+1} the multiplicity of $(B_{i,\mathrm{red}}, p_i)$ at p_i. (Note that $B_{i,\mathrm{red}}$ is the reduced *total* transform of B_{red}, instead of the *strict* transform.) One can check that if B is a compact curve, then

$$p_a(B_{r,\mathrm{red}}) = p_a(B_{\mathrm{red}}) - \frac{1}{2}\sum_{i=1}^{r}(m_i-1)(m_i-2). \tag{3.1}$$

Suppose B has only one singular point, $p = p_0$. Let $k_p = k_p(B)$ (resp. $\mu_p = \mu_p(B)$) be the number of local branches (resp. Milnor number) of (B_{red}, p). Then

$$\mu_p = \sum_{i=1}^{r}(m_i-1)(m_i-2) + k_p - 1. \tag{3.2}$$

1) $m_i = 2$ for all i if and only if (B_{red}, p) is a node.
2) $m_i \le 3$ for all i if and only if (B_{red}, p) is an ADE singular point ([5], Ch.II, §8).

If $q \in B_{r,\mathrm{red}}$ is a double point, and the two local components of (B_r, q) have multiplicities a_q and b_q, then we define $[a_q, b_q] := \frac{\gcd(a_q,b_q)^2}{a_q b_q}$, and

$$\alpha_p = \sum_{i=1}^{r}(m_i-2)^2, \qquad \beta_p = \sum_{q\in B_r}[a_q, b_q], \tag{3.3}$$

where q runs over all of the double points of $B_{r,\mathrm{red}}$. These two invariants are independent of the resolution.

In [13], we prove that $\mu_p \ge \alpha_p + \beta_p$. Actually, we need more precise inequalities of this kind.

Example 3.1. The invariants of an ADE singularity (B_{red}, p) are as follows:

	A_{2k-1}	A_{2k}	D_{2k+2}	D_{2k+3}	E_6	E_7	E_8
μ_p	$2k-1$	$2k$	$2k+2$	$2k+3$	6	7	8
α_p	$k-1$	k	k	$k+1$	3	3	4
β_p	I_k	$\frac{3k}{2k+1}$	II_k	III_k	1	IV	$\frac{4}{5}$
β_p^-	$\ge 1-\frac{1}{k}$	$\ge \frac{6k-1}{4k+2}$		$\ge \frac{1}{2}$	$\ge \frac{11}{12}$	$\ge \frac{1}{3}$	$\ge \frac{11}{15}$

$$\begin{cases} \mathrm{I}_k = 1 - \frac{1}{k} + [k(n+m),\ n] + [k(n+m),\ m], \\ \mathrm{II}_k = \frac{k(n,m+l)^2}{n(n+k(m+l))} + [n+k(m+l),\ m] + [n+k(m+l),\ l], \\ \mathrm{III}_k = \frac{1}{2} + [m,\ 2((2k+1)m+n)] + \frac{(2k+1)(n,2m)^2}{2n((2k+1)m+n)}, \\ \mathrm{IV} = \frac{1}{3} + \frac{2(3m,n)^2}{3n(2m+n)} + \frac{(m,3n)^2}{3m(2m+n)}. \end{cases}$$

Where n (resp. m or l) is the multiplicity of a local branch of (F, p). n corresponds to a smooth branch. We have

$$\mathrm{I}_k \leq 1, \quad \mathrm{II}_k \leq 1, \quad \mathrm{III}_k \leq \frac{3(k+1)}{2k+3}, \quad \mathrm{IV} \leq \frac{4}{5}.$$

Lemma 3.2. 1) *$\mu_p \geq \alpha_p + \beta_p$, with equality iff the singularity is of types A_1 or A_2.*

2) *$\mu_p \geq \alpha_p + \beta_p + 1$ except for the singularities of types A_k for $k \leq 4$.*
3) *$\mu_p \geq \alpha_p + \beta_p + 2$ except for the singularities of types A_k $(k \leq 6)$ and D_5.*
4) *If $2(\mu_p - \alpha_p - \beta_p) + \alpha_p + 3\beta_p^- < 6$, then p is of types A_1, A_2, A_3 and D_4.*
If $2(\mu_p - \alpha_p - \beta_p) + \alpha_p + 3\beta_p^- < 5$, then p is of types A_1, A_2 and A_3.
If $2(\mu_p - \alpha_p - \beta_p) + \alpha_p + 3\beta_p^- < \frac{7}{2}$, then p is a node.

Proof. For an ADE singular point p, the inequalities can be checked directly from the computation above.

If p is not an ADE singular point, then at least one $m_i \geq 4$, so $\alpha_p \geq 4$. We claim that $\mu_p \geq \alpha_p + \beta_p + 2$. In Definition 2.1, we assume that $\sigma = \sigma_1 \circ \sigma_2$, where $\sigma_1 : X' \to X$ consists of blowing-ups at the non-ADE singular points $p_0, \cdots, p_{r'-1}$ such that $B' = \sigma_1^* B$ admits at worst ADE singular points. Then we have

$$\mu_p - \alpha_p - \beta_p = \sum_{i=1}^{r'} (m_i - 3) + \sum_{p' \in B'} (\mu_{p'} - \alpha_{p'} - \beta_{p'}). \tag{3.4}$$

Because p is not an ADE singular point, at least one of m_i $(i \leq r')$ is larger than 3. If two of these m_i's are larger than 3, then $\mu_p \geq \alpha_p + \beta_p + 2$. Without loss of generality, we assume that $m_1 = 4$ and $r' = 1$. Namely $m_1 = 4$ and $m_i \leq 3$ for all $i \geq 2$. We can also assume that $\mu_{p'} < \alpha_{p'} + \beta_{p'} + 1$ for any singular point p' of B'_{red}.

Now we consider the ADE singular points of B'. Because the exceptional curve is one of the branches of the singular points p' of B'_{red}, each singular point p' has at least two branches. According to 1), the singular points p' of B'_{red} are of types A_1 or A_3. Note that if p' is of type A_3, then $\mu_{p'} - \alpha_{p'} - \beta_{p'} = \frac{1}{2}$. Thus if B' admits at least two A_3, then $\mu_p \geq \alpha_p + \beta_p + 2$ holds true.

If B' admits only one A_3, then we can assume that (B, p) is defined by $(x - y)^a(x+y)^b(x^2 - y^3)^c = 0$. Now it is easy to check that $\mu_p = 10$, $\alpha_p = 5$ and $\beta_p \leq 2$. So $\mu_p \geq \alpha_p + \beta_p + 2$.

If B' admits no A_3, then B' admits 4 A_1. Hence we can assume that (B, p) is defined by $x^a y^b (x-y)^c (x+y)^d = 0$. We have $\mu_p = 9$, $\alpha_p = 4$ and $\beta_p \leq 1$. Thus $\mu_p \geq \alpha_p + \beta_p + 2$. □

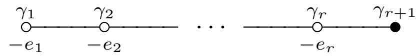
Lemma 3.3. *A curve singularity p satisfying $\sum_{i=1}^r m_i(m_i - 2) \leq 5$ must be of types A_1, A_2, A_3 and D_4.*

Proof. The condition implies that $m_i \leq 3$ for any i and there exists at most one i such that $m_i = 3$, so p is an ADE singular point. Now one can check the result directly. □

We define β_F as the sum of β_p. One can easily check that β_F is independent of the resolution; thus F, $\sigma^* F$ and $\bar{F}$ have the same β-invariant.

3.2. **Invariants β^- and β^+.**

Definition 3.4. Let $\bar{F}$ be the minimal normal crossing model of F, and let $G(\bar{F})$ be the dual graph of $\bar{F}$. An H-J branch of rational curves in $G(\bar{F})$ is



where $\overset{\gamma_i}{\underset{-e_i}{\circ}}$ denotes a smooth rational curve Γ_i with $\Gamma_i^2 = -e_i$ whose multiplicity in $\bar{F}$ is γ_i. $\bullet$ denotes either a curve $\Gamma \not\cong \mathbb{P}^1$ or a smooth rational curve meeting at 3 or more points with the other components. We call Γ_1 an *end point* of $G(\bar{F})$.

