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ON SURFACE SINGULARITIES OF MULTIPLICITY THREE*

JUN LU† AND SHENG-LI TAN†

Dedicated to Prof. Stephen S.-T. Yau on his 60th birthday

Abstract. Let P be a normal singularity of multiplicity $d = 2$ or 3 of a complex surface X. It is well-known that X is locally an irreducible finite cover $\pi : X \to Y$ of degree d over a smooth surface Y, and the singularity (X, P) can be resolved by the canonical resolution $X_k \to X_{k-1} \to \cdots \to X_0 = X$, which is the pullback of the embedded resolution of the corresponding singularity $p = \pi(P)$ of the branch locus. Let F be the maximal ideal cycle of this resolution. We will prove that F has a unique decomposition $F = Z_1 + \cdots + Z_d$ with $Z_1 \geq Z_2 \geq \cdots \geq Z_d \geq 0$, where Z_i is a fundamental cycle or zero. We show that $w = p_a(Z_1) + \cdots + p_a(Z_d)$ is an invariant of (X, P) that can also be computed from the multiplicity of the branch locus at p. (X, P) is a rational singularity iff all of the singular points in the canonical resolution satisfies $w \leq d - 1$. In order to get the minimal resolution from the canonical one, we need to blow down some exceptional curves, the number of blowing-downs is exactly that of fundamental cycles Z in the canonical resolution satisfying $p_a(Z) = 0$ and $Z^2 = -1$.

Key words. Jung's resolution, canonical resolution, fundamental cycle, surface singularity, triple cover.

AMS subject classifications. Primary 14E15; Secondary 14B05, 32S45.

1. Introduction.

Fundamental cycle (defined by M. Artin [2]), maximal ideal cycle (defined by S. S.-T. Yau [17]) and canonical cycle of a resolution are important invariants of a surface singularity. It is well-known that a surface singular point of multiplicity 2 or 3 admits a canonical resolution. Therefore, we get a sequence of maximal ideal cycles and fundamental cycles. Our first purpose is to compute explicitly the maximal ideal cycle of the canonical resolution. Then we will prove that each maximal ideal cycle can be decomposed as a sum of fundamental cycles. We use the sequence of fundamental cycles to define a sequence of numerical invariants. As an application, we will give a new criterion for the singularity to be rational. In order to get the minimal resolution from the canonical resolution, we need to blow down some (-1)-exceptional curves. We will prove that the number of curves blown down in the exceptional set is equal to that of the (-1)-fundamental cycles in the sequence. The main idea is to try to classify surface singularities by their branch loci.

There are several equivalent definitions of the *multiplicity* $\mathrm{mult}_P(X)$ of a singular point P of X. We recall one of them (see [6], p.22). $\mathrm{mult}_P(X)$ is the minimal degree of all finite local covers $\pi : X \to Y$ over a smooth local surface Y.

$$m_P = \mathrm{mult}_P(X) = \min\{\deg \pi \mid \pi : X \to Y \text{ is finite}\}$$

The classical method is to present (X, P) as a finite cover $\pi : X \to Y$ of degree $d = d_P$ over an open set (Y, p) of $\mathbb{C}^2$ at the origin p such that $\pi^{-1}(p) = P$. Then p must be a singular point of the branch locus B_π of π in Y. The most useful resolution

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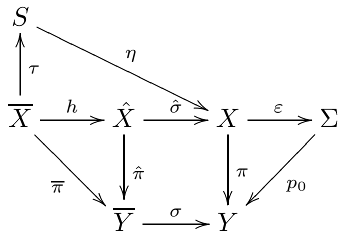
†Department of Mathematics, and Laboratory of Pure Mathematics and Mathematical Practice, East China Normal University, Dongchuan RD 500, Shanghai 200241, P. R. of China ({jlu; sltan}@math.ecnu.edu.cn).

of a normal surface singularity (X, P) is the Jung's resolution as follows.



By a sequence of the blowing-ups $\sigma : \overline{Y} \to Y$ of Y at the singular points of the branch locus, the pullback of B_π in $\overline{Y}$ is a normal crossing divisor. The normalization $\hat{X}$ of $X \times_Y \overline{Y}$ is a finite cover $\hat{\pi} : \hat{X} \to \overline{Y}$ whose branch locus $B_{\hat{\pi}}$ is a normal crossing divisor contained in the pullback of B_π in $\overline{Y}$. Then $\hat{X}$ admits at worst Hirzebruch-Jung singularities which can be resolved directly by the method of Hirzebruch, $h : \overline{X} \to \hat{X}$. Finally, we get the resolution morphism $\overline{\sigma} := \hat{\sigma} \circ h : \overline{X} \to X$ which is called *Jung's resolution* of the singular point (X, P) (see [10]). τ is the contraction of the (-1)-curves in the exceptional set of $\overline{\sigma}$, we get the minimal resolution $\eta : S \to X$.

By Noether's Normalization Theorem, X is the normal model of a local surface $\Sigma \subset \mathbb{C}^3$ defined by an equation of degree d,

$$z^d + a_1(x, y)z^{d-1} + \cdots + a_{d-1}(x, y)z + a_d(x, y) = 0. \tag{1}$$

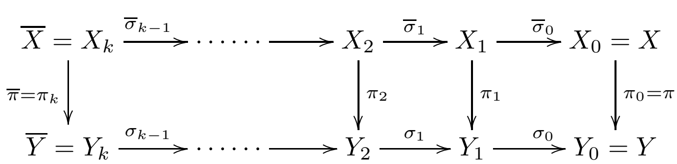
π is the composition of the normalization map $\varepsilon : X \to \Sigma$ with the projection $p_0 : \Sigma \to Y$.

The *maximal ideal cycle* M of a resolution $\overline{\sigma} : (\overline{X}, E_P) \to (X, P)$ is defined as the greatest divisor contained in any divisor of type $\operatorname{div}(\overline{\sigma}^* g)$, where g is any nonzero holomorphic function on X with $g(P) = 0$. Namely,

$$M = \gcd\{ \operatorname{div}(\overline{\sigma}^* g) \mid 0 \neq g \in m_P \subseteq \mathcal{O}_{X,P} \}.$$

If $d = 2$ or 3, then there exists $\sigma = \sigma_0 \circ \cdots \circ \sigma_{k-1}$ such that the branch locus of $\hat{\pi}$ is a smooth curve, thus $\hat{X}$ is a smooth surface, i.e., $\overline{X} = \hat{X}$. This fact is proved by Horikawa for $d = 2$, by Ashikaga [1] for hypersurface triple singularities, and by the second author for the general case with $d = 3$ [12, 13]. This resolution $\overline{\sigma} : \overline{X} = \hat{X} \to X$ of (X, P) is usually called the *canonical resolution.* We explain in detail the process of the canonical resolution.

Let $P_0 = P$ and $p_0 = p = \pi(P)$.



We know that p_0 is a singular point of the branch locus B_{π_0}. Let $\sigma_0 : Y_1 \to Y_0$ be the blowing-up of Y_0 at p_0, let X_1 be the normalization of $X_0 \times_{Y_0} Y_1$, and let $\overline{\sigma}_0$ and π_1 be the induced morphisms. If X_1 is smooth, we stop. Otherwise, we let P_1 be a singular point of X_1, and $p_1 = \pi_1(P_1)$. Note that $d = 2$ or 3. It implies that $\pi_1^{-1}(p_1) = \{ P_1 \}$ or $\{ P_1, P_1' \}$. In the later case, $d = 3$, $d_{P_1} = 2$, $d_{P_1'} = 1$ and P_1' is a smooth point of X_1.

Let $\sigma_1 : Y_2 \to Y_1$ be the blowing up of Y_1 at p_1, let X_2 be the normalization of $X_1 \times_{Y_1} Y_2$, and let $\overline{\sigma}_1$ and π_2 be the induced morphisms. If X_2 is smooth, we stop. Otherwise, we let P_2 be a singular point of X_2 and let $p_2 = \pi_2(P_2)$. We have $\pi_2^{-1}(p_2) = \{ P_2 \}$ or $\{ P_2, P_2' \}$. P_2' must be a smooth point of X_2.

Repeat this process, and after a finite number of steps, X_k must be a smooth surface. Therefore, we get a sequence of surface singular points $\{P = P_0, P_1, \cdots, P_{k-1}\}$, which are called the *infinitely near singular points* of (X, P).

We denote by $E = E_0, E_1, \cdots, E_{k-1}$ the exceptional curves of $\sigma_0, \sigma_1 \cdots, \sigma_{k-1}$, respectively, and by $\mathcal{E}_i$ (resp. $\overline{E}_i$) the total (resp. strict) transformation of E_i in $\overline{Y}$. In the canonical resolution, we let F_i (resp. F_i') be the connected components of $\overline{\pi}^*(\mathcal{E}_i)$ corresponding to P_i (resp. P_i'), i.e.,

$$\overline{\pi}^*(\mathcal{E}_i) = \begin{cases} F_i\ , & \text{if } \pi_i^{-1}(p_i) = \{ P_i \}, \\ F_i + F_i'\ , & \text{if } \pi_i^{-1}(p_i) = \{ P_i\, , P_i' \}, \end{cases}$$

where P_i is an infinitely near singular point of (X, P), so $d_{P_i} \geq 2$. P_i' is always a smooth point of the surface, and $d_{P_i'} = 1$. Let

$$E = E_0, \quad \mathcal{E} = \mathcal{E}_0, \quad \overline{E} = \overline{E}_0, \quad F = F_0, \quad F' = F_0'.$$

Because P_i' is a smooth point of X_i, it is easily to see that F_i' is the fundamental cycle of the first kind, i.e., a (-1)-cycle,

$$p_a(F_i') = 0, \qquad F_i'^2 = -1.$$

F_i and F_i' are disjoint. In fact, we can ignore the smooth points P_i' and the (-1)-cycles F_i'.

In order to compute the maximal ideal cycle, we will decompose F_i as a sum of fundamental cycles.

THEOREM 1.1. *Let $\pi : (X, P) \to (Y, p)$ be a local normal finite cover of degree $d = d_P \leq 3$ over a smooth surface Y such that π is totally ramified over $p = \pi(P)$. Suppose (X, P) is a singularity with multiplicity $m_P \geq 2$. $\overline{\sigma} : \overline{X} \to (X, P)$ is the canonical resolution. $\mathcal{E}$ is the total transform of the first exception curve of the blowing ups $\sigma : \overline{Y} \to Y$. In this case, $F = \overline{\pi}^*(\mathcal{E})$ is connected.*

Then there are $\ell = \ell_P \leq d_P$ fundamental cycles $Z = Z_1 > Z_2 > \cdots > Z_\ell > 0$ such that

$$F = Z_1 + Z_2 + \cdots + Z_\ell\ , \quad Z_i Z_j = 0 \ \text{ for } \ \mathrm{i} \neq \mathrm{j}. \tag{2}$$

Note that $m_P < d_P$ implies $m_P = 2$ and $d_P = 3$.

THEOREM 1.2. *With the same notation and assumption as in the previous theorem. Let M be the maximal ideal cycle under the canonical resolution.*

1. *If $2 \leq m_P = d_P \leq 3$, then $M = F = Z_1 + Z_2 + \cdots + Z_\ell$.*
2. *If $m_P = 2$, $d_P = 3$ and $\ell_P = 2$, then $M = Z_1$.*
3. *If $m_P = 2$, $d_P = 3$ and $\ell_P = 3$, then $M = Z_1$ or $M = Z_1 + Z_2$.*

DEFINITION 1.1. We call $\ell = \ell_P$ in the decomposition (2) as the *length* of (X, P). Z_j is called the *j-th fundamental cycle* of (X, P). Z_1 is the *fundamental cycle* in the usual sense. Let

$$w_P := p_a(Z_1) + \cdots + p_a(Z_{\ell_P}) + d_P - \ell_P. \tag{3}$$

Note that if we set $Z_{\ell_P+1} = \cdots = Z_{d_P} = 0$, then $p_a(Z_j) = 1$ for $j > \ell_P$, and

$$w_P = p_a(Z_1) + \cdots + p_a(Z_{d_P}). \tag{4}$$

In the canonical resolution of (X, P), we get a sequence of infinitely near singular points $\{ P = P_0, P_1, \cdots, P_{k-1} \}$. Thus we get a sequence of numerical invariants $w = w_0, w_1, \cdots, w_{k-1}$, where $w_i = w_{P_i}$.