Note that the r rational curves can be contracted to a Hirzebruch-Jung singularity of type (n, q) with defining equation $z^n = xy^{n-q}$ ([5], Ch. III, §5), where n and q are respectively the determinants of the matrices $[e_1, \cdots, e_r]$ and $[e_2, \cdots, e_r]$. n and q can also be determined by the multiplicities γ_i as follows.

According to (2.1) and $\gamma_0 = 0$, we see that γ_1 divides γ_i for any i. Using the notation of ([5], Ch. III, §5), $\gamma_1 < \gamma_2 < \cdots < \gamma_r$, $\gamma_i = \mu_i \gamma_1$ for any i, so $1 = \mu_1 < \mu_2 < \cdots < \mu_{r+1}$.

$$n = \mu_{r+1} = \frac{\gamma_{r+1}}{\gamma_1}, \quad q' = \mu_r = \frac{\gamma_r}{\gamma_1},$$

and q is the unique solution of the equation

$$qq' \equiv 1 \pmod{n}, \quad 1 \leq q < n.$$

Since μ_i and μ_{i+1} are coprime, the contribution of the branch to $\beta_F = \beta_{\bar{F}}$ is

$$\beta' = \frac{1}{\mu_1\mu_2} + \frac{1}{\mu_2\mu_3} + \cdots + \frac{1}{\mu_r\mu_{r+1}}. \tag{3.5}$$

There is a relation ([5], Ch. III, §5, eq. (6))

$$\lambda_k \mu_{k+1} - \lambda_{k+1}\mu_k = n, \tag{3.6}$$

i.e.,

$$\frac{\lambda_k}{\mu_k} - \frac{\lambda_{k+1}}{\mu_{k+1}} = n\frac{1}{\mu_k\mu_{k+1}}. \tag{3.7}$$

Note that $\lambda_1 = q$ and $\lambda_{r+1} = 0$. Take the sum of (3.7) from $k = 1$ to r; we have

$$\beta' = \frac{1}{n}\left(\frac{\lambda_1}{\mu_1} - \frac{\lambda_{r+1}}{\mu_{r+1}}\right) = \frac{q}{n}. \tag{3.8}$$

Lemma 3.5. *The contribution of the H-J branch to β_F is $\frac{q}{n}$.*

Definition 3.6. $\beta_F^- = \sum \beta'$ is the total contribution of all H-J branches in $G(\bar{F})$.

Note that $\gamma_2 = e_1\gamma_1$; the contribution of an H-J branch to β_F^- is at least $[\gamma_1, \gamma_2] = \frac{1}{e_1}$.

Example 3.7. If $e_1 = \cdots = e_{r-1} = 2$ and $e_r \geq 2$, then $n = r(e_r - 1) + 1$, $q = n - (e_r - 1) = (r-1)(e_r - 1) + 1$, and the contribution of this H-J branch to β_F^- is

$$\beta' = \frac{(r-1)(e_r-1)+1}{r(e_r-1)+1} = 1 - \frac{e_r - 1}{r(e_r-1)+1}. \tag{3.9}$$

Theorem 3.8 (Gang Xiao [18]). *Assume that $n \equiv 0 \pmod{M_F}$. Let $\bar{F}$ be the minimal normal crossing model of F. Consider the construction of the n-th root model of $\bar{F}$ as in §2.2. Then a curve in X' is contracted by τ if and only if it comes from an H-J branch in $\bar{F}$.*

The theorem above is contained in the proof of Prop. 1 of [18].

From the previous theorem, β_F^- is just $c_{-1}(F)$ defined in [13] by the remark of ([13], p. 666), i.e., $n\beta_F^-$ is the number of (-1)-curves contracted by τ. Let $\beta_F^+ = \beta_F - \beta_F^-$. Then

$$\beta_F = \beta_F^+ + \beta_F^-.$$

3.3. **Formulas for the Chern numbers of a fiber.** Let $\mu_F = \sum_p \mu_p(F_{\mathrm{red}})$ be the sum of the Milnor numbers of the singularities of F_{red}.

Let $N_F = g - p_a(F_{\mathrm{red}})$. One can prove that $0 \le N_F \le g$. $N_F = 0$ iff F is reduced or $g = 1$ and F is of type ${}_m\mathrm{I}_b$. $N_F = g$ iff F is a tree of smooth rational curves.

The topological characteristic of F is equal to $2N_F + \mu_F + 2 - 2g$.

Then we have the following formulas for the computation of the Chern numbers of F:

$$\begin{cases} c_1^2(F) = 4N_F + F_{\mathrm{red}}^2 + \alpha_F - \beta_F^-, \\ c_2(F) = 2N_F + \mu_F - \beta_F^+, \\ 12\chi_F = 6N_F + F_{\mathrm{red}}^2 + \alpha_F + \mu_F - \beta_F. \end{cases} \tag{3.10}$$

From the blow-up formulas, we only need to compute the Chern numbers of the minimal normal crossing model $\bar{F}$.

4. Proof of Theorem 1.1

4.1. **Dedekind's reciprocity law.** We denote by (p, q) the greatest common divisor of two integers p and q. The following notation is from Dedekind's Reciprocity Law. Take

$$\chi(p,q) = \frac{1}{12}\left(\frac{q}{p} + \frac{p}{q} + \frac{(p,q)^2}{pq}\right) - \frac{1}{4}.$$

One can easily check the following identities:

$$\chi(p,p) = 0, \qquad \chi(p,q) = \chi(p,p+q) + \chi(p+q,q). \tag{4.1}$$

If p and q are coprime, then Dedekind's sum is defined as follows:

$$s(p,q) = \sum_{i=0}^{q-1} \left(\left(\frac{pi}{q}\right)\right)\left(\left(\frac{i}{q}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}, \end{cases}$$

and $[x]$ is the largest integer $\le x$. $((x))$ is an odd function since $((-x)) = -((x))$ and is periodic with period 1.

If p and q are not coprime, then we define $s(p,q) := s\left(p/(p,q), q/(p,q)\right)$. Therefore, $s(-p,q) + s(p,q) = 0$, and $s(p+kq,q) = s(p,q)$ for all integers k. In particular, if $p + p'$ is divisible by q, then

$$s(p,q) + s(p',q) = 0. \tag{4.2}$$

The well-known *Dedekind's Reciprocity Law* says

$$s(p,q)+s(q,p)=\chi(p,q). \tag{4.3}$$

4.2. **Compute χ_F from the normal crossing model $\bar{F}$.** Let F be a singular fiber and $\bar{F}=\sum_{i=1}^k n_iC_i$ be the normal crossing model of F, where the C_i's are all irreducible components. Take $M_F=\mathrm{lcm}(n_1,\cdots,n_k)$.

Theorem 4.1. *Let $N_{\bar{F}}=g-p_a(\bar{F}_{\rm red})$. Then*

$$\chi_F=\frac{1}{2}N_{\bar{F}}-\sum_{i<j}\chi(n_i,n_j)C_iC_j.$$

Proof. Note that χ_F is a birational invariant, so

$$\chi_F=\chi_{\bar{F}}=\frac{1}{2}N_{\bar{F}}+\frac{1}{12}(\mu_{\bar{F}}-\beta_{\bar{F}}+\bar{F}_{\rm red}^2).$$

By definition,

$$\mu_{\bar{F}}=\sum_{i<j}C_iC_j,\quad \beta_{\bar{F}}=\sum_{i<j}\frac{(n_i,n_j)^2}{n_in_j}C_iC_j,\quad \bar{F}_{\rm red}^2=\sum_{i<j}2C_iC_j+\sum_{i=1}^kC_i^2.$$

Since $C_i\bar{F}=0$, $C_i^2=-\sum_{j\neq i}\frac{n_j}{n_i}C_iC_j$, we have $\sum_{i=1}^kC_i^2=-\sum_{i<j}\left(\frac{n_i}{n_j}+\frac{n_j}{n_i}\right)C_iC_j$. Thus

$$\mu_{\bar{F}}-\beta_{\bar{F}}+\bar{F}_{\rm red}^2=\sum_{i<j}\left(3-\frac{(n_i,n_j)^2}{n_in_j}-\frac{n_j}{n_i}-\frac{n_i}{n_j}\right)C_iC_j=-12\sum_{i<j}\chi(n_i,n_j)C_iC_j.$$