We would like to point out that $w_0, w_1, \cdots, w_{k-1}$ are invariants of the singularity p of the branch locus B_π.

If $d = d_P = 2$, then $w_i = [m_i/2]$ in Horikawa's notation, where $m_i's$ are the multiplicities $\mathrm{mult}_{p_i}(B_{\pi_i})$ of the singular points of the branch locus. Then we have the well-known formula for the geometric genus of (X, P).

$$p_g(X, P) = \sum_{i=0}^{k-1} \frac{1}{2} w_i(w_i - 1).$$

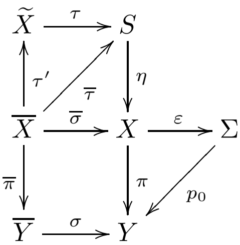
If $d = d_P = 3$, then the second author found a similar computation formula for w_i by using the singularities of the branch locus, (see § 4.3, or [13], Theorem 6.4.) In fact, (3) or (4) is a unified formula for the invariants w_i defined respectively by Horikawa and the second author. It gives a direct relationship between the singularities of the branch locus and the singularities of the surface. Therefore, we can use the invariants $w_0, w_1, \cdots, w_{k-1}$ of the branch locus to classify the surface singularity (X, P).

THEOREM 1.3. *Suppose the surface singularity $(X, P) \to (Y, p)$ is a finite cover of degree $d = 2$ or 3 totally ramified over p. Let $P = P_0, P_1, \cdots, P_{k-1}$ be the infinitely near singular points obtained in the canonical resolution. Then*

1. *$w_i \geq 1$ for $i = 0, 1, \cdots, k-1$.*
2. *(X, P) is a rational singular point iff $w_i \leq d - 1$ for any i.*
3. *If (X, P) is rational, then the multiplicity of (X, P) is $w_0 + 1$.*

In order to get the minimal resolution $\eta : S \to (X, P)$, we need to blow down the (-1)-curves in the exceptional set of the canonical resolution, $\overline{\tau} : \overline{X} \to S$.

The first step is to contract the obvious (-1)-cycles F_i'. We get a smooth surface $\widetilde{X}$, $\tau' : \overline{X} \to \widetilde{X}$. Then we need to contract the (-1)-curves in $F_0, F_1, \cdots, F_{k-1}$, $\tau : \widetilde{X} \to S$, we get the minimal resolution $\eta : S \to (X, P)$.



DEFINITION 1.2. $J = \eta \circ \tau : \widetilde{X} \to X$ is usually called the *Jung's resolution* of (X, P).

Because τ consists of blowing ups at smooth points, the total transformations $\mathcal{D}_1, \cdots, \mathcal{D}_r$ of the exceptional curves $D_1, \cdots, D_r$ of τ are (-1)-cycles in $\widetilde{X}$. The number $r = r(J)$ of curves contracted by τ is determined by the Jung's resolution J.

The following fundamental problems remain open (see [10]).

PROBLEM 1.1. Adapt Jung's resolution to get embedded resolution of germs of surfaces Σ in $\mathbb{C}^3$

PROBLEM 1.2. Use Jung's resolution to get obstructions on the topology of local surfaces with isolated singularities in $\mathbb{C}^3$.

PROBLEM 1.3. Fix the topology of (X, P). Is $r = r(J)$ bounded from above? Compute it in terms of the weighted dual graph of the minimal good resolution of (X, P) (which encodes the topology of (X, P), as ensured by a theorem of Neumann).

Suppose the decompositions of $F_0, F_1, \cdots, F_{k-1}$ are as follows.

$$\begin{aligned} F_0 &= Z_1 + \cdots + Z_{\ell_0}\ , \\ F_1 &= Z_1' + \cdots + Z_{\ell_1}'\ , \\ F_2 &= Z_1'' + \cdots + Z_{\ell_2}''\ , \\ &\vdots \\ F_{k-1} &= Z_1^{(k-1)} + \cdots + Z_{\ell_{k-1}}^{(k-1)}. \end{aligned}$$

We will prove that all of the components $Z_j^{(i)}$ are different. Let

$$\mathcal{F}und(J) := \left\{ \, Z_j^{(i)} \, \middle| \text{ for any } i,\ j \, \right\}$$

THEOREM 1.4. *$\mathcal{D}_1, \cdots, \mathcal{D}_r$ are exactly the (-1)-cycles in the set $\mathcal{F}und(J)$ of fundamental cycles.*

$$\{\ \mathcal{D}_1, \cdots, \mathcal{D}_r\ \} = \{\ Z \in \mathcal{F}und(J) \mid p_a(Z) = 0,\ Z^2 = -1\ \}.$$

So the number $r = r(J)$ of curves contracted by τ can be computed from the decompositions. This result allows us to determine the curves contracted by τ from the singularities of the branch locus. For a surface singularity of multiplicity 2, Xiao determined the number r of the curves contracted by τ using a different method. Xiao proves that r is equal to the number of singularities of types $(2w+1 \to 2w+1)$ of the branch locus. In fact, such a singular point has positive contribution to the geometric genus. As a consequence, we get

$$r(J) \leq p_g(X, P).$$

We will try to get a similar classification for surface singularities of multiplicity 3.

2. Fundamental cycles and the canonical resolution.

2.1. Fundamental cycles. Let (X, P) be an isolated surface singularity and $\overline{\sigma} : (\overline{X}, E_P) \to (X, P)$ be a resolution, where E_P is the set of exceptional curves. There is a unique divisor Z supported on E_P such that $Z\Gamma \leq 0$ for any component Γ in E_P, and Z is minimal with respect to this property (see [2]) or [7, Sec.4.5]). Such a minimal cycle Z is called the *fundamental cycle* of E_P, or of the resolution.

In general, for a connected subset E' of E_P, we can also define a fundamental cycle Z' whose support is E', i.e., Z' is a minimal cycle satisfying $Z'\Gamma \le 0$ for any curve Γ in E'. We simply say that Z' is a fundamental cycle.

We can obtain Z by the computational sequence: Let Z_1 be any fundamental cycle of its support (e.g., Z_1 is one component of the exceptional set), choose a component Γ_1 such that $Z_1\Gamma_1 > 0$, and let $Z_2 = Z_1 + \Gamma_1$. Choose a component Γ_2 such that $Z_2\Gamma_2 > 0$ and let $Z_3 = Z_2 + \Gamma_2, \cdots$, after a finite number of steps, we get $Z = Z_\ell$. It is well-known that $p_a(Z_i) = h^1(\mathcal{O}_{Z_i})$ and

$$p_a(Z) = p_a(Z_\ell) \ge p_a(Z_{\ell-1}) \ge \cdots \ge p_a(Z_1) \ge 0.$$

LEMMA 2.1. *If Z' and Z are two fundamental cycles, and the support of Z' is contained in that of Z, then $Z' \le Z$ and $p_a(Z') \le p_a(Z)$.*

LEMMA 2.2. *Let A and B be positive cycles supported on E_P. Suppose A is connected and $BC \le 0$ for any component C in A. Then either the support of A is contained in that of B, or A and B are disjoint.*

Proof. Let $A = A_1 + A_2$ be the decomposition such that the support of A_1 is contained in B, and A_2 has no common component with B. Then we have $A_2B \ge 0$. By assumption, $A_2B \le 0$, so $A_2B = 0$ and A_2 are disjoint from B. Hence A_2 is disjoint from A_1. Because A is connected, we get either $A_1 = 0$ or $A_2 = 0$. This is what we desired. □

LEMMA 2.3. *Suppose A and B are two distinct positive cycles supported on E_P. If $A^2 = B^2 = -1$, then $AB = 0$. Furthermore, if A and B are not disjoint, then $A \ge B$ or $B \ge A$.*

Proof. Because $A \ne B$, $A \pm B$ is not zero, we have

$$(A \pm B)^2 = -1 - 1 \pm 2AB < 0,$$

so $AB = 0$.

If A and B are not disjoint, then they have at least one common component. Let C be the biggest common positive cycle such that $A \ge C$ and $B \ge C$. Let $A' = A - C$ and $B' = B - C$. Then A' and B' have no common component. So $A'B' \ge 0$. Now we claim that either $A' = 0$ or $B' = 0$. Otherwise, $A'^2 \le -1$ and $B'^2 \le -1$.

$$-2 = (A - B)^2 = (A' - B')^2 = A'^2 + B'^2 - 2A'B' \le -2,$$

so $A'B' = 0$, $A'^2 = -1$ and $B'^2 = -1$. Note that $A' \ne B$, $B' \ne A$, and $A^2 = B^2 = A'^2 = B'^2 = -1$, by the proof of the first part, we get $AB' = A'B = 0$. So

$$C^2 = (A - A')(B - B') = AB + A'B' - AB' - AB' = 0,$$

it implies $C = 0$, a contradiction. □

2.2. Maximal ideal cycle, canonical cycle and $(-n)$-cycles. The *maximal ideal cycle* M of E_P is defined as the greatest divisor contained in every divisor of type $\operatorname{div}(\overline{\sigma}^*g)$, where g is any nonzero holomorphic function on X with $g(P) = 0$. Namely,

$$M = \gcd\{\, \operatorname{div}(\overline{\sigma}^*g) \mid 0 \ne g \in m_P \subseteq \mathcal{O}_{X,P}\, \}.$$

It is easy to see that $Z \leq \operatorname{div}(\overline{\sigma}^* g)$ for any g in m_P, so $Z \leq M$. The equality holds in some important cases: for example, rational singularities and elliptic Gorenstein singularities (see [7, Theorem 4.17 and 4.23]). If $m_P\mathcal{O}_Y = \mathcal{O}_Y(-M)$, then $\operatorname{mult}_P(X) = -M^2$ (see [7, Page 85]).

Let K be the unique $\mathbb{Q}$-divisor supported on the exceptional set E_P such that $K\Gamma + \Gamma^2 = 2p_a(\Gamma) - 2$ for any component Γ of E_P. K is called the *canonical cycle* of (X, P).

An effective divisor D supported on some exceptional curves in E_P is called a *$(-n)$-cycle* if D is the fundamental cycle of its support and

$$p_a(D) = 0, \qquad D^2 = -n.$$

By Artin's theorem [2], (-1)-cycle can be contracted to a smooth point of a surface. It is well-known that the arithmetic genus $p_a(Z) \geq 0$ for any fundamental cycle Z.

2.3. Canonical resolution of double and triple covers. In what follows, we will try to compute the cycles defined in the previous section for the canonical resolution $\overline{\sigma} : \overline{X} \to (X, P)$ explained in the introduction.

Note that in the Picard group of $\overline{Y}$, $\{\overline{E}_0, \cdots, \overline{E}_{k-1}\}$ and $\{\mathcal{E}_0, \cdots, \mathcal{E}_{k-1}\}$ generate the same subgroup.

$$\mathbb{Z}\overline{E}_0 + \cdots + \mathbb{Z}\overline{E}_{k-1} = \mathbb{Z}\mathcal{E}_0 + \cdots + \mathbb{Z}\mathcal{E}_{k-1}\,. \tag{5}$$

As generators, $\mathcal{E}_0, \cdots, \mathcal{E}_{k-1}$ are convenient for computation. For any i and $j \neq i$, we have

$$\mathcal{E}_i^2 = -1, \quad \mathcal{E}_i \cdot \mathcal{E}_j = 0. \tag{6}$$

$\mathcal{E}_i$ is a (-1)-fundamental cycle. For $i > j$, either $\mathcal{E}_i$ and $\mathcal{E}_j$ are disjoint, or $\mathcal{E}_i < \mathcal{E}_j$. For any exceptional curve Γ of σ in $\overline{Y}$,

$$\Gamma \cdot \mathcal{E}_i = \begin{cases} -1, & \text{if } \Gamma = \overline{E}_i\,, \\ 0, & \text{if } \Gamma \neq \overline{E}_i \text{ is contained in } \mathcal{E}_i\,, \\ \geq 0, & \text{otherwise.} \end{cases} \tag{7}$$

Lemma 2.4. *In the subgroup $\mathbb{Z}\overline{E}_0 + \cdots + \mathbb{Z}\overline{E}_{k-1}$ of $\operatorname{Pic}(\overline{\mathrm{Y}})$, $\mathcal{E}_0, \cdots, \mathcal{E}_{k-1}$ are the only effective divisors whose self-intersection numbers are -1.*

Proof. Let D be an effective divisor in this subgroup, by (5), we can write $D = n_0\mathcal{E}_0 + \cdots + n_{k-1}\mathcal{E}_{k-1}$ for some integers $n_0, \cdots, n_{k-1}$. If $D^2 = -1$, then we have

$$-1 = -n_0^2 - \cdots - n_{k-1}^2,$$

so there exists an i such that $n_i = \pm 1$, $n_j = 0$ for $j \neq i$. Because D is effective, $D = \mathcal{E}_i$. □

Lemma 2.5. *Let Z be a fundamental cycle containing the support of a (-1)-cycle $\mathcal{D}$ on a smooth surface X, and let $\gamma : X \to S$ be the contraction map of the curves in $\mathcal{D}$ to a smooth surface S. Then either $Z = \mathcal{D}$ or $Z = \gamma^*\gamma_*(Z)$. In particular, if $Z \neq \mathcal{D}$, then $Z\mathcal{D} = 0$.*

Proof. Suppose $Z \neq \mathcal{D}$. From the uniqueness of the fundamental cycle, we see that $\mathrm{Supp}(\mathcal{D}) \subsetneq \mathrm{Supp}(Z)$. We see that at least one curve C in Z is not contracted by γ. For such a curve C,

$$C \cdot \gamma^*\gamma_*(Z) = \gamma^*\gamma_*(C) \cdot \gamma^*\gamma_*(Z) = \gamma^*\gamma_*(C) \cdot Z \leq 0.$$

If C' is contracted by γ, then $C' \cdot \gamma^*\gamma_*(Z) = 0$. From the minimality of fundamental cycle, we have that $Z \leq \gamma^*\gamma_*(Z)$, so $\gamma^*\gamma_*(Z) = Z + A$ for some effective divisor A whose support consists of curves contracted by γ. Hence

$$0 = \gamma^*\gamma_*(Z)A = ZA + A^2 \leq A^2,$$

we have $A^2 = 0$ and $A = 0$. Thus $Z = \gamma^*\gamma_*(Z)$ and $Z\mathcal{D} = 0$. □

2.4. Computation of the maximal ideal cycle. The following theorem is known for surface singularities of multiplicity 2.

THEOREM 2.1. *Let (X, P) be a normal surface singularity, let $\pi : (X, P) \to (Y, p)$ be a finite cover of degree $d = d_P \leq 3$ over a smooth surface Y totally ramified over $p = \pi(P)$, and let $\overline{\sigma} : \overline{X} \to X$ be the canonical resolution. Let $F = \overline{\pi}^*\mathcal{E} = \overline{\pi}^*\mathcal{E}_0$. If the multiplicity m_P of (X, P) is equal to the local degree d_P, then the maximal ideal cycle M of (X, P) under $\overline{\sigma}$ is equal to F.*

Proof. Because the case when $d = 2$ is well-known, we assume that $d = 3$. By [9], X is a local surface in $\mathbb{C}^4$ defined by the following equations.

$$\begin{cases} z^2 = az + bw + 2A, \\ zw = -dz - aw - B, \\ w^2 = cz + dw + 2C, \end{cases} \tag{8}$$

where $a, b, c, d \in \mathcal{O}_{Y,p}$ and $A = a^2 - bd$, $B = ad - bc$ and $C = d^2 - ac$. Suppose $m_p \subset \mathcal{O}_{Y,p}$ is generated by x and y, then $m_P \subset \mathcal{O}_{X,P}$ is generated by x, y, z and w. Since π is totally ramified over p, we have $A(p) = B(p) = C(p) = 0$.

If $b(p) \neq 0$, then b is invertible, from the defining equations, we can eliminate w. Namely, (X, P) is a hypersurface singularity defined by

$$z^3 + sz + t = 0, \tag{9}$$

where $s = -3A$ and $t = bB - 2aA$. Because $\mathrm{mult}_P(X) = 3$, we have $\nu_p(s) \geq 2$ and $\nu_p(t) \geq 3$. Now from

$$\overline{\sigma}^*(z)^3 + \overline{\pi}^*(\sigma^*(s))\overline{\sigma}^*(z) + \overline{\pi}^*(\sigma^*(t)) = 0,$$

we see that for any element $g \in m_p$ with $\nu_p(g) = 1$, we have

$$\nu_\Gamma(\overline{\sigma}^*(z)) \geq \nu_\Gamma(\overline{\pi}^*\sigma^*(g))$$

for any exceptional curve Γ of $\overline{\sigma}$. In particular,

$$\nu_\Gamma(\overline{\sigma}^*(z)) \geq \nu_\Gamma(\overline{\pi}^*\sigma^*(x)), \quad \nu_\Gamma(\overline{\sigma}^*(z)) \geq \nu_\Gamma(\overline{\pi}^*\sigma^*(y)). \tag{10}$$

In this case, m_P is generated by x, y and z. Hence

$$\begin{aligned} F &= \gcd\{\,\mathrm{div}(\overline{\sigma}^*(\pi^*(x))),\ \mathrm{div}(\overline{\sigma}^*\pi^*(y)),\ \mathrm{div}(\overline{\sigma}^*(z))\,\} \\ &= \gcd\{\,\mathrm{div}(\overline{\pi}^*(\sigma^*(x))),\ \mathrm{div}(\overline{\pi}^*\sigma^*(y)),\ \mathrm{div}(\overline{\sigma}^*(z))\,\} \\ &= \gcd\{\,\mathrm{div}(\overline{\pi}^*(\sigma^*(x))),\ \mathrm{div}(\overline{\pi}^*\sigma^*(y))\,\} \\ &= \overline{\pi}^*(\mathcal{E}_0). \end{aligned}$$

If $c(p) \neq 0$, the proof is similar.

Now suppose $b(p) = c(p) = 0$, from $A(p) = B(p) = C(p) = 0$, we have also $a(p) = 0$ and $d(p) = 0$. Now from the defining equation (8), we can get (10) and

$$\nu_\Gamma(\overline{\sigma}^*(w)) \geq \nu_\Gamma(\overline{\pi}^*\sigma^*(x)), \quad \nu_\Gamma(\overline{\sigma}^*(w)) \geq \nu_\Gamma(\overline{\pi}^*\sigma^*(y)). \tag{11}$$

We obtain similarly that $F = \overline{\pi}^*(\mathcal{E}_0)$. □

2.5. Decomposition of F. Let $\pi : (X, P) \to (Y, p)$ be a normal finite cover of degree $d \leq 3$ over a smooth surface Y such that π is totally ramified over $p = \pi(P)$, let $\overline{\sigma} : \overline{X} \to X$ be the canonical resolution.

$$\begin{array}{ccccccccc}
\overline{X} = X_k & \xrightarrow{\overline{\sigma}_{k-1}} \cdots\cdots \longrightarrow & X_2 & \xrightarrow{\overline{\sigma}_1} & X_1 & \xrightarrow{\overline{\sigma}_0} & X_0 = X \\
\downarrow{\scriptstyle \overline{\pi}=\pi_k} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0=\pi} \\
\overline{Y} = Y_k & \xrightarrow{\sigma_{k-1}} \cdots\cdots \longrightarrow & Y_2 & \xrightarrow{\sigma_1} & Y_1 & \xrightarrow{\sigma_0} & Y_0 = Y
\end{array}$$

We have defined the local degree $d = d_P$, the length $\ell = \ell_P$ and the multiplicity m_P of (X, P). $F_P = \overline{\pi}^*\mathcal{E}_P$, and M_P is the maximal ideal cycle of (X, P) under the canonical resolution $\overline{\sigma}$.

Note that we have a sequence of infinitely near singular points $P = P_0, P_1, \cdots, P_{k-1}$ with local degrees $d = d_0, d_1, \cdots, d_{k-1}$, local lengths $\ell = \ell_0, \ell_1, \cdots, \ell_{k-1}$ and multiplicities $m = m_0, m_1, \cdots, m_{k-1}$. The corresponding sequence of connected cycles are denoted by $F = F_0, F_1, \cdots, F_{k-1}$.

Theorem 2.2. *Suppose $d = d_0 \leq 3$. Then there are $\ell = \ell_0 \leq d$ fundamental cycles $Z_1 > Z_2 > \cdots > Z_{\ell_0} > 0$ such that*

$$F = F_0 = Z_1 + Z_2 + \cdots + Z_{\ell_0}, \quad Z_i Z_j = 0 \ \text{ for } \ \mathrm{i} \neq \mathrm{j}. \tag{12}$$

The decomposition is unique. In particular, we have

$$p_a(Z_1) \geq p_a(Z_2) \geq \cdots \geq p_a(Z_{\ell_0}) \geq 0. \tag{13}$$

Proof. We have proved that $F_0 = \overline{\pi}^*(\mathcal{E}_0)$. So $F_0^2 = -d_0 \geq -3$. Note that for any component Γ of F_0, we have $\Gamma F_0 \leq 0$. The fundamental cycle $Z_1 = Z$ supported on F_0 is the minimal effective divisor satisfying $Z\Gamma \leq 0$. So $Z_1 \leq F_0$ and $A := F_0 - Z_1$ is an effective divisor, which implies that $AZ_1 \leq 0$.

$$-3 \leq F_0^2 = Z_1^2 + 2Z_1A + A^2 \leq Z_1^2.$$

If $A = 0$, the proof is completed. We assume that $A \neq 0$. Thus $A^2 < 0$. Since $F_0^2 \geq -3$, we have

$$-3 \leq A^2 + Z_1^2 + 2AZ_1 \leq -2 + 2AZ_1,$$

so $AZ_1 \geq 0$. Because $AZ_1 \leq 0$, we have $AZ_1 = 0$, which implies $\Gamma Z_1 = 0$ for each component $\Gamma \leq A$. Thus

$$\Gamma \cdot A = \Gamma \cdot F_0 - \Gamma \cdot Z_1 = F_0 \cdot \Gamma = \mathcal{E}_1 \cdot \overline{\pi}_*\Gamma \leq 0.$$

From $F_0^2 = Z_1^2 + A^2$, we get $A^2 = -1$ or -2.

Let Z_2 be the fundamental cycle on the support of A. Then $Z_2 \leq A$, so $Z_2 Z_1 = 0$. Let $B = A - Z_2$. If $B = 0$, we are done. Suppose $B \neq 0$. Then $B^2 \leq -1$ and $BZ_2 \leq 0$. From

$$-2 \leq A^2 = Z_2^2 + 2BZ_2 + B^2 \leq -2 + 2BZ_2 \leq -2,$$

we see that $BZ_2 = 0$, $B^2 = Z_2^2 = -1$, $A^2 = -2$. Since $BZ_2 = 0$, we get $\Gamma Z_2 = 0$ for any component $\Gamma \leq B$, and

$$\Gamma B = \Gamma A - \Gamma Z_2 = \Gamma A \leq 0.$$

Let Z_3 be the fundamental cycle of the support of B. Then we have $Z_3 Z_2 = Z_3 Z_1 = 0$. Take $C = B - Z_3$. Then $CZ_3 \leq 0$.

$$-1 = B^2 = Z_3^2 + 2CZ_3 + C^2 \leq -1 + C^2 \leq -1,$$

so $C^2 = 0$, which implies $C = 0$.