Hence $\chi_F=\frac{1}{2}N_{\bar{F}}-\sum\limits_{i<j}\chi(n_i,n_j)C_iC_j$. □

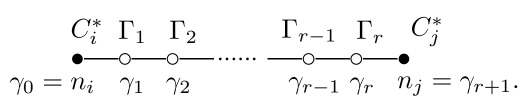
4.3. **Duality theorem for χ.**

Theorem 4.2. *F^* is the dual fiber of F. Then $\chi_F+\chi_{F^*}=N_{\bar{F}}=N_{\bar{F}^*}$.*

Proof. We use the notation in §2.3. We have seen that the normal crossing model $\bar{F}^*$ of F^* is of the following type:

$$\bar{F}^*=\sum_{i=1}^kn_iC_i^*+\sum_p\Gamma_p^*,$$

where p runs over all double points of $\bar{F}$, and $\Gamma_p^*=\gamma_1\Gamma_1+\cdots+\gamma_r\Gamma_r$ is as follows:



By 1) of Lemma 2.6, where if $i=1,\cdots,r$, then γ_i divides $\gamma_{i-1}+\gamma_{i+1}$, we have

$$s(\gamma_{i-1},\gamma_i)+s(\gamma_{i+1},\gamma_i)=0,\qquad \text{for } i=1,\cdots,r.$$

By 2) of Lemma 2.6, we have

$$s(\gamma_1,\gamma_0)=-s(\gamma_{r+1},\gamma_0),\qquad s(\gamma_r,\gamma_{r+1})=-s(\gamma_0,\gamma_{r+1}).$$

Hence

$$\begin{aligned}\sum_{i=1}^{r+1}\chi(\gamma_{i-1},\gamma_i)\Gamma_{i-1}\Gamma_i &= \sum_{i=1}^{r+1}(s(\gamma_{i-1},\gamma_i)+s(\gamma_i,\gamma_{i-1}))\\ &= s(\gamma_1,\gamma_0)+s(\gamma_r,\gamma_{r+1})+\sum_{i=1}^{r}(s(\gamma_{i-1},\gamma_i)+s(\gamma_{i+1},\gamma_i))\\ &= -s(\gamma_{r+1},\gamma_0)-s(\gamma_0,\gamma_{r+1})=-\chi(n_i,n_j).\end{aligned}$$

Thus

$$\mu_{\bar{F}^*}-\beta_{\bar{F}^*}+\bar{F}^{*2}_{\text{red}}=-(\mu_{\bar{F}}-\beta_{\bar{F}}+\bar{F}^2_{\text{red}}).$$

By Lemma 2.7, $p_a(\bar{F}^*{}_{\text{red}})=p_a(\bar{F}_{\text{red}})$, so $N_{\bar{F}^*}=N_{\bar{F}}$. We get $\chi_F+\chi_{F^*}=N_{\bar{F}}$. □

4.4. **Upper and lower bounds on χ.**

Theorem 4.3. $\frac{1}{6}N_{\bar{F}}\le\chi_F\le\frac{5}{6}N_{\bar{F}}$. *If F is not semistable, then* $\frac{1}{6}\le\chi_F\le\frac{5g}{6}$.

Proof. By the adjunction formula, $2N_F=K_X(F-F_{\text{red}})-F^2_{\text{red}}$. By the resolution of the singularities of F, we have $p_a(F_{\text{red}})=p_a(\bar{F}_{\text{red}})-\sum_i\frac{1}{2}(m_i-1)(m_i-2)$, so $2N_F=2N_{\bar{F}}-\sum_i(m_i-1)(m_i-2)$, where $m_i\ge 2$ are the multiplicities of singularities occurring in the partial resolutions of F. By definition, $\alpha_F=\sum_i(m_i-2)^2$. From formulas (1.3),

$$\begin{aligned}12\chi_F &= 6N_F+F^2_{\text{red}}+\alpha_F+\mu_F-\beta_F\\ &= 2N_F+(2N_F+F^2_{\text{red}})+(\mu_F-\alpha_F-\beta_F)+(2N_F+\alpha_F)\\ &= 2N_F+(F-F_{\text{red}})K_X+(\mu_F-\alpha_F-\beta_F)+2N_{\bar{F}}+\sum_i(m_i-2)(m_i-3).\end{aligned}$$

Since F is minimal, $(F-F_{\text{red}})K_X\ge 0$. $\mu_F-\alpha_F-\beta_F\ge 0$ is proved in Lemma 3.2. Hence $12\chi_F\ge 2N_{\bar{F}}$.

Similarly, $12\chi_{F^*}\ge 2N_{\bar{F}^*}=2N_{\bar{F}}$. On the other hand, $\chi_F+\chi_{F^*}=N_{\bar{F}}$, so $12\chi_F\le 10N_{\bar{F}}$. □

Corollary 4.4. $\chi_F=\frac{1}{6}N_{\bar{F}}$ *(resp.* $\chi_F=\frac{5}{6}N_{\bar{F}}$*) if and only if F (resp. F^*) is a reduced curve whose singularities are at worst ordinary cusps or nodes.*

Proof. It follows from Lemma 3.2. □

4.5. **Applications.**

Theorem 4.5. 1) *If f is non-trivial, then* $\chi_f\le\frac{g}{2}\left(2b-2+\frac{8}{3}s\right)$.

2) *If f is isotrivial, then* $\chi_f\le\frac{5gs}{6}$.

Proof. 1) We first assume that f is non-isotrivial. Let $F_1,\cdots,F_s$ be all of the singular fibers. There exists some semistable reduction $\pi:\tilde{C}\to C$ such that

(i) π is ramified uniformly over the s critical points of f, and the ramification index of π at any ramified point is exactly e.

(ii) e is divisible by M_{F_i} for all i, and it can be arbitrarily large.

In fact, if $b=g(C)>0$, the existence follows from Kodaira-Parshin's construction; if $b=0$, then $s\ge 3$. Thus one can construct a base change totally ramified over the s points. The existence is induced to the case $b>0$.

Let $\tilde{f}: \tilde{S} \to \tilde{C}$ be the semistable model and $\tilde{s}$ be the number of singular fibers of $\tilde{f}$. Let $\tilde{b} = g(\tilde{C})$ and $d = \deg \pi$. One has

$$2\tilde{b} - 2 = d(2b-2) + d\left(1 - \frac{1}{e}\right)s, \qquad \tilde{s} \le \frac{ds}{e}.$$

Hence we have

$$\chi_f - \frac{g}{2}\left(2b - 2 + \frac{8}{3}s\right) = \frac{1}{d}\left(\chi_{\tilde{f}} - \frac{g}{2}(2\tilde{b} - 2 + \tilde{s})\right) + \frac{g}{2d}\left(\tilde{s} - \frac{ds}{e}\right) + \sum_{i=1}^{s}\left(\chi_{F_i} - \frac{5g}{6}\right).$$

$\chi_{\tilde{f}} \le \frac{g}{2}(2\tilde{b} - 2 + \tilde{s})$ is the Arakelov inequality, so one gets inequality (1).

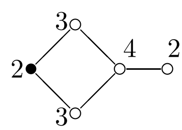
2) It is obvious. If f is also non-trivial, then 3) of Theorem 1.2 implies 2). □

5. Proof of Theorem 1.3

5.1. **Fibers with high c_1^2.** We try to prove Theorem 1.3, which implies Theorem 1.2, 2). To describe a fiber F, we usually consider the dual graph of its normal crossing model $\bar{F}$. We use $\circ$ to denote a (-2)-curve, and $\bullet$ a smooth rational curve but not a (-2)-curve. The number beside is the multiplicity of the curve in $\bar{F}$. The self-intersection number of each component $\bullet$ can be determined by using Zariski's lemma.

The following fiber F of genus g satisfies $c_1^2(F) = 4g - \frac{11}{2}$, $c_2(F) = 2g + \frac{5}{2}$, $\chi_F = \frac{g}{2} - \frac{1}{4}$.