Therefore, $Z_i Z_j = 0$ $(i \neq j)$. By Lemma 2.1, $Z_i \geq Z_{i+1}$. Because $Z_i \cdot Z_{i+1} = 0$, we have $Z_i > Z_{i+1}$. Hence

$$d_0 = -F_0^2 = -Z_1^2 - Z_2^2 - \cdots - Z_{\ell_0}^2 \geq \ell_0.$$

The inequalities (13) are well-known facts about fundamental cycles (see Lemma 2.1). ∎

Note that $-F_P$ is nef with respect to the exceptional curves. From the theorem, $Z_i^2 = Z_i F$, thus we have

$$Z_1^2 \leq Z_2^2 \leq \cdots \leq Z_\ell^2.$$

Since $F^2 = Z_1^2 + \cdots + Z_\ell^2 = -d$, we see that if $Z_1^2 = -1$, then $\ell = d$ and $Z_i^2 = -1$ for any i.

Theorem 2.3. *With the notations as in the previous theorem, we have*

1. *If $d_P = 3$, $m_P = 2$ and $\ell_P = 2$, then $M_P = Z_1$.*
2. *If $d_P = 3$, $m_P = 2$ and $\ell_P = 3$, then $M_P = Z_1$ or $M_P = Z_1 + Z_2$.*

Proof. We use the well-known fact that $-M_P^2 \leq m_P = 2$. Since ${F_P}^2 = -3$, we see that $M_P \neq F_P$. Note that $M_P \Gamma \leq 0$ for any component in the exceptional set. So $M_P \geq Z_1$. If $M_P = Z_1$, then (1) and (2) are proved.

Now we assume that $M_P = Z_1 + D$ for some effective nonzero divisor D.

$$-2 \leq {M_P}^2 = Z_1^2 + D^2 + 2Z_1 D \leq -1 - 1,$$

we have $M_P^2 = -2$, $Z_1^2 = D^2 = -1$ and $Z_1 D = 0$. So $\ell_P = 3$, $F_P = Z_1 + Z_2 + Z_3$, and $Z_1^2 = Z_2^2 = Z_3^2 = -1$.

Now we prove that $D \neq Z_3$. Otherwise, $M_P = Z_1 + Z_3$, $(Z_2 - Z_3)M_P = -Z_3^2 = 1$, which contradicts the nefness of $-M_P$ on the exceptional set. By Lemma 2.3, we get that $DZ_3 = 0$.

Suppose $D \neq Z_2$, by Lemma 2.3, we have $DZ_2 = 0$. Then $DF_P = DZ_1 + DZ_2 + DZ_3 = 0$. So $F_P M_P = F_P Z_1 = -1$.

On the other hand, take a generic smooth curve C on Y passing through p, and we let $g = 0$ is its defining equation. Then $\pi^*(g)$ is a holomorphic function on X vanishing on P. One can see that $\mathrm{div}(\overline{\sigma}^*(\pi^*(g)) = \mathrm{div}(\overline{\pi}^*(\sigma^*(g))) = F + \overline{C}$, where

$\overline{C}$ is the strict transform of the curve C on $\overline{X}$. Thus $F_P \geq M_P$, which implies that $1 = -1 - (-2) = (F_P - M_P)M_P \leq 0$, a contradiction.

Therefore, $D = Z_2$ and $M_P = Z_1 + Z_2$. This completes the proof. □

Suppose the decompositions of $F_0, F_1, \cdots$ are as follows.

$$\begin{aligned} F_0 &= Z_1 + \cdots + Z_{\ell_0} \ , \\ F_1 &= Z'_1 + \cdots + Z'_{\ell_1} \ , \\ F_2 &= Z''_1 + \cdots + Z''_{\ell_2} \ , \\ &\vdots \\ F_{k-1} &= Z_1^{(k-1)} + \cdots + Z_{\ell_{k-1}}^{(k-1)}. \end{aligned}$$

Corollary 2.1. *$Z_j^{(i)} = Z_{j'}^{(i')}$ if and only if $i = i'$ and $j = j'$.*

Proof. Suppose that $Z_j^{(i)} = Z_{j'}^{(i')} > 0$. Let $\overline{\pi}^*\mathcal{E}_i = Z_j^{(i)} + A$, $\overline{\pi}^*\mathcal{E}_{i'} = Z_j^{(i)} + B$. From Theorem 2.2, if $i = i'$, then $j = j'$. Now we assume that $i' \neq i$, so $\mathcal{E}_{i'} \cdot \mathcal{E}_i = 0$. Then $A - B = \overline{\pi}^*\mathcal{E}_i - \overline{\pi}^*\mathcal{E}_{i'}$ implies

$$(A - B)^2 = (\overline{\pi}^*\mathcal{E}_i)^2 + (\overline{\pi}^*\mathcal{E}_{i'})^2 = -2d_0 \leq -4.$$

If $Z_j^{(i)} \cdot Z_j^{(i)} = -d_0$, then $A = B = 0$, and $(A - B)^2 = 0$, a contradiction. If $Z_j^{(i)} \cdot Z_j^{(i)} = -d_0 + 1$, then $A^2 = B^2 = -1$, hence $AB = 0$ by Lemma 2.3. So $(A - B)^2 = -2$, a contradiction. Therefore $Z_j^{(i)} \cdot Z_j^{(i)} \geq -d_0 + 2$. It implies $d_0 = 3$ and $Z_j^{(i)} \cdot Z_j^{(i)} = -1$. From Theorem 2.2, $A^2 = B^2 = -2$ and $Z_{j'}^{(i')} \cdot B = Z_j^{(i)} \cdot B = 0$.

Without a loss of generality, we assume that $\mathcal{E}_i > \mathcal{E}_{i'}$. So $\overline{\pi}^*\mathcal{E}_i \cdot B = \mathcal{E}_i \cdot \overline{\pi}_* B = 0$. Thus $AB = \overline{\pi}^*\mathcal{E}_i \cdot B - Z_j^{(i)} \cdot B = 0$. On the other hand, $(A - B)^2 = -6$ implies $AB = 1$, a contradiction. □

We call (d_P, ℓ_P) the *type* of the singularity (X, P) or the local finite cover $(X, P) \to (Y, p)$. For the infinitely near singular points $P = P_0, P_1, \cdots, P_{k-1}$ in the canonical resolution, we have a sequence of types:

$$(d_0, \ell_0),\ (d_1, \ell_1),\ \cdots,\ (d_{k-1}, \ell_{k-1}).$$

In what follows, we are trying to find the relationships between the types and fundamental cycles of (X, P) and (X_1, P_1).

2.6. The case $\ell_0 = d_0$.

Corollary 2.2. *Let Γ be an irreducible component of Z_{ℓ_0} such that $Z_{\ell_0}\Gamma < 0$. Then we have $\overline{\pi}^*\overline{E}_0 \geq \ell_0\Gamma$. Therefore, if $\ell_0 = \ell_{P_0} \geq 2$, then $\overline{E}_0$ lies in the branch locus.*

Moreover, if $\ell_{P_0} = d_0$, then $\overline{\pi}$ is totally ramified over $\overline{E}_0$ and the multiplicity of Γ in each Z_i is 1. In particular, $\overline{\pi}^*\overline{E}_0 = d_0\Gamma$.

Proof. Since $Z_\ell Z_i = 0$ and $Z_{\ell_0} < Z_i$ for any $i < \ell_0$, $\Gamma Z_i = 0$. Thus

$$\mathcal{E}_0 \cdot \overline{\pi}_*\Gamma = \overline{\pi}^*\mathcal{E}_0 \cdot \Gamma = Z_{\ell_0}\Gamma < 0.$$

So $\overline{\pi}(\Gamma) = \overline{E}_0$. Since $\Gamma \leq Z_i$ for each i, $\overline{\pi}^*\mathcal{E}_0 = Z_1 + \cdots + Z_{\ell_0} \geq \ell_0\Gamma$. Hence $\overline{\pi}^*\overline{E}_0 \geq \ell_0\Gamma$, i.e., π_1 is ramified over E_0. Furthermore, if $\ell_0 = d_0$, then $\overline{\pi}^*\overline{E}_0 = d_0\Gamma$, i.e., π_1 is totally ramified over E_0. In particular, the multiplicity of Γ in each Z_i is 1. ∎

THEOREM 2.4. *Suppose $\ell_0 = d_0 \le 3$. Then each fundamental cycle Z_i in $F_0 = Z_1 + \cdots + Z_{\ell_0}$ satisfies $Z_i^2 = -1$ and π_1 is totally ramified over E_0. Suppose P_1 is any singular point of X_1. Then $\pi_1^{-1}(p_1) = \{ P_1 \}$ and $\ell_1 = \ell_{P_1} \ge d_0 - 1$.*

1. If $\ell_1 = d_0 - 1$, then $Z_1' = Z_1 - Z_{\ell_0}$. Therefore we have

$$Z_1'^2 = -2, \qquad Z_1'Z_1 = -1, \qquad Z_1'Z_{\ell_0} = 1.$$

2. If $\ell_1 = d_0$, then $Z_i'Z_j = 0$ for any i and j, and

$$Z_{\ell_1}' < Z_{\ell_1-1}' < \cdots < Z_1' < Z_{\ell_0} < Z_{\ell_0-1} < \cdots < Z_2 < Z_1.$$

3. X_1 admits a singular point, say P_1, of type (1).

Proof. Since $-Z_i^2 \ge 1$ $(i = 1, \cdots, \ell_0 = d_0)$ and

$$d_0 = -Z_1^2 - Z_2^2 - \cdots - Z_{d_0}^2,$$

we have $-Z_i^2 = 1$ for all i.

Let $D = Z_1 - Z_{\ell_0}$. Let Γ be a component of Z_{ℓ_0} with $Z_{\ell_0}\Gamma < 0$. By Corollary 2.2, D does not contain Γ. Since $Z_{\ell_0}^2 = -1$, Z_{ℓ_0} is connected and Γ is the only component with $\Gamma \cdot Z_{\ell_0} < 0$. For any component $C \ne \Gamma$ of Z_1, $Z_{\ell_0}C \ge 0$ and $Z_1C \le 0$. We see that $DC \le 0$. Applying Lemma 2.2 to $A = Z_1'$ and $B = D$, we see that either $D \ge Z_1'$ or D and Z_1' are disjoint.

We claim that $CZ_1' \le 0$ for any irreducible exceptional component $C \ne \Gamma$. Otherwise, $CZ_1' > 0$ implies Z_1' doesn't contain C and hence $\overline{\pi}_*C \cdot \mathcal{E}_1 = C \cdot \overline{\pi}^*\mathcal{E}_1 > 0$. Thus $\overline{\pi}(C) = \overline{E}_0$ by (7), which implies $C = \Gamma$ by Corollary 2.2, a contradiction.

(1) Assume that $D \ge Z_1'$. Let $D' = D - Z_1'$. Since D' does not contain Γ, the support of D' is contained in Z_1'. Thus $D'Z_1' \le 0$, and

$$-2 = D^2 = D'^2 + Z_1'^2 + 2D'Z_1'.$$

If $D'Z_1' < 0$, then $D'^2 = Z_1'^2 = 0$, a contradiction. So $D'Z_1' = 0$.

Now we claim that $D' = 0$, i.e., $D = Z_1'$. Otherwise, $D'^2 = Z_1'^2 = -1$. Hence $Z_1'D = Z_1'D' + Z_1'^2 = -1$. Note that Z_1', Z_1 and Z_{ℓ_0} are distinct positive cycles with $Z_1'^2 = Z_1^2 = Z_{\ell_0}^2 = -1$, we get by Lemma 2.3 that $Z_1'Z_1 = Z_1'Z_{\ell_0} = 0$. Therefore $Z_1'D = Z_1'Z_1 - Z_1'Z_{\ell_0} = 0$, which implies that $Z_1'^2 = DZ_1' - D'Z_1' = 0$, a contradiction.

Moreover we have $Z_1'^2 = D^2 = -2$, $Z_1'Z_1 = Z_1^2 = -1$, $Z_1'Z_{\ell_0} = -Z_{\ell_0}^2 = 1$ and $Z_1'Z_i = 0$ for $i \ne 1, \ell_0$. This is case 1).

(2) Assume that D and Z_1' are disjoint. Since $Z_1 = D + Z_{\ell_0} \ge Z_1'$, $Z_{\ell_0} \ge Z_1'$. By Theorem 2.2, one gets $Z_1'Z_i = 0$ for any $i < \ell_0$. Note that $\ell_0 = d_0 \ge 2$, we get $Z_1Z_1' = 0$, so Γ is not contained in Z_1'. Since $Z_{\ell_0}^2 = -1$, we know that Γ is the unique component of Z_{ℓ_0} with $Z_{\ell_0}\Gamma < 0$, and for any other component C of Z_{ℓ_0}, we have $Z_{\ell_0}C = 0$. From $Z_{\ell_0} \ge Z_1'$, we see that $Z_1'Z_{\ell_0} = 0$.

In order to prove $\ell_1 = d_0$, it is enough to show $Z_1'^2 = -1$. Consider the effective divisor $L = Z_{\ell_0} - Z_1' - \Gamma$. Γ is not a component of L. For any component C of L, we claim that $CZ_1' \le 0$. Indeed, if $CZ_1' > 0$, then C is not a component of Z_1' because Z_1' is a fundamental cycle. Because $C \ne \Gamma$, we know that C is the exceptional curve of any other singular point $P_i \ne P_1$ of X_1, in this case, C is disjoint with Z_1', a contradiction. Thus $CZ_1' \le 0$. Hence $LZ_1' = (Z_{\ell_0} - Z_1' - \Gamma)Z_1' \le 0$, i.e., $-Z_1'^2 \le \Gamma Z_1'$.

By (7), $\Gamma \cdot \overline{\pi}^*\mathcal{E}_1 = \overline{\pi}_*\Gamma \cdot \mathcal{E}_1 = \overline{E}_0 \cdot \mathcal{E}_1 = 1$, i.e., $\Gamma Z'_1 + \cdots + \Gamma Z'_{\ell_1} = 1$. Because Γ is not a component of Z'_i, we have $\Gamma Z'_1 \leq 1$. Now we get $1 \leq -Z'^2_1 \leq 1$, and $Z'^2_1 = -1$. This is case 2).

(3) Note that $Z_1 Z_{\ell_0} = 0$ and Γ is contained in Z_{ℓ_0}, we get $Z_1\Gamma = 0$. Since $Z_1^2 = -1$, we can find an irreducible component $\Gamma_1 \neq \Gamma$ such that $\Gamma_1 Z_1 = -1$. Then we know that Γ_1 is not contained in Z_{ℓ_0}.

On the other hand, suppose any singularity on X_1 is of type (2), then all of the new exceptional curves are contained in Z_{ℓ_0}, which contradicts the existence of Γ_1. Therefore, X_1 admits at least one singular point of Type (1). ◻

Corollary 2.3. *Assume $\ell_0 = d_0$. Let Γ_j $(j = 1, \cdots, d_0)$ be the unique irreducible components such that $\Gamma_j Z_j = -1$. Then $\Gamma_j Z_{j+1} = 1$ and $\Gamma_j Z_k = 0$ $(k \neq j, j+1)$. For any other irreducible component $C \neq \Gamma_j$, we have $CZ_j = 0$.*

Proof. Since $Z_j^2 = -1$, one can find a unique Γ_j such that $\Gamma_j Z_j = -1$. By the proof of Theorem 2.4, $\overline{\pi}^*\overline{E}_0 = d_0\Gamma_{d_0}$.

Since $\Gamma_j Z_j = -1$ and $Z_j Z_k = 0$ (for $k \neq j$), Γ_j does not lie in Z_{j+1}. Thus $\Gamma_j Z_k = 0$ for $k < j$ and $\Gamma_j Z_k \geq 0$ for $k > j$. Since $\Gamma_j \neq \Gamma_{d_0}$, $\Gamma_j \cdot \overline{\pi}^*\mathcal{E}_0 = 0$. Thus

$$\Gamma_j Z_j + \sum_{k>j} \Gamma_j Z_k = 0.$$

Note that $\Gamma_j Z_k \geq 0$ and $\Gamma_j Z_j = -1$, we have $\Gamma_j Z_{j+1} = 1$ and $\Gamma_j Z_k = 0$ for $k > j+1$.

Let $C \neq \Gamma_1, \cdots, \Gamma_{d_0}$. If $CZ_k < 0$ for some k, then $C = \Gamma_k$, a contradiction. So $CZ_k \geq 0$ for any k. Since $C \neq \Gamma_{d_0}$, $C \cdot \overline{\pi}^*\mathcal{E}_0 = 0$, i.e.,

$$CZ_1 + \cdots + CZ_{d_0} = 0.$$

Hence $CZ_k = 0$ for any k. ◻

2.7. The case $\ell_0 = d_0 - 1$.

Theorem 2.5. *Suppose $\ell_0 = d_0 - 1 \leq 2$. Suppose P_1 be any singular point of X_1. Then $-1 \leq Z'_1 Z_1 \leq 0$.*

1. *If $\ell_1 = d_0$, then $Z'_1 Z_1 = \cdots = Z'_1 Z_{\ell_0} = 0$ and*

$$Z'_{\ell_1} < \cdots < Z'_1 < Z_{\ell_0} < \cdots < Z_1.$$

2. *If $\ell_1 \leq d_0 - 1$, then $Z'_1 Z_1 = 2 - d_0$. Moreover, if $d_0 = 3$, X_1 admits at most two such singular points P_1 and P_2.*
3. *Assume $d_0 = 3$. X_1 has two singular points P_1 and P_2 if and only if there are two components Γ_1 and Γ_2 such that $Z_1\Gamma_1 = Z_1\Gamma_2 = -1$ and π_1 is totally ramified over E_1. In this case, $\pi_1^{-1}(p_1) = \{ P_1 \}$ and $\pi_2^{-1}(p_2) = \{ P_2 \}$, we have*

$$Z_1 = 2Z_2 + Z'_1 + Z''_1,$$

where Z'_1 (resp. Z''_1) is the fundamental cycle of the exceptional set corresponding to P_1 (resp. P_2). Moreover, $Z'_1\Gamma_1 = Z''_1\Gamma_2 = -3$.

Proof. If $d_0 = 2$, then $\ell_0 = 1$, i.e., $\overline{\pi}^*\mathcal{E}_0 = Z_1$. So $Z'_1 Z_1 = Z'_1 \cdot \overline{\pi}^*\mathcal{E}_0 = 0$. Everything is trivial. In what follows, we assume $d_0 = 3$.

Since $Z'_1 \leq Z_1$, $Z'_1 Z_1 \leq 0$. From $(Z_2 + Z'_1)^2 < 0$ and $Z_2^2 = -1$, we get $Z'_1 Z_2 \leq 1$. Thus $Z'_1 Z_1 = -Z'_1 Z_2 \geq -1$. The equality implies $Z'^2_1 \leq -2$.

Let Γ be a component as in Corollary 2.2. Since $\Gamma\cdot\overline{\pi}^*\mathcal{E}_1=1$, $\Gamma Z_1'=1$.

(1) Assume $Z_1'Z_2=0$, i.e., $Z_1'Z_1=0$. Let C be any irreducible component of Z_1'. Since $C\neq\Gamma$, $CZ_2\geq 0$. Hence $CZ_2=0$. Note that $\Gamma Z_1'=1$ and $\Gamma\leq Z_2$. By Lemma 2.2, it implies $Z_1'<Z_2$. Hence $Z_2-\Gamma-Z_1'\geq 0$.

Similar to the proof of Theorem 2.4, one can prove that $CZ_1'\leq 0$ for any irreducible exceptional component C of $Z_2-\Gamma-Z_1'$. So $(Z_2-\Gamma-Z_1')Z_1'\leq 0$. It implies that $Z_1'^2=-1$, i.e., $\ell_1=3$. This is the case 1).

(2) Assume $Z_1'Z_2=1$, i.e., $Z_1'Z_1=-1$. By the above discussion, $Z_1'^2\leq -2$. This is the case (2).

Let $p_1,\cdots,p_s$ be the singular points in E_0 obtained in the canonical resolution. Then one can find an irreducible component Γ_i in the exceptional set corresponding of P_i such that $\Gamma_iZ_1=-1$ since $Z_1'Z_1=-1$. So

$$-2=Z_1^2\leq\sum_{i=1}^{s}Z_1\Gamma_i\leq -s.$$

i.e., $s\leq 2$.

(3) Assume that $s=2$. Note that $Z_1'Z_1=Z_1''Z_1=-1$ and $Z_1'Z_2=Z_1''Z_2=1$. One has

$$(Z_1-2Z_2-Z_1'-Z_1'')^2=6+Z_1'^2+Z_1''^2\leq 0.$$

Since $Z_1'^2\geq -3$ and $Z_1''^2\geq -3$, we get $Z_1'^2=Z_1''^2=-3$ and $Z_1=2Z_2+Z_1'+Z_1''$. Thus $Z_1\geq 2Z_2\geq 2\Gamma$ and hence $\overline{\pi}^*\overline{E}_0\geq 3\Gamma$, i.e., $\overline{\pi}^*\overline{E}_0=3\Gamma$, and π_1 is totally ramified over E_0. Since $\Gamma_1Z_1=-1$ and $\Gamma_1Z_2=1$, $Z_1'\Gamma_1=Z_1\Gamma_1-2Z_2\Gamma_1=-3$. Similarly, we have $Z_1''\Gamma_2=-3$.

Conversely, we assume that there are two components Γ_1 and Γ_2 such that $Z_1\Gamma_1=Z_1\Gamma_2=-1$ and $\overline{\pi}^*\overline{E}_0=3\Gamma$, we claim that there are two singular points p_1 and p_2 in E_0.

Indeed, $\Gamma\neq\Gamma_1,\Gamma_2$, i.e., $E_0\neq\overline{\pi}(\Gamma_1)$ and $\overline{\pi}(\Gamma_2)$. One can find a singular point p_1 in E_0 such that $\Gamma_1\leq Z_1'$. If Z_1' also contains Γ_2, then $Z_1Z_1'\leq Z_1\Gamma_1+Z_1\Gamma_2=-2$, which contradicts to (2). Hence Z_1' does not contain Γ_2 and one can find another singular point p_2 in E_0 such that $\Gamma_2\leq Z_1''$. Thus $Z_1Z_1'=Z_1Z_1''=-1$. ∎

2.8. The case $\ell_0=d_0=3$. In this case, we will always assume that P_1 is a singular point of X_1 of type $(3,2)$. What we are going to consider is the type of P_2.

$$P_2\to P_1\to P_0$$

THEOREM 2.6. *Assume that $\ell_0=d_0=3$ and π_2 is totally ramified over E_1. Suppose $p_2=E_0\cap E_1$. $\pi_2^{-1}(p_2)=\{P_2\}$. Let Γ_1, Γ_2 and Γ_3 be the irreducible components satisfying $\Gamma_iZ_i=-1$ as in Corollary 2.3. Then*

1. *$Z_2'\Gamma_1=Z_2'\Gamma_2=Z_1'\Gamma_3=1$, $Z_1'\Gamma_1=Z_1'\Gamma_2=-1$ and $Z_2'\Gamma_3=0$.*
2. *Z_3 and Z_2' are disjoint.*
3. *$Z_2-Z_3=Z_1''+Z_2'$.*
4. *$F_2=Z_1''$, $Z_1''\Gamma_2=-3$ and $\overline{E}_2$ does not lie in the branch locus.*

Proof. Let $D=Z_2-Z_3$.

(1) By Corollary 2.3 and $Z_1'=Z_1-Z_3$, one has $Z_1'\Gamma_1=Z_1'\Gamma_2=-1$ and $Z_1'\Gamma_3=1$. Since $Z_1'Z_2'=0$ and $Z_1'>Z_2'$, Z_2' does not contain Γ_k. From Corollary 2.2 and E_1 is totally ramified, $\overline{E}_2\neq\overline{\pi}(\Gamma_1),\overline{\pi}(\Gamma_2)$. Hence $\overline{\pi}^*\mathcal{E}_1\cdot\Gamma_1=\overline{\pi}^*\mathcal{E}_1\cdot\Gamma_2=0$. So $Z_2'\cdot\Gamma_1=Z_2'\cdot\Gamma_2=1$. Since $1=\Gamma_3\cdot\overline{\pi}^*\mathcal{E}_1$ and $Z_1'\cdot\Gamma_3=1$, $Z_2'\Gamma_3=0$.