Example 5.1. $F = (g-1)F_0$, where F_0 is a curve of genus 2 whose dual graph is as follows:



Lemma 5.2 (Artin [1])**.** *Let D be an effective divisor on a surface. Suppose $D^2 < 0$ and $D\Gamma_i \le 0$ for any component Γ_i of D. Then D is a negative curve, i.e., the intersection matrix $(\Gamma_i\Gamma_j)$ is negative definite.*

In what follows, we always assume that F satisfies $c_1^2(F) > 4g - \frac{11}{2}$, namely,

$$4p_a(\bar{F}_{\rm red}) - \bar{F}_{\rm red}^2 + \beta_F^- + \sum_{i=1}^{r} m_i(m_i - 2) < \frac{11}{2}. \tag{5.1}$$

Note that each term on the left hand side of (5.1) is non-negative.

Lemma 5.3. 1) *$m_i \le 3$ for all i and at most one m_i is equal to 3. So $F_{\rm red}$ admits at most one singular point p which is not a node. In fact, p is of types A_2, A_3 or D_4.*

2) *$\bar{F}_{\rm red}^2 \le -1$.*

3) *$p_a(\bar{F}_{\rm red}) = 0$, so $\bar{F}$ is a tree of smooth rational curves.*

4) *$p_a(F_{\rm red}) \le 1$, with equality iff one singular point p of $F_{\rm red}$ is not a node as in 1).*

Proof. 1) follows from the inequality $\sum_{i=1}^{r} m_i(m_i-2) < 11/2$ and Lemma 3.3.

2) (5.1) implies that $p_a(\bar{F}_{\rm red}) \le 1$, i.e., $K\bar{F}_{\rm red} + \bar{F}_{\rm red}^2 \le 0$. If $\bar{F}_{\rm red}^2 = 0$, then by Zariski's lemma, $\bar{F} = n\bar{F}_{\rm red}$ for some positive integer n. Since $K\bar{F}_{\rm red} \le 0$, we see that $2g-2 = K\bar{F} = nK\bar{F}_{\rm red} \le 0$, a contradiction. So $\bar{F}_{\rm red}^2 \le -1$.

3) Note that $p_a(\bar{F}_{\rm red}) \le 1$. Suppose that $p_a(\bar{F}_{\rm red}) = 1$. Then $\sum_{i=1}^{r} m_i(m_i-2) \le 3/2$, so all $m_i = 2$ and $F_{\rm red} = \bar{F}_{\rm red}$ is a nodal curve. We also see that $-F_{\rm red}^2 < 3/2$, so $F_{\rm red}^2 = -1$, $KF_{\rm red} = 1$, and F consists of one (-3)-curve and some (-2)-curves. Now from (5.1), we get $\beta_F^- < \frac{1}{2}$.

If one (-2)-curve E in F meets with the other components at only one point, then E is the end point of some H-J branch, and the contribution of E to β_F^- is at least $\frac{1}{2}$, a contradiction. Hence any (-2)-curve is a point in some loops in the dual graph of F. Because $p_a(F_{\rm red}) = 1$, there is only one loop in the dual graph. Hence the dual graph of F consists of one loop. Now we see that $F_{\rm red}\Gamma \le 0$ for each irreducible component Γ. Combining with $F_{\rm red}^2 < 0$, we know that F is a negative curve (Lemma 5.2), a contradiction.

4) By Lemma 5.3 and (3.1), we have $p_a(F_{\rm red}) = p_a(\bar{F}_{\rm red}) + \sum_i \frac{1}{2}(m_i-1)(m_i-2) \le 1$. □

5.2. **The case $p_a(F_{\rm red}) = 1$.**

Proposition 5.4. *If F is not a nodal curve, then $F_{\rm red}$ has one singular point of type A_3. The normal crossing model of F is of type* 21.

Proof. In this case, F has a unique singularity p of types A_2, A_3, or D_4. $p_a(F_{\rm red}) = 1$; one has $-F_{\rm red}^2 + \beta_F^- < \frac{5}{2}$. Since $p_a(\bar{F}_{\rm red}) = 0$, the dual graph of $\bar{F}$ is a tree of rational curves.

Case A_2. Suppose that p is of type A_2. Then the contribution of p to $\beta_F^- \ge \frac{5}{6}$, so $-F_{\rm red}^2 < \frac{5}{3}$. We have $-F_{\rm red}^2 = F_{\rm red}K_X = 1$, and $\beta_F^- < \frac{3}{2}$. Let C_1 be the irreducible component passing through p. Then $K_X C_1 = 1$ and $F_{\rm red} - C_1$ is composed of some ADE curves. Suppose that $F_{\rm red} - C_1$ contains at least two (-2)-curves as the end points in the dual graph of F. Then their contributions to β_F^- is at least 1. So $\beta_F^- \ge 1 + \frac{5}{6} > \frac{3}{2}$, a contradiction. So only one (-2)-curve is an end point. On the other hand, from $p_a(\bar{F}_{\rm red}) = 0$, we see that F contains no loop. Hence F is an H-J chain with an end point C_1. It implies F is a negative curve, a contradiction.

Case A_3. Assume that p is of type A_3. The contribution of p to $\beta_F^- \ge \frac{1}{2}$, and so $-F_{\rm red}^2 < 2$. Now we have $-F_{\rm red}^2 = F_{\rm red}K_X = 1$ and $\beta_F^- < \frac{3}{2}$. F consists of some (-2)-curves and one curve C_1 passing through p. Note that $\bar{F}$ is a tree of rational curves, so no node is a singular point of C_1, namely C_1 is smooth except at p. If C_1 is singular at p, then there is no (-2)-curve passing through p. Similar to the discussion above, only one (-2)-curve is the end point. Now we know that F is a chain of (-2)-curves and C_1, so F is a negative curve, a contradiction. Hence C_1 is smooth at p and there is a (-2)-curve C_2 tangent to C_1 at p. Because $K_X C_1 = 1$, C_1 is a (-3)-curve.

There is a (-2)-curve C_2 tangent to C_1 at p. $F_{\rm red} - C_1 - C_2$ consists of ADE curves. Because only one (-2)-curve is the end point, we know that $\Gamma = F_{\rm red} - C_1 - C_2$ is just a curve of type A_n.

If C_1 intersects Γ, then $F_{\rm red} = \Gamma + C_1 + C_2$ is a chain. One can prove that F is a negative curve by Lemma 5.2, a contradiction. So C_1 is disjoint with Γ. $C_2 + \Gamma$ is a connected curve of type A_{n+1}.

By using Zariski's lemma, one can determine the multiplicities of all irreducible components in F and the number of (-2)-curves. Finally, we get the fiber of type 21.

Case D_4. Suppose that p is of type D_4. Because $\bar{F}$ is a tree of rational curves, the three local branches of F at p come from three different components, C_1, C_2 and C_3 of F. At least one component, say C_1, is not a (-2)-curve since $g \geq 2$. Suppose that C_2 is not a (-2)-curve. Then $F_{\rm red}K_X \geq 2$. Recall that $F_{\rm red}K_X = -F_{\rm red}^2 \leq 2$; one has $F_{\rm red}K_X = -F_{\rm red}^2 = 2$. Thus $\beta_F^- < \frac{1}{2}$ and $C_1K_X = C_2K_X = 1$, namely, C_1 and C_2 are (-3)-curves. Hence $F_{\rm red} - C_1 - C_2$ consists of ADE-curves whose contributions to $\beta_F^- \geq \frac{1}{2}$, a contradiction. Therefore C_2 and C_3 must be (-2)-curves. Similarly, we can also prove that C_1 is not a (-4)-curve, hence it is a (-3)-curve. One can also prove that the other curves in F are (-2)-curves. Now we have $-F_{\rm red}^2 = KF_{\rm red} = 1$, and $\beta_F^- < \frac{3}{2}$.

The normal crossing model $\bar{F}$ of F is obtained by blowing up F at p. Since the intersection matrix of C_1, C_2 and C_3 is negative definite, $\Gamma = F_{\rm red} - C_1 - C_2 - C_3$ consists of $s \geq 1$ connected ADE-curves $\Gamma_1, \cdots, \Gamma_s$. From $\beta_F^- < \frac{3}{2}$, we see that at most two end points are (-2)-curves, so $s \leq 2$. Let $r_i - 1$ be the number of irreducible components of Γ_i.