(2) Since $Z_3^2 = Z_2'^2 = -1$, by Lemma 2.3, either Z_3 and Z_2' are disjoint or $Z_3 > Z_2'$ or $Z_3 < Z_2'$. However, the last case is impossible since it implies $\Gamma_3 \leq Z_2'$. Suppose that $Z_3 > Z_2'$. Since $Z_2'\Gamma_1 = 1$ and Γ_1 does not lie Z_3, $Z_3\Gamma_1 \geq Z_2'\Gamma_1 = 1$. which contradicts Corollary 2.3. So Z_3 and Z_2' are disjoint. In particular, Γ_3 and Z_2' are disjoint.

(3) From Theorem 2.5, if $Z_1''Z_2' = 0$, then $Z_1'' < Z_2'$. Since $Z_1''\Gamma_3 > 0$, $Z_2'\Gamma_3 > 0$, which contradicts (2). Hence $Z_1''Z_2' = 1$ by Theorem 2.5. Thus $Z_1''Z_1' = -1$.

Since $Z_1'\Gamma_1 = Z_1'\Gamma_2 = Z_1''Z_1' = -1$, Z_1'' contains one of Γ_1 and Γ_2. If Z_1'' contains Γ_1, then $Z_3Z_1'' = 0$ by Corollary 2.3. Moreover Corollary 2.3 implies $Z_3C \geq 0$ for any irreducible component $C \leq Z_1''$. By Lemma 2.3, Z_3 and Z_1'' are disjoint, which contradicts the fact $\Gamma_3 Z_1'' > 0$. Thus Z_1'' contains Γ_2. Hence $Z_1''Z_3 = 1$, $Z_1''Z_2 = -1$ by Corollary 2.3. So $Z_1''D = -2$.

By Lemma 2.3, $Z_2Z_2' = Z_3Z_2' = 0$, hence $DZ_2' = 0$. Now we obtain

$$(D - Z_1'' - Z_2')^2 = 3 + Z_1''^2 \geq 0.$$

So $D = Z_1'' + Z_2'$ and $Z_1''^2 = -3$, i.e., $\overline{\pi}^*\mathcal{E}_2 = Z_1''$.

(4) Since $D\Gamma_2 = Z_2\Gamma_2 - Z_3\Gamma_2 = -2$ and $Z_2' \cdot \Gamma_2 = 1$, $Z_1'' \cdot \Gamma_2 = -3$. Thus $\overline{\pi}_*\Gamma_2 \cdot \mathcal{E}_2 = \Gamma_2 \cdot \overline{\pi}^*\mathcal{E}_2 = \Gamma_2 \cdot Z_1'' = -3$, which implies $\overline{\pi}_*\Gamma_2 = 3\overline{E}_2$. Thus E_2 is not in branch locus. □

Corollary 2.4. *Under the assumptions of Theorem 2.6, if π_2 is totally ramified over E_1, then there is a singular point p_3 ($\neq p_2$) in E_1 such that $Z_1'''\Gamma_1 = -3$.*

$$Z_1' = 2Z_2' + Z_1'' + Z_1'''.$$

Proof. It follows from Theorem 2.5 and Theorem 2.6. □

2.9. (-1)**-curves in the canonical resolution.** Let $\tau : \widetilde{X} \to S$ be the contraction map of those (-1)-curves in the exceptional set of Jung's resolution $J : \widetilde{X} \to (X, P)$. We get a minimal resolution $\eta : S \to X$.

Similar to σ, we let $D_1, \cdots, D_r$ be the exceptional curves of τ and denote by $\mathcal{D}_i$ the total transform of D_i in $\widetilde{X}$. Then we know that

$$\mathbb{Z}\widetilde{D}_1 + \cdots + \mathbb{Z}\widetilde{D}_r = \mathbb{Z}\mathcal{D}_1 + \cdots + \mathbb{Z}\mathcal{D}_r.$$

For each i and $j \neq i$, $\mathcal{D}_i^2 = -1$ and $\mathcal{D}_i \cdot \mathcal{D}_j = 0$.

Theorem 2.7. *$\mathcal{D}_1, \cdots, \mathcal{D}_r$ are exactly the (-1)-cycles in the set $\mathcal{F}und(J)$ of fundamental cycles.*

$$\{ \mathcal{D}_1, \cdots, \mathcal{D}_r \} = \{ Z \in \mathcal{F}und(J) \mid p_a(Z) = 0, \ Z^2 = -1 \}$$

Proof. If $Z_j^{(i)}$ is a (-1)-cycle, then the curves in $Z_j^{(i)}$ are contracted by τ. So $Z_j^{(i)}$ is a divisor in $\mathbb{Z}\mathcal{D}_1 + \cdots + \mathbb{Z}\mathcal{D}_r$. Since $Z_j^{(i)} \cdot Z_j^{(i)} = -1$, by Lemma 2.4, we have $Z_j^{(i)} = \mathcal{D}_\ell$ for some ℓ.

Conversely, suppose some $\mathcal{D}_\ell \neq Z_j^{(i)}$ for any i and j. Note that $\mathrm{Supp}(\overline{\pi}_*\mathcal{D}_\ell) \subset \mathcal{E}_0$. Suppose i is the maximal integer such that

$$\mathrm{Supp}(\overline{\pi}_*\mathcal{D}_\ell) \subset \mathcal{E}_i.$$

Thus the strict transform of E_i in $\overline{Y}$ must be contained in $\overline{\pi}_*\mathcal{D}_\ell$, hence $\overline{\pi}_*\mathcal{D}_\ell \cdot \mathcal{E}_i < 0$. Thus

$$\mathcal{D}_\ell Z_1^{(i)} + \cdots + \mathcal{D}_\ell Z_{\ell_i}^{(i)} = \mathcal{D}_\ell \cdot \overline{\pi}^*\mathcal{E}_i = \overline{\pi}_*\mathcal{D}_\ell \cdot \mathcal{E}_i < 0.$$

implies $D_\ell Z_j^{(i)} < 0$ for some j. Furthermore, by Lemma 2.3, one has $Z_j^{(i)} \cdot Z_j^{(i)} \leq -2$ and hence $j = 1$.

By assumption, the support of $\mathcal{D}_\ell$ is contained in $\overline{\pi}^*\mathcal{E}_i$, hence in $Z_1^{(i)}$, by Lemma 2.5 and our assumption, we get $\mathcal{D}_\ell Z_1^{(i)} = 0$, a contradiction. □

Remark 2.1. By Theorem 2.2, F_i has a decomposition as (12) for $i = 0, 1, \cdots, k-1$.

$$F_i = Z_1^{(i)} + \cdots + Z_{\ell_i}^{(i)}.$$

For convenience, we will also write the decomposition as follows.

$$F_i = Z_1^{(i)} + \cdots + Z_{d_i}^{(i)},$$

where

$$Z_1^{(i)} > \cdots > Z_{\ell_i}^{(i)} > 0 \quad \text{and} \quad Z_{\ell_i+1}^{(i)} = \cdots = Z_{d_i}^{(i)} = 0.$$

3. Numerical invariants of double points. In this section, we will give new proofs of some results on double points (X, P) by our method.

3.1. Decomposition of cycles and invariants . In this section, $d_0 = 2$. Let π_0 be determined by the double cover data (B_0, δ_0), i.e., $B_0 \equiv 2\delta_0$.

Denote by m_i the multiplicity of the branch locus B_i of π_i at p_i, w_i is the integral part of $m_i/2$.

$$m_i = \mathrm{mult}_{p_i}(B_i), \qquad w_i = \left[\frac{m_i}{2}\right].$$

$\overline{\sigma}$ is minimal iff $m_i \geq 2$, equivalently $w_i \geq 1$ for any i. The double cover data $(\overline{B}, \overline{\delta})$ of $\overline{\pi}$ satisfies

$$\overline{\delta} = \sigma^*(\delta_0) - \sum_{i=0}^{k-1} w_i \mathcal{E}_i \ . \tag{14}$$

By the formulas for double covers (see [3], Ch.III, §7), we can compute the rational canonical divisor of the canonical resolution $\overline{\sigma}$.

$$K = \sum_{i=0}^{k-1}(1 - w_i) \cdot \overline{\pi}^*\mathcal{E}_i\,. \tag{15}$$

Note that $w_i \geq 1$ for any i.

Lemma 3.1. *For the canonical resolution $\overline{\sigma}$, $-K$ is an effective divisor. $K\overline{E}_s < 0$ iff $\overline{E}_s$ is a (-1)-curve.*

Note that $\pi_i^{-1}(p_i) = \{P_i\}$, and $F_i = \overline{\pi}^*\mathcal{E}_i$ is connected. By Theorem 2.2, each $\overline{\pi}^*\mathcal{E}_i$ has a unique decomposition

$$F_i = \overline{\pi}^*\mathcal{E}_i = Z_1^{(i)} + \cdots + Z_{\ell_i}^{(i)}.$$

Corollary 3.1. *$\ell_{i-1} \leq 2$.*

I) *If* $\ell_{i-1}=1$, *then* $\left(Z_1^{(i)}\right)^2=-2$.

II) *If* $\ell_{i-1}=2$, *then* $\left(Z_1^{(i)}\right)^2=\left(Z_2^{(i)}\right)^2=-1$, *and* $\overline{E}_{i-1}$ *is in the branch locus.*

Furthermore, $Z_{j,i}=Z_{j',i'}$ iff $i=i'$ and $j=j'$ by Corollary 2.1.

COROLLARY 3.2. $K\cdot\overline{\pi}^*\mathcal{E}_i=2w_i-2$, $p_a(\overline{\pi}^*\mathcal{E}_i)=w_i-1$ *and*

$$w_i=p_a\left(Z_1^{(i)}\right)+\cdots+p_a\left(Z_{\ell_i}^{(i)}\right)+d_i-\ell_i,$$

Proof. Let $Z_j^{(i)}=0$ for $j>\ell_i$. By the adjunction formula and (15),

$$\begin{aligned}p_a\left(Z_1^{(i)}\right)+p_a\left(Z_2^{(i)}\right)&=1+\frac{1}{2}\left(\left(Z_1^{(i)}\right)^2+KZ_1^{(i)}\right)+1+\frac{1}{2}\left(\left(Z_2^{(i)}\right)^2+KZ_2^{(i)}\right)\\&=2+\frac{1}{2}\left((\overline{\pi}^*\mathcal{E}_i)^2+K\cdot\overline{\pi}^*\mathcal{E}_i\right)\\&=2+\frac{1}{2}(-2-2(1-w_i))=w_i.\end{aligned}$$

∎

COROLLARY 3.3. $Z_1=Z_1^{(0)}$ *is the fundamental divisor of the canonical resolution of* (X,P) *and* $Z_1\geq Z_j^{(i)}$ *for any* i *and* j.

Proof. By definition, $Z_1=Z_1^{(0)}$ is the fundamental cycle of the canonical resolution. Note that the support of $Z_j^{(i)}$ is contained in Z_1. We write $Z_1=Z'+Z''$, where Z' has the same support as $Z_j^{(i)}$, and Z' has no common component with Z''. Then for any curve Γ in $Z_j^{(i)}$, we have $\Gamma Z'\leq\Gamma Z_1\leq 0$, by definition, $Z'\geq Z_j^{(i)}$, so $Z_1\geq Z_j^{(i)}$. ∎

3.2. A relation between the invariants. From Theorem 2.4, we have

THEOREM 3.1. *Assume that* $\ell_0=2$ *and* $p_1\to p_0$.
I) *If* $\ell_1=1$, *then* $Z_1'=Z_1-Z_2$. *In this case,*

$$w_1=p_a(Z_1)-p_a(Z_2)+1,$$
$$w_0=p_a(Z_1)+p_a(Z_2).$$

II) *If* $\ell_1=2$, *then*

$$Z_1>Z_2>Z_1'>Z_2',$$

and the intersection number of any two distinct cycles in this chain is zero. In particular, exactly one point p_1 *on* E_0 *is of type* I).

EXAMPLE 3.1. Let (X,P) be defined by $z^2=y(x^4+y^6)$. There are two infinitely closed singular points p_1,p_2 in E_0 where p_1 is of type I) and p_2 is of type II). $Z_1'=Z_1-Z_2$ and $Z_1''^2=-1$. $w_0=2$, $w_1=1$, $w_2=1$.

3.3. (−1)-curves in the canonical resolution. Let $\tau : \overline{X} \to S$ be the contraction map of those (−1)-curves in the exceptional set of the canonical resolution. We get a minimal resolution $\eta : S \to X$.

From Theorem 1.4, $\mathcal{D}_1, \cdots, \mathcal{D}_r$ are exactly the set of (−1)-cycles $Z_j^{(i)}$.

$$\{ \mathcal{D}_1, \cdots, \mathcal{D}_r \} = \left\{ Z_j^{(i)} \;\middle|\; p_a(Z_j^{(i)}) = 0,\ (Z_j^{(i)})^2 = -1 \right\}$$

THEOREM 3.2. *For any ℓ, $\mathcal{D}_\ell = \overline{D}_\ell$ consists of only one irreducible (−1)-curve $\overline{D}_\ell$. Namely $\overline{D}_1, \cdots, \overline{D}_r$ are r disjoint (−1)-curves.*

Proof. From $w_i \geq 1$, we see that $Z_1^{(i)}$ can not be a (−1)-cycle. Suppose $\mathcal{D}_\ell = Z_2^{(i)}$ for some i. Since $\overline{D}_\ell \cdot Z_2^{(i)} = \overline{D}_\ell \cdot \mathcal{D}_\ell = -1$ and $\overline{D}_\ell \cdot Z_1^{(i)} = 0$, we have

$$\overline{\pi}_*(\overline{D}_\ell) \cdot \mathcal{E}_i = \overline{D}_\ell \cdot \overline{\pi}^*(\mathcal{E}_i) = \overline{D}_\ell \cdot (Z_1^{(i)} + Z_2^{(i)}) = -1, \tag{16}$$

so $\overline{\pi}_*(\overline{D}_\ell) = \overline{E}_i$ and $\overline{\pi}_*(\overline{D}_\ell)$ is in the branch locus.

We will prove that $\overline{E}_i$ is a (−2)-curve, i.e., $\overline{D}_\ell$ is a (−1)-curve.

Suppose that $\overline{D}_\ell$ is not a (−1)-curve. There is another component $\overline{D}_s$ in $\mathcal{D}_\ell$ such that $\overline{D}_\ell \overline{D}_s = 1$. Let $\mathcal{D}_s$ be the (−1)-cycle such that $\mathcal{D}_s \overline{D}_s = -1$. It is easy to see $\mathcal{D}_s < \mathcal{D}_\ell$ and $\overline{D}_\ell \mathcal{D}_s = 1$.

We can find a singular point p_s with a decomposition $\overline{\pi}^*\mathcal{E}_s = Z_{1,s} + Z_{2,s}$ such that $\mathcal{D}_s = Z_{2,s}$. Since $\mathcal{D}_s < \mathcal{D}_\ell$, $\mathcal{E}_s < \mathcal{E}_i$ by the choice of $\mathcal{E}_s$. Since $\overline{E}_i$ does not lie in $\mathcal{E}_s$, $\overline{D}_\ell Z_{1,s} \geq \overline{D}_\ell \mathcal{D}_s = 1$. Hence $E\mathcal{E}_s = \overline{D}_\ell \overline{\pi}^* \mathcal{E}_s \geq \overline{D}_\ell Z_{1,s} + \overline{D}_\ell \mathcal{D}_s \geq 2$, a contradiction. □

COROLLARY 3.4. *Each (−1)-curve $\overline{D}_\ell$ comes from a singular point (B_{i-1}, p_{i-1}) of type $(2k+1 \to 2k+1)$. Namely, (B_{i-1}, p_{i-1}) is a curve singularity of multiplicity $2k+1$, whose strict transform $\overline{B}_{i-1}$ under σ_i admits only one singular point at $p_i = \overline{B}_{i-1} \cap E_{i-1}$ with multiplicity $2k+1$ and E_{i-1} intersects $\overline{B}_{i-1}$ at p_i transversely.*

REMARK 3.1. In [15, 16], the singular point p_{i-1} of type $(2k+1 \to 2k+1)$ is just the one such that $Z_2^{(i)}$ is a (−1)-curve. Thus the number of (−1)-curves in the exceptional set of the canonical resolution is equal to the number of singular points of the branch locus with types $(2k+1 \to 2k+1)$ for some positive integers k.

3.4. A criterion for rational double points. As an application, we obtain two well-known criteria for double points to be rational via the canonical resolution. Recall that (X, P) is rational iff the fundamental cycle Z of one resolution satisfies $p_a(Z) = 0$.

THEOREM 3.3. *A double point (X, P) is rational if and only if $w_i = 1$ for all i.*

Proof. We have seen that the fundamental cycle of the canonical resolution is $Z_1 = Z_1^{(0)}$.

Suppose (X, P) is a rational double point. Then $p_a(Z_1) = 0$ and $Z_1^2 = -2$, so $Z_2 = 0$, $KZ_1 = 0$, and $w_0 = p_a(Z_1) + p_a(Z_2) = 1$. Since $Z_1 \geq Z_j^{(i)}$, if $Z_j^{(i)} \neq 0$, then we have $0 \leq p_a(Z_j^{(i)}) \leq p_a(Z_1) = 0$, so $p_a(Z_j^{(i)}) = 0$. Note that $Z_1^{(i)} \neq 0$, we have

$$1 \leq w_i = p_a(Z_1^{(i)}) + p_a(Z_2^{(i)}) = p_a(Z_2^{(i)}) \leq 1,$$

hence $w_i = 1$ and $Z_2^{(i)} = 0$ for any i.

Conversely, suppose $w_i = 1$ for any i. By the formula (15), $K = 0$.

$$(Z_j^{(i)})^2 = (Z_j^{(i)})^2 + KZ_j^{(i)} = 2p_a(Z_j^{(i)}) - 2 \geq -2$$

is an even number, so $Z_1^2 = -2$, which implies $Z_2 = 0$. Now we have $p_a(Z_1) = 0$. By definition, (X, P) is a rational double point. □

COROLLARY 3.5. *The canonical resolution $\overline{\sigma}$ of a rational double point is a minimal resolution.*

According to the classification of curve singularity, the condition in the above criterion is equivalent to that the branch curve B_π admits a singular point of type ADE at p (see [3, p.78]).

4. Local invariants of singularities of multiplicity 3. For the reader's convenience, we will introduce some basic facts on triple covers. See [12], [13] or [8] for the details. See also [1] and [14] for different methods for the study of triple covers defined by a cubic equation.

4.1. Singularities of a triple covering surface. Let Y_0 be a smooth algebraic surface over $\mathbb{C}$, and let $\pi_0 : X_0 \to Y_0$ be a normal triple cover. Then X_0 is the normalization of a surface Σ defined by a cubic equation in a line bundle $[\mathcal{L}]$:

$$z^3 + sz + t = 0,$$

where $\mathcal{L}$ is an invertible sheaf, $s \in H^0(Y_0, \mathcal{L}^2)$, $0 \neq t \in H^0(Y_0, \mathcal{L}^3)$ and z is the fiber coordinate of $[\mathcal{L}]$.

If $s = 0$, then the triple cover is cyclic and everything is known (see for instance [13, Section 1.4] or [11]). In what follows, we assume that $s \neq 0$. Let

$$a = \frac{4s^3}{\gcd(s^3, t^2)}, \quad b = \frac{27t^2}{\gcd(s^3, t^2)}, \quad c = \frac{4s^3 + 27t^2}{\gcd(s^3, t^2)}.$$

Then a, b and c are coprime sections of an invertible sheaf such that $a + b = c$. In fact, the triple cover data $(s, t, \mathcal{L})$ is equivalent to the data (a, b, c) with $a + b = c$. The latter is more convenient for the canonical resolution.

Assume that we have the following factorization of sections (according to the decompositions of their divisors)

$$a = 4a_1a_1^2a_0^3, \quad b = 27b_1b_0^2, \quad c = c_1c_0^2,$$
$$s = a_1a_1^2b_1a_0, \quad t = a_1a_1^2b_1^2b_0.$$

where a_1, a_2, b_1, c_1 are square-free and are pairwise coprime.

$$A_i = \mathrm{div}(a_i), \quad B_i = \mathrm{div}(b_i), \quad C_i = \mathrm{div}(c_i).$$

1. π_0 is totally (resp. simply) ramified over $D_2 = A_1 + A_2$ (resp. $D_1 = B_1 + C_1$). $R = D_1 + 2D_2$ is called the branch locus.
2. Let π_0 be totally ramified over a singular point p_0 of $D_1 + D_2$. Then X_0 is smooth over p_0 if and only if (D_1, p_0) is a cusp (i.e., locally defined by $y^2 + f(x, y)^3 = 0$, $f(0, 0) = 0$), and D_2 does not pass through p_0.
3. A local triple cover is Galois iff $D_1 = 0$. (This is not the case for global triple covers.)

DEFINITION 4.1. A singular point of the branch locus satisfying the above condition (2) is called a *good cusp* of the branch locus.

4.2. Canonical resolution. Suppose $X = X_0 \to Y = Y_0$ is a triple cover. As in the double cover case, we have the following canonical resolution of the singularity (X, P). We use the same notations as in the double cover case.

$$\begin{array}{ccccccccc} \overline{X} = X_k & \xrightarrow{\overline{\sigma}_{k-1}} & \cdots\cdots & \longrightarrow & X_2 & \xrightarrow{\overline{\sigma}_1} & X_1 & \xrightarrow{\overline{\sigma}_0} & X_0 = X \\ \downarrow{\scriptstyle \overline{\pi}=\pi_k} & & & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0=\pi} \\ \overline{Y} = Y_k & \xrightarrow{\sigma_{k-1}} & \cdots\cdots & \longrightarrow & Y_2 & \xrightarrow{\sigma_1} & Y_1 & \xrightarrow{\sigma_0} & Y_0 = Y \end{array}$$

The corresponding data $(a^{(i+1)}, b^{(i+1)}, c^{(i+1)})$ of π_{i+1} is obtained from

$$(\sigma_i^* a^{(i)}, \sigma_i^* b^{(i)}, \sigma_i^* c^{(i)})$$

by eliminating the common factors.

4.3. Local invariants of the canonical resolution. We denote by $A^{(i)}$, $B^{(i)}$ and $C^{(i)}$ the divisors of $a^{(i)}$, $b^{(i)}$ and $c^{(i)}$, respectively. Let

$$r_i = \min\left\{m_{p_i}(A^{(i)}),\ m_{p_i}(B^{(i)}),\ m_{p_i}(C^{(i)})\right\},$$

where $m_p(D)$ is the multiplicity of a divisor D at p. Let

$$m_i = \left[\frac{m_{p_i}(D_1^{(i)})}{2}\right], \tag{17}$$

$$n_i = \begin{cases} m_{p_i}(D_2^{(i)}), & \text{if } r_i \equiv m_{p_i}(A^{(i)}) \pmod 3; \\ m_{p_i}(D_2^{(i)}) - 1, & \text{otherwise.} \end{cases} \tag{18}$$

Let

$$w_i = m_i + n_i \tag{19}$$

By some computation, we get the branch locus of $\overline{\pi}$,

$$\begin{cases} \overline{D}_1 = \sigma^*(D_1) - 2\sum_{i=0}^{k-1} m_i \mathcal{E}_i, \\ \overline{D}_2 = \sigma^*(D_2) - \sum_{i=0}^{k-1} n_i \mathcal{E}_i. \end{cases} \tag{20}$$

In the global case, we have

$$\begin{cases} p_a(\overline{D}_1) &= p_a(D_1) - \sum_{i=0}^{k-1} m_i(2m_i - 1), \\ p_a(\overline{D}_2) &= p_a(D_2) - \frac{1}{2}\sum_{i=0}^{k-1} n_i(n_i - 1). \end{cases} \tag{21}$$

5. Fundamental cycles of singularities of multiplicity 3. The case $\pi_i^{-1}(p_i) = \{P_i, P_i'\}$ can be reduced to the double cover case, since $d_{P_i} = 2$ and $d_{P_i'} = 1$. We always assume that $d_{i-1} = 3$.