Suppose $s = 2$. Since at most two end points are (-2)-curves, Γ_1 and Γ_2 are of types A_{r_1-1} and A_{r_2-1} respectively. In $\bar{F}$, $C_1^2 = -4$, $C_2^2 = C_3^2 = -3$. Γ_i meets C_j at one point, so we obtain an H-J branch of type $[2, 2, \cdots, 2, e_{r_i}]$, where $e_{r_i} = -C_j^2$.

Symmetrically, we only need to consider two cases: I) Γ_1 meets C_2 and Γ_2 meets C_1; II) Γ_1 meets C_2 and Γ_2 meets C_3.

In case I), from Zariski's lemma, one can find an equality $\frac{2}{3} = \frac{r_1}{2r_1+1} + \frac{r_2}{3r_2+1}$, i.e., $1 = \frac{3}{2r_1+1} + \frac{2}{3r_2+1}$. We claim that there are no non-negative integers r_1 and r_2 satisfying this equation. Indeed, for $r_1 = 0$, 1 or 2, this equation has no non-negative integral solution r_2. So we can assume that $r_1 \geq 3$. Similarly, we can also assume that $r_2 \geq 2$. Now the right hand side is less than 1. So case I) does not occur.

In case II), we have similarly $\frac{3}{4} = \frac{r_1}{2r_1+1} + \frac{r_2}{2r_2+1}$, i.e., $\frac{1}{2} = \frac{1}{2r_1+1} + \frac{1}{2r_2+1}$. It is obvious that this equation has no integral solutions. So case II) cannot occur.

Suppose $s = 1$. If Γ_1 is of type A_{r_1-1}, by Zariski's lemma, we have either $\frac{12}{5} = \frac{2r_1+1}{r_1}$ or $3 = 3 + \frac{1}{r_1}$. These equations have no integral solutions. So this case does not occur.

Finally, we assume that Γ_1 is not of type A_{r_1-1}. Now we see that there are two end points which are (-2)-curves, so the contribution of them to β_F^- is at least 1. On the other hand, the contribution of the two components disjoint from Γ_1 are at least $\frac{1}{4} + \frac{1}{3} = \frac{7}{12}$. So $\beta_F^- \geq 1 + \frac{7}{12} > \frac{3}{2}$, a contradiction.

Up to now, we have proved that the case D_4 does not occur. □

5.3. **The case $p_a(F_{\rm red}) = 0$.** From now on, we always assume that $F_{\rm red}$ is a tree of smooth rational curves, namely, $p_a(F_{\rm red}) = 0$. Hence (5.1) becomes $-F_{\rm red}^2 + \beta_F^- < \frac{11}{2}$. Namely,

$$F_{\rm red}K_X + \beta_F^- < \frac{7}{2}. \tag{5.2}$$

Lemma 5.5. *We have $F_{\rm red}K_X = 1$ and $F_{\rm red}^2 = -3$. Namely, $F_{\rm red}$ consists of a (-3)-curve and some (-2)-curves. So $\beta_F^- < \frac{5}{2}$.*

Proof. Suppose that $F_{\rm red}K_X \geq 2$. Let s be the number of (-2)-curves as the end points in the dual graph of F. $\beta_F^- < \frac{3}{2}$ implies $s \leq 2$. Assume that the dual graph of F contains r end points. Obviously $r \geq 3$.

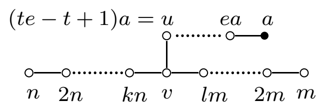
We first claim that $r = 3$, $s = 1$ and $F_{\rm red}K_X = 2$.

Indeed, there are at least $r - s$ end points which are not (-2)-curves. So $F_{\rm red}K_X \geq r - s$ and $\beta_F^- < \frac{7}{2} + s - r$. On the other hand, $\beta_F^- > \frac{s}{2}$. So $s \geq 2r - 6$. Note that $s \leq 2$; we get $r \leq 4$. If $r = 4$, then $s = 2$. Then we see that $1 < \beta_F^- < \frac{3}{2}$ and $F_{\rm red}K_X = 2$. It also implies that two of the end points are (-3)-curves. Thus $\beta_F^- \geq 2(\frac{1}{2} + \frac{1}{3}) > \frac{3}{2}$, a contradiction. So $r = 3$.

If $F_{\rm red}K_X = 3$, then $\beta_F^- < \frac{1}{2}$. So any end point is a (-3)-curve. Thus $\beta_F^- \geq 1$, a contradiction. Hence $F_{\rm red}K_X = 2$. It implies $s \geq r - F_{\rm red}K_X = 1$.

Suppose that $s = 2$. Since $r = 3$ and F is a tree of rational curves, F has two H-J chains of type A_n and one H-J chain whose end point is a $(-e)$-curve, $e = 3$ or 4. We have seen in §3.2 that the multiplicities in an H-J branch increase strictly from the end point to the other side.

Suppose $e = 4$. From $F_{\rm red}K_X = 2$, we see that all other components are (-2)-curves. The dual graph of F is as follows:



where $(k+1)n = (l+1)m = ((t+1)e - t)a = v$ $(1 \leq k \leq l)$ and $kn + lm + u = 2v$ by Zariski's lemma, so we have $\frac{k}{k+1} + \frac{l}{l+1} + \frac{u}{v} = 2$. It is easy to see that

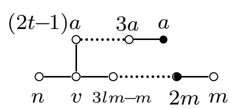
$$\frac{k}{k+1} + \frac{l}{l+1} + \frac{1}{4} \leq \beta_F^- < \frac{3}{2}.$$

So either $k = l = 1$, or $k = 1$ and $l = 2$. Now we see that $\frac{u}{v} = 1$ or $\frac{5}{6}$, a contradiction.

If $e = 3$, then there exists another (-3)-curve E. In fact, E cannot be in the center, otherwise $3v = kn + lm + u < v + v + v$, a contradiction. E cannot be in the vertical branch. Otherwise, we have

$$\frac{k}{k+1} + \frac{l}{l+1} + \frac{1}{3} \leq \beta_F^- < \frac{c3}{2},$$

which implies $k = l = 1$, i.e., $n = m$ and $v = 2n$. Since $kn + lm + u = 2v$, we have $u = v$, a contradiction with $v > u$. Hence E must be a component of the horizontal branch. Without loss of generality, we assume that E is on the right branch. Consider the contribution to β_F^-; we have $k = 1$ and E intersects with the (-2)-curve at the end. The dual graph of F is as follows:



We have $v = 2n = (2t+1)a = (3l+2)m$, and $n + (2t-1)a + (3l-1)m = 2v$. It implies

$$\frac{1}{2} + \frac{2t-1}{2t+1} + \frac{3l-1}{3l+2} = 2\,,$$

i.e.,

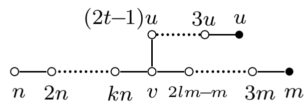
$$\frac{2}{2t+1}+\frac{3}{3l+2}=\frac{1}{2}\,.$$

This equation has only one solution, $t=3$ and $l=4$. Now we can compute $\beta_F^-=\frac{11}{7}>\frac{3}{2}$, a contradiction.

We have proved that $s=2$ cannot occur. So $s=1$. The claim is proved.

Finally, we need to exclude the case in the claim.

F has exactly two H-J branches whose end points are (-3)-curves. The remaining H-J branch is of type A_n which contains k vertexes. The dual graph is as follows:



where $(2l+1)m=(2t+1)u=(k+1)n=v$ $(l\le t)$ and $\frac{2l-1}{2l+1}+\frac{2t-1}{2t+1}+\frac{k}{k+1}=2$ by Zariski's lemma, i.e., $\frac{1}{2l+1}+\frac{1}{2t+1}=\frac{k}{2k+2}$. Then we have

$$\begin{aligned}\beta_F^-&=\frac{l}{2l+1}+\frac{t}{2t+1}+\frac{k}{k+1}\\&=1-\frac{1}{2}\left(\frac{1}{2l+1}+\frac{1}{2t+1}\right)+\frac{k}{k+1}\\&=1-\frac{1}{4}\cdot\frac{k}{k+1}+\frac{k}{k+1}=1+\frac{3}{4}\cdot\frac{k}{k+1}<\frac{3}{2};\end{aligned}$$

we get $k=1$. It is easy to see that the equation $\frac{1}{2l+1}+\frac{1}{2t+1}=\frac{1}{4}$ has no positive integral solutions l and t. So the case in the claim is excluded. Hence $F_{\rm red}K_X=2$ is impossible.

The lemma is finally proved. □

Now F consists of one (-3)-curve C_0 and some connected ADE curves $\Gamma_1,\cdots,\Gamma_r$. Let Z_i be the fundamental cycle supported on Γ_i. Then $Z_i^2=-2$. See ([5], Ch. III, §3) for the list of Z_i.

Since $(C_0+Z_i)^2\le 0$, $1\le C_0Z_i\le 2$. If $C_0Z_i=2$, then Z_i cannot be of type A_n. Otherwise Z_i is reduced and $C_0Z_i=2$ implies that F is not a tree. Hence Z_i must be of types E_k or D_n.

Lemma 5.6. *If $C_0Z_i=2$ for some i, then $g=2$ and F is of types* $10\sim 16$.

Proof.

Step 1. There is at most one Z_i such that $C_0Z_i=2$. Otherwise if Z_i and Z_j satisfy $C_0Z_i=C_0Z_j=2$, then $(C_0+Z_1+Z_2)^2=1$, a contradiction. Without loss of generality, we assume $C_0Z_1=2$ and $C_0Z_i=1$ for all $i\ge 2$.

Step 2. Suppose $r\ge 3$. One can check that

$$(2C_0+2Z_1+Z_2+Z_3)^2=0.$$

Note that F is simply connected and cannot be a multiple fiber ([18], p. 389). So $F=2C_0+2Z_1+Z_2+Z_3$. Let C_2 be an irreducible component of Z_2 such that $C_2Z_2<0$. From $FC_2=0$ we get $C_2Z_2=-2C_0C_2$. Since $(Z_2-C_2)^2\le 0$, we have $C_2Z_2\ge -2$, so $C_2Z_2=-2$ and $(C_2-Z_2)^2=0$, i.e., $Z_2=C_2$ is just one (-2)-curve. Similarly, Z_3 is also a (-2)-curve. Recall that supp(Z_1) is a curve of types D_n or E_k, and C_0 meets with Z_1 at the component E with $EZ_1<0$. Because $\beta_F^-<\frac{5}{2}$, Z_1

cannot be of type D_n. Now one can check that the possibilities are just the fibers of types 11, 12 and 13.

Step 3. Suppose $r = 2$. Let C_1 be the irreducible component of Z_1 such that $C_1Z_1 < 0$. Since Z_1 is not a curve of type A_n, one can check from the list that $C_1Z_1 = -1$. Then $(2C_0 + 2Z_1 + Z_2 - C_1)^2 = -4C_0C_1$. If $C_0C_1 = 0$, $F = 2C_0 + 2Z_1 + Z_2 - C_1$. By Zariski's lemma, $0 = FZ_1 = -C_1Z_1$, a contradiction. So $C_0C_1 = 1$.

Let C_2 be an irreducible component of Z_2 such that $C_0C_2 = 1$. Since $C_0Z_2 = 1$, the multiplicity of C_2 in Z_2 is 1. If $Z_2C_2 < 0$, then one can check that Z_2 is of type A_n, C_2 is at the end of Z_2 and $C_2Z_2 = -1$. Consider $D = C_0 + Z_1 + Z_2$; one can check that $D\Gamma \le 0$ for each irreducible Γ of D, e.g., $C_1D = 0$ and $C_2D = 0$. $D^2 = -1$. By Lemma 5.2, D is a negative curve, a contradiction. Hence $Z_2C_2 = 0$.

Now we have

$$(2C_0 + 2Z_1 + Z_2 + C_2)^2 = 0.$$

Thus $F = 2C_0 + 2Z_1 + Z_2 + C_2$. Since $Z_2C_2 = 0$, Z_2 cannot be irreducible. There is another component C_3 of Z_2 such that $Z_2C_3 < 0$. Since $0 = FC_3 = Z_2C_3 + C_2C_3$, we see that $Z_2C_3 = -1$, and $C_2C_3 = 1$. Check each type of ADE fundamental cycles; one find that Z_2 must be of type D_n. From $\beta_F^- < \frac{5}{2}$, we see that the dual graph of F has at most four (-2)-curves as its end points, so Z_1 cannot be of type D_n. Now we obtain that F is of types 14, 15 and 16.

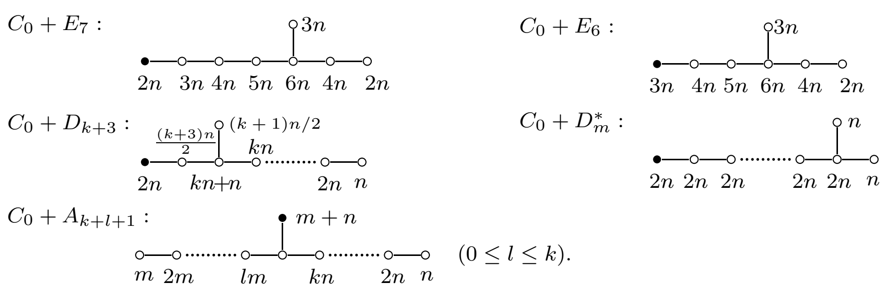
Step 4. Suppose $r = 1$. Let C_1 be the irreducible component of Z_1 such that $C_0C_1 = 1$. If $C_1Z_1 < 0$, $C_0 + Z_1$ is a negative cycle by Lemma 5.2, a contradiction. So $C_1Z_1 = 0$. Let C_2 be another irreducible component of Z_1 such that $Z_1C_2 < 0$; one can check that $Z_1C_2 = -1$.

If $C_1C_2 = 0$, then

$$(2C_0 + 2Z_1 + C_1 - C_2)^2 = 0.$$

So $F = 2C_0 + 2Z_1 + C_1 - C_2$. Thus $0 = FC_0 = -1 - C_0C_2$, a contradiction. Hence $C_1C_2 = 1$. By checking each type of ADE fundamental cycles, we see that Z_1 is of type D_n. Note that C_2 is unique in Z_1. We claim that C_1 is not at the end of D_n. Otherwise, by Lemma 5.2, $C_0 + Z_1 + C_1$ is a negative cycle, a contradiction. So the position of C_1 is determined. Now we easily see that F is just the fiber of type 10. □

From now on we always assume that $C_0Z_i = 1$ for all i. So C_0 meets with a component whose multiplicity in Z_i is 1. From Zariski's lemma, one can determine the multiplicities of the irreducible components of Z_i in F whenever the multiplicity of C_0 is determined. The following are all possible partial dual graphs of C_0 and Z_i in F:



Recall that $F_{\rm red}-C_0$ consists of r connected components $\Gamma_1,\cdots,\Gamma_r$ and $\beta_F^- < \frac{5}{2}$. Let C_i be the irreducible component of Γ_i meeting with C_0, and let n_i be the multiplicity of C_i in F. Let β_i be the contribution of Γ_i to β_F^-. If $r \geq 2$, then we have

$$3=\sum_{i=1}^{r}\frac{n_i}{n_0}, \qquad \beta_F^-=\sum_{i=1}^{r}\beta_i<\frac{5}{2}. \tag{5.3}$$

Lemma 5.7. 1) *$r \leq 3$, and if $r=3$, then F is the fiber of type* 18.

2) *If $r=2$ and all Γ_i are not of type A_n, then F is of types* 5, 19, 20 *and* 22.

3) *If $r=2$ and Γ_1 is of type A_n, then F is of types* 1, 2, 4, 6, 9 *and* 17.

4) *If $r=1$, then F is of types* 3, 7 *and* 8.

Proof. 1) Note first that if all Γ_i are of type A_n and form some H-J branches, then by a straightforward computation we have $\beta_F^-=\sum_{i=1}^r \frac{n_i}{n_0}=3>\frac{5}{2}$, a contradiction. Since $\beta_F^-<\frac{5}{2}$, $r\leq 4$. If $r=4$, then all Γ_i are of type A_n and form H-J branches, which is impossible. Hence $r \leq 3$.