We denote by K the canonical divisor of the canonical resolution.

Corollary 5.1. *Suppose $d_i = 3$. Then $K \cdot \overline{\pi}^* \mathcal{E}_i = 2w_i - 3$, $p_a(\overline{\pi}^*\mathcal{E}_i) = w_i - 2$.*

$$w_i = p_a(Z_1^{(i)}) + \cdots + p_a(Z_{\ell_i}^{(i)}) + d_i - \ell_i \geq 1.$$

Proof. Denote by $\overline{R}_i$ the ramification locus over $\overline{D}_i$. By Hurwitz formula, $K_{\overline{X}} = \overline{\pi}^*(K_{\overline{Y}}) + 2\overline{R}_2 + \overline{R}_1$. $K_{\overline{Y}} = \sigma^*(K_Y) + \sum_j \mathcal{E}_j$. We have

$$\begin{aligned}\overline{\pi}^*\mathcal{E}_i \cdot K_{\overline{X}} &= \mathcal{E}_i \cdot \overline{\pi}_*(K_{\overline{X}}) = \mathcal{E}_i \cdot (3K_{\overline{Y}} + 2\overline{D}_2 + \overline{D}_1)\\ &= \mathcal{E}_i \cdot \big(\sigma^*(3K_Y + D_1 + 2D_2) + \sum_j (3 - 2m_j - 2n_j)\mathcal{E}_j\big)\\ &= 2w_i - 3.\end{aligned}$$

The other formulas can be obtained easily. □

COROLLARY 5.2. *Assume that π is totally ramified over $p = \pi(P)$, i.e., $d = d_P = 3$.*

1. *(X, P) is a rational double point if and only if $w_0 = 1$ and $\ell_0 = 2$. In this case, Z_2 is a (-1)-cycle.*
2. *(X, P) is a rational triple point if and only if $w_0 = 2$ and $\ell_0 = 1$.*
3. *If $w_0 = 1$ and (X, P) is not rational, then (X, P) is weakly elliptic and $\ell_0 = 3$, $p_a(Z_1) = 1$. Moreover, Z_2 and Z_3 are (-1)-cycles.*

Proof. If (X, P) is rational double point, $p_a(Z_1) = 0$ and $Z_1^2 = -2$. From Theorem 2.2, $Z_2^2 = -1$, $Z_3 = 0$ and $w_0 - 1 = p_a(Z_1) + p_a(Z_2)$. Since $Z_2 \le Z_1$, $p_a(Z_2) = 0$ and $w_0 = 1$.

The other parts can be proved similarly. □

COROLLARY 5.3. *Under the assumptions in Theorem 2.6, we have*

$$\begin{aligned} w_2 &= p_a(Z_2) - p_a(Z_3) - p_a(Z_2') + 2,\\ w_1 &= p_a(Z_1) - p_a(Z_3) + p_a(Z_2') + 1,\\ w_0 &= p_a(Z_1) + p_a(Z_2) + p_a(Z_3).\end{aligned} \tag{22}$$

COROLLARY 5.4. *Under the assumptions of Corollary 2.4, we have*

$$w_3 = p_a(Z_1) - p_a(Z_2) - p_a(Z_2') + 2. \tag{23}$$

6. **(-1)-curves in the canonical resolution of a triple cover.** We will give a proof of the following result of [4, Lemma 5.3].

COROLLARY 6.1. *Let Γ be an exceptional curve in $\overline{X}$ which is contracted by τ and $E = \overline{\pi}(\Gamma)$. Let $\mathcal{D}$ be a (-1)-cycle such that $\mathcal{D}\Gamma = -1$ and i is the maximal integer such that*

$$Supp(\overline{\pi}_*\mathcal{D}) \subset \mathcal{E}_i.$$

Then Γ occurs in one of the following cases:

1. *$E = \overline{E}_i$ is a (-3)-curve contained in $\overline{D}_2$ and $\overline{\pi}^*E = 3\Gamma$;*
2. *$E = \overline{E}_i$ is a (-2)-curve contained in $\overline{D}_1$ and $\overline{\pi}^*E = 2\Gamma + \overline{\Gamma}$;*
3. *E is not a component of branch locus, but E intersects with two (-3)-curves in $\overline{D}_2$;*
4. *$\pi_i^{-1}(p_i) = \{\, P_i,\, P_i' \,\}$ and Γ is a component of F_i'.*

Proof. In what follows we always assume Γ does not occur in case (4). By Theorem 1.4, $\mathcal{D} = Z_j^{(i)}$ for some j.

Claim 1. *If E lies in the branch locus and $\overline{\pi}^*E \geq 2\Gamma$, then $E = \overline{E}_i$.*

Suppose that $E \neq \overline{E}_i$. If $\ell_i = 2$, then $\mathcal{D} = Z_2^{(i)}$. Since $\Gamma Z_1^{(i)} = 0$, $E\cdot\mathcal{E}_i = \Gamma\cdot\overline{\pi}^*\mathcal{E}_i = Z_2^{(i)}\Gamma < 0$, i.e., $E = \overline{E}_i$, a contradiction. Therefore $\ell_i = 3$ and $Z_1^{(i)} > Z_2^{(i)} > Z_3^{(i)} > 0$. If $\mathcal{D} = Z_3^{(i)}$, $E \cdot \mathcal{E}_i = \Gamma \cdot \overline{\pi}^*\mathcal{E}_i = Z_3^{(i)} \cdot \Gamma < 0$, i.e., $E = \overline{E}_i$, a contradiction. So we have $\mathcal{D} = Z_2^{(i)}$. From Theorem 2.6, E does not lie in the branch locus, a contradiction.

Claim 2. *Γ is a (-1)-curve if and only if E lies in branch locus and $\overline{\pi}^*E \geq 2\Gamma$.*

If Γ is a (-1)-curve, then $\Gamma = \mathcal{D} = Z_j^{(i)}$ $(j \geq 2)$. By Lemma 2.3 and Lemma 2.5, $\Gamma \cdot \overline{\pi}^*\mathcal{E}_i = \Gamma^2 = -1$, hence Γ is a component of $\overline{\pi}^*\overline{E}_i$. $\overline{\pi}^*\mathcal{E}_i \geq Z_1^{(i)} + Z_j^{(i)} \geq 2\Gamma$. Namely, $E = \overline{\pi}(\Gamma)$ lies in the branch locus.

Conversely, we assume that E lies in the branch locus and $\overline{\pi}^*E \geq 2\Gamma$. From Claim 1, $E = E_i$. Suppose that Γ is not a (-1)-curve. There is another component Γ' in $\mathcal{D}$ such that $\Gamma\Gamma' = 1$. Let $\mathcal{D}'$ be the (-1)-cycle such that $\mathcal{D}'\Gamma' = -1$. It is easy to see $\mathcal{D}' < \mathcal{D}$ and $\Gamma\mathcal{D}' = 1$.

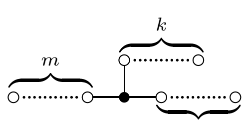
From Theorem 1.4, we can find a singular point p_s with $\overline{\pi}^*\mathcal{E}_s = Z_1^{(s)} + Z_2^{(s)} + Z_3^{(s)}$ such that $\mathcal{D}' = Z_j^{(s)}$ for some $j > 1$. Since $\mathcal{D}' < \mathcal{D}$, we have $\mathcal{E}_s < \mathcal{E}_i$ by the choice of $\mathcal{E}_s$. Note that $E = \overline{E}_i$ does not lie in $\mathcal{E}_s$, $\Gamma Z_1^{(s)} \geq \Gamma\mathcal{D}' = 1$. Hence $E \cdot \mathcal{E}_s = \Gamma \cdot \overline{\pi}^*\mathcal{E}_s \geq \Gamma Z_1^{(s)} + \Gamma\mathcal{D}' \geq 2$, a contradiction.

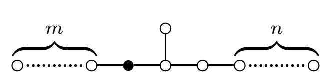
Claim 3. *If Γ is not a (-1)-curve, then E occurs in case* (3).

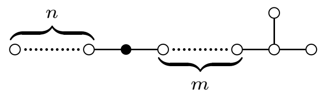
From Claim 2 and our assumption, if Γ is not a (-1)-curve, then $\ell_i = 3$, $Z_1^{(i)} > Z_2^{(i)} > Z_3^{(i)} > 0$ and $\mathcal{D} = Z_2^{(i)}$. By Claim 2 and Corollary 2.2, $\overline{E}_i$ is a (-3)-curve, $\overline{\pi}^*\overline{E}_i = 3Z_3^{(i)}$ and $Z_3^{(i)}$ is a (-1)-curve. Let p_{i+1} be a singular point in E_i $(\subseteq Y_{i+1})$ satisfying Theorem 2.4 (I) and $p_{i+2} = E_{i+1} \cap E_i$ $(\subseteq Y_{i+2})$. From Theorem 2.6 (3), $p_a(Z_1^{(i+2)}) = 0$ and $Z_2^{(i+1)}$ is a (-1)-cycle. Hence $w_{i+2} = p_a(Z_1^{(i+2)}) + 2 = 2$ and E_{i+2} does not lie in the branch locus by Theorem 2.4, and $\overline{E}_{i+1}$ is either a (-3)-curve in $\overline{D}_2$ or a (-2)-curve in $\overline{D}_1$ by Claim 2.

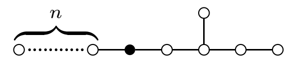
Suppose that $\overline{E}_{i+1}$ is a (-2)-curve in $\overline{D}_1$. Since $\overline{E}_{i+1}$ is (-2)-curve, the branch locus will become smooth after blowing-up p_{i+2}. Furthermore, from $w_{i+2} = 2$, the strict transform of the branch locus of π in Y_{i+2} is smooth at p_{i+2} and does not tangent to E_i, E_{i+1}. Thus p_i is a good cusp, namely, X_i is smooth over p_i, which contracts our assumption. So $\overline{E}_{i+1}$ is a (-3)-curve over which $\overline{\pi}$ is totally ramified. $w_{i+2} = 2$ implies that the strict transform of the branch locus of π_{i+2} in Y_{i+2} does not pass through p_{i+2}. Therefore $\Gamma = \overline{\pi}^*E_{i+2}$ meets exactly E_{i+1} and E_i. ☐

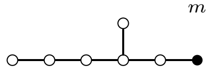
7. A criterion for rational singularities of multiplicity two or three . [2] listed all rational triple points associated with the Dynkin graphs which are denoted by $A_{n,m,k}$, $B_{n,m}$, $C_{n,m}$, $D_{n,5}$, $E_{6,0}$, $E_{7,0}$, $E_{0,7}$, $F_{n,6}$, $G_{n,0}$ in [5],

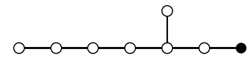
$A_{n,m,k}$: 

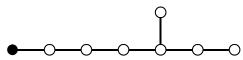
$B_{m,n}$: 

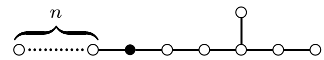
$C_{m,n}$: 

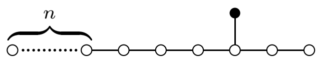
$D_{n,5}$: 

$E_{6,0}$: 

$E_{7,0}$: 

$E_{0,7}$: 

$F_{n,6}$: 

$G_{n,0}$: 

where $\circ$ is a (-2)-curve, $\bullet$ is a (-3)-curve.

THEOREM 7.1. *Suppose the surface singularity $(X, P) \to (Y, p)$ is a finite cover of degree $d = 2$ or 3 totally ramified over p. Let $P = P_0, P_1, \cdots, P_{k-1}$ be the infinitely near singular points obtained in the canonical resolution. Then*

1. *$w_i \geq 1$ for $i = 0, 1, \cdots, k-1$.*
2. *(X, P) is a rational singular point iff $w_i \leq d - 1$ for any i.*
3. *If (X, P) is rational, then the multiplicity of (X, P) is $w_0 + 1$.*

Proof. (1) (3) follows from Corollary 5.1 and Corollary 5.2.

(2) Suppose