Assume $r=3$. $\beta_F^-<\frac{5}{2}$ implies $F_{\rm red}-C_0$ contains two H-J branches of type A_n, say Γ_1 and Γ_2. By (5.3), we have

$$\beta_F^-=\frac{n_1}{n_0}+\frac{n_2}{n_0}+\beta_3=3-\frac{n_3}{n_0}+\beta_3.$$

If Γ_3 is of type E_7, E_6 or D_m^*, then one can check that $\beta_F^-=\frac{8}{3}$, $\frac{17}{6}$ and 3 respectively, which contradicts the condition $\beta_F^-<\frac{5}{2}$. If Γ_3 is of type A_{k+l+1} as above, then $k\geq 1$ and $l \geq 1$ (since Γ_3 is not an H-J branch of type A_n). On the other hand, $\frac{n_1}{n_0}\geq\frac{1}{2}$ and $\frac{n_2}{n_0}\geq\frac{1}{2}$, so $\frac{n_3}{n_0}\leq 2$. $n_3=(l+1)m=(k+1)n$, so $\frac{n_3}{n_0}=\frac{(k+1)(l+1)}{(k+1)+(l+1)}\leq 2$ $(l\leq k)$; we obtain that $l=1$. Hence

$$\beta_F^-=3+\frac{k}{k+1}+\frac{1}{2}-\frac{2(k+1)}{k+3}=\frac{5}{2}+\frac{3k+1}{(k+1)(k+3)}>\frac{5}{2},$$

a contradiction.

Finally, assume that Γ_3 is of type D_{k+3} as above. $\beta_3=\frac{1}{2}+\frac{k}{k+1}$, so

$$\beta_F^-=3+\frac{1}{2}+\frac{k}{k+1}-\frac{k+3}{4}<\frac{5}{2};$$

we get $k\geq 5$. On the other hand, $\frac{n_3}{n_0}\leq 2$, i.e., $\frac{k+3}{4}\leq 2$ and $k\leq 5$. Hence $k=5$, $2n_1=2n_2=n_0=2n$. Because F cannot be a multiple fiber, $n=1$. This is just the fiber of type 18.

2), 3) and 4) can be proved by similar calculations. □

5.4. **Applications.** The local canonical class inequality has some interesting applications. It has been used to establish the canonical class inequality for non-semistable fibrations. Now we give a new proof of the following well-known result.

Corollary 5.8. *Let $f : X \to \mathbb{P}^1$ be a non-trivial fibration of genus $g \geq 1$. Then f admits at least* 2 *singular fibers.*

Proof. If f is smooth, then it is trivial. Now we assume that f admits only one singular fiber F. In this case, f is isotrivial. So

$$c_1^2(X)=-8(g-1)+c_1^2(F), \quad c_2(X)=-4(g-1)+c_2(F). \tag{5.4}$$

If $g\geq 2$, we proved in [16] that $c_1^2(X)+8(g-1)=K_f^2\geq 4(g-1)$. By (5.4), we have $c_1^2(F)\geq 4g-4$, a contradiction.

If $g = 1$, then $12\chi(\mathcal{O}_X) = c_2(X) = c_2(F)$. So $c_2(F)$ is divided by 12. We know that $c_2(F) = 0$, and $F = nE$ for some smooth elliptic curve E and $n \geq 2$. Hence $\chi(\mathcal{O}_X) = 0$. By the formula for the canonical class, we have $K_X \sim -(n+1)E$. Hence X is birationally ruled, $p_g(X) = 0$ and $q(X) = 1$. The Albanese map $\alpha : X \to B$ is the ruling. Let F' be a fiber of α. Then $2 = -K_X F' = (n+1)EF' \geq n+1 \geq 3$, a contradiction.

This proves that f admits at least 2 singular fibers. □

5.5. **Chern numbers of the fibers in Theorem 1.3.** In order to prove 2) of Theorem 1.2, we need to compute the Chern numbers for all 22 singular fibers in Theorem 1.3.

F	1	2	3	4	5	6	7	8	9	10	11
g	6	4	3	3	3	3	2	2	2	2	2
c_1^2	$\frac{130}{7}$	$\frac{54}{5}$	7	$\frac{48}{7}$	$\frac{98}{15}$	$\frac{20}{3}$	$\frac{16}{5}$	3	3	3	$\frac{8}{3}$
c_2	30	26	21	18	$\frac{268}{15}$	20	16	15	15	9	$\frac{34}{3}$
χ	$\frac{85}{21}$	$\frac{46}{15}$	$\frac{7}{3}$	$\frac{29}{14}$	$\frac{61}{30}$	$\frac{20}{9}$	$\frac{8}{5}$	$\frac{3}{2}$	$\frac{3}{2}$	1	$\frac{7}{6}$

F	12	13	14	15	16	17	18	19	20	21	22
g	2	2	2	2	2	2	2	2	2	2	2
c_1^2	$\frac{11}{4}$	$\frac{17}{6}$	$\frac{8}{3}$	$\frac{11}{4}$	$\frac{17}{6}$	$\frac{14}{5}$	$\frac{8}{3}$	$\frac{8}{3}$	$\frac{8}{3}$	$\frac{13}{5}$	$\frac{31}{12}$
c_2	$\frac{49}{4}$	$\frac{79}{6}$	$\frac{34}{3}$	$\frac{49}{4}$	$\frac{79}{6}$	14	$\frac{40}{3}$	$\frac{40}{3}$	$\frac{52}{3}$	7	$\frac{197}{12}$
χ	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{7}{6}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{7}{5}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{4}{5}$	$\frac{19}{12}$

6. Proof of Theorem 1.4

In this section, we will classify all singular fibers satisfying $2c_2(F) - c_1^2(F) < 6$.

Lemma 6.1. *If* $2c_2(F) - c_1^2(F) \neq 0$*, then* $2c_2(F) - c_1^2(F) \geq 3$*.*

Proof. We have

$$2c_2(F) - c_1^2(F) = 2(\mu_F - \beta_F - \alpha_F) + \alpha_F + 3\beta_F^- - F_{\rm red}^2 < 3. \tag{6.1}$$

In particular, we have $\sum_{p\in F}\left(2(\mu_p - \beta_p - \alpha_p) + \alpha_p + 3\beta_p^-\right) < 3 + F_{\rm red}^2 \leq 3$. By Lemma 3.2, 4), F is a nodal curve, and so $\alpha_F = 0$. If $F_{\rm red}^2 = 0$, then $F = nF_{\rm red}$ and $2c_2(F) - c_1^2(F) = 0$. If $F_{\rm red}^2 \leq -1$, then $\mu_F - \beta_F < 1$. Note that if a node p satisfies $\beta_p \neq 1$, then $\beta_p \leq \frac{1}{2}$, and $\mu_p - \beta_p \geq \frac{1}{2}$. Hence $\mu_F - \beta_F < 1$ implies that at most one node p satisfies $\beta_p \neq 1$. So $F = nA + mB$, A and B are reduced nodal curves and $AB = 1$. By Zariski's lemma, $0 = AF = nA^2 + mAB = nA^2 + m$; similarly, $n + mB^2 = 0$. Hence $m = n$, and $F_{\rm red}^2 = 0$, a contradiction. This proves the lemma. □

Proof of Theorem 1.4. We can assume that $2c_2(F) - c_1^2(F) \geq 3$:

$$3 \leq 2(\mu_F - \beta_F - \alpha_F) + \alpha_F + 3\beta_F^- - F_{\rm red}^2 < 6. \tag{6.2}$$

In particular, we have $\sum_{p\in F}\left(2(\mu_p-\beta_p-\alpha_p)+\alpha_p+3\beta_p^-\right) < 6+F_{\rm red}^2 \le 6$. By Lemma 3.2, 4), F admits at most one singular point p which is not a node, and p is of type A_2, A_3 or D_4.

If $F_{\rm red}^2=0$, then $F=nF_{\rm red}$. Then one can compute all the local invariants directly, and we get the cases 2) $\sim$ 5). In what follows we assume that $F_{\rm red}^2\le -1$. So $\sum_{p\in F}\left(2(\mu_p-\beta_p-\alpha_p)+\alpha_p+3\beta_p^-\right)<5$, and p is at worst of A_3.

Let s be the number of nodes in $F_{\rm red}$ satisfying $\beta_q<1$. For such a node q, the two components of F at q have distinct multiplicities. So $\beta_q\le\frac{1}{2}$ and $\mu_q-\beta_q\ge\frac{1}{2}$.

If the non-nodal singular point p exists, then p is of type A_2 or A_3. Also, $2(\mu_p-\beta_p-\alpha_p)+\alpha_p+3\beta_p^-$ is at least $\frac{7}{2}$ if p is of type A_2 or A_3, so we get $\frac{7}{2}+s<5$, i.e., $s\le 1$. As in the proof of the previous lemma, $s=1$ is impossible by Zariski's lemma. So $s=0$, i.e., the multiplicities of the two local branches of any node are the same. Hence p is of type A_3.

From Zariski's lemma, one has a decomposition $F=n(A+2B)$, where A and B are connected reduced nodal curves and smooth at p, $A\cap B=\{p\}$, $A^2=-4$, $B^2=-1$, $AB=2$. We get case 8).

From now on we always assume $F_{\rm red}$ is a nodal curve. By assumption, $s\neq 0$. As in the proof of the previous lemma, $s=1$ is also impossible. So $2\le s<6+F_{\rm red}^2\le 5$ by (6.2), i.e., $2\le s\le 4$.

Let $F=\gamma_1\Gamma_1+\cdots+\gamma_r\Gamma_r$, $r\ge 2$, where the Γ_i's are reduced with $\Gamma_i^2=-e_i\le -1$ (not necessarily irreducible) and have no pairwise common components. $\gamma_1,\cdots,\gamma_r$ are pairwise distinct. Then we have

$$r-1\le\sum_{i<j}\Gamma_i\Gamma_j=s,\quad F_{\rm red}^2=2s-e_1-\cdots-e_r.$$

If $r-1=s$, then $\Gamma_1,\cdots,\Gamma_r$ form a chain. Assume that this is a chain such as the one before Lemma 2.6, where $\gamma_0=\gamma_{r+1}=0$. So the linear equation (2.1) holds true. Since $\gamma_0=0$, we can see from the equation that γ_1 divides γ_i for any i. Symmetrically, from $\gamma_{r+1}=0$, we know that γ_r divides γ_i for all i. So $\gamma_1=\gamma_r$, which contradict our assumption. So $r\le s$.

Suppose $s=2$. Then $r=2$. Now one can prove that $F=nA+2nB$, $AB=2$, $A^2=-4$ and $B^2=-1$. We get cases 6) and 7).

Suppose $s=3$. Then $F_{\rm red}^2\ge -2$ and $r=2$ or 3.

If $r=2$, then one can prove that $F=\gamma_1\Gamma_1+3\gamma_1\Gamma_2$, $\Gamma_1\Gamma_2=3$. Hence $\mu_F-\beta_F=3-1=2$. Now we have $2(\mu_F-\beta_F)\ge 6+F_{\rm red}^2$, a contradiction.

If $r=3$ and $\Gamma_1\Gamma_2=\Gamma_2\Gamma_3=\Gamma_3\Gamma_1=1$, then one can prove that $F=\gamma_1\Gamma_1+3\gamma_1\Gamma_2+2\gamma_1\Gamma_3$. Hence $\mu_F-\beta_F=3-1=2$ and $2(\mu_F-\beta_F)\ge 6+F_{\rm red}^2$, a contradiction.

If $r=3$, $\Gamma_1\Gamma_2=2$, $\Gamma_2\Gamma_3=1$ and $\Gamma_3\Gamma_1=0$, then we have $-e_1\gamma_1+2\gamma_2=0$, $2\gamma_1-e_2\gamma_2+\gamma_3=0$ and $\gamma_2-e_3\gamma_3=0$. Since $\gamma_2\neq\gamma_3$, we have $e_3\ge 2$. We obtain that $e_1=e_3(e_1e_2-4)$, which implies $e_2=1$. $e_1=e_3=5$, or $e_1=6$ and $e_3=3$, or $e_1=8$ and $e_3=2$. Hence $F_{\rm red}^2\le -4$, a contradiction.

Suppose $s=4$. Then $F_{\rm red}^2=-1$. Let $F=\sum_{i=1}^k n_iC_i$, where C_i is irreducible. Let $q_1,\cdots,q_4$ be the nodes satisfying $\frac{1}{2}\ge\beta_{q_1}\ge\cdots\ge\beta_{q_4}$. By (6.2), we have

$$2\sum_{k=1}^{4}(\mu_{q_k}-\beta_{q_k})<5, \tag{6.3}$$

which implies $\beta_{q_1} = \beta_{q_2} = \frac{1}{2}$. Let C_{i_1} and C_{i_2} be the components passing through q_i ($i \leq 4$) and let $n_{i1} \leq n_{i2}$. From $\beta_{q_1} = \beta_{q_2} = \frac{1}{2}$, we have $n_{i2} = 2n_{i1}$ for $i = 1$ and 2, and thus $\chi(n_{11}, n_{12}) = \chi(n_{21}, n_{22}) = 0$.

From the proof of Theorem 4.1, we have

$$\sum_{k\leq 4}(\mu_{q_k} - \beta_{q_k}) + F_{\text{red}}^2 = -12\sum_{i\leq 4}\chi(n_{i1}, n_{i2}).$$

Since $F_{\text{red}}^2 = -1$ and $\beta_{q_1} = \beta_{q_2} = \frac{1}{2}$,

$$\mu_{q_3} - \beta_{q_3} + \mu_{q_4} - \beta_{q_4} = 6 - \left(\frac{n_{31}}{n_{32}} + \frac{n_{32}}{n_{31}}\right) - \left(\frac{n_{41}}{n_{42}} + \frac{n_{42}}{n_{41}}\right) - \beta_{q_3} - \beta_{q_4},$$

i.e.,

$$4 = \left(\frac{n_{31}}{n_{32}} + \frac{n_{32}}{n_{31}}\right) + \left(\frac{n_{41}}{n_{42}} + \frac{n_{42}}{n_{41}}\right) \geq 2 + 2 = 4.$$

Thus $n_{31} = n_{32}$ and $n_{41} = n_{42}$, so $\beta_{q_3} = \beta_{q_4} = 1$, a contradiction.

Up to now we have completed the proof. □

Corollary 6.2. *If the semistable model of F is smooth and F is not the multiple of a smooth curve, then $2c_2(F) - c_1^2(F) \geq 6$.*

Corollary 6.3 ([10])**.** *If $f : X \to C$ is an isotrivial family of curves, then $K_X^2 \neq 8\chi(\mathcal{O}_X) - 1$.*

Proof. In this case, the modular invariants of f are zero. Hence

$$2c_2(X) - c_1^2(X) = \sum_i \left(2c_2(F_i) - c_1^2(F_i)\right) \geq 0.$$

Suppose $K_X^2 \neq 8\chi(\mathcal{O}_X)$, i.e., $2c_2(X) \neq c_1^2(X)$. Then at least one singular fiber satisfies the condition of Corollary 6.2. Hence $2c_2(X) - c_1^2(X) \geq 6$ and , equivalently, $K_X^2 \leq 8\chi(\mathcal{O}_X) - 2$. □

Corollary 6.4. *If F is not semistable, then $c_2(F) \geq \frac{11}{6}$ and $\chi_F \geq \frac{1}{6}$. One of the equalities holds if and only if F is a reduced curve with one ordinary cusp and some nodes.*

Proof. If $2c_2(F) - c_1^2(F) \geq 6$, equivalently, $8\chi_F - c_1^2(F) \geq 2$, then $c_2(F) > 3$ and $\chi_F > \frac{1}{4}$. So we can assume that $2c_2(F) - c_1^2(F) < 6$. By Theorem 1.4, we have 8 types of singular fibers. We see that only the type 2) fiber with $N_F = 0$ has the minimal $c_2(F)$ and χ_F. This proves the corollary. □

Questions.

1) What is the upper bound of $c_2(F)$? We conjecture $c_2(F) \leq \frac{55g}{6}$.

2) Is $\frac{1}{6}$ the lower bound of $c_1^2(F)$ for a minimal non-semistable fiber F?

3) Is the inequality $c_1^2(F) \geq \chi_F$ true for any minimal singular fiber F? (If F is a singular fiber in an isotrivial family, then one can easily prove that $c_1^2(F) \geq \frac{4(g-1)}{g}\chi_F$.)

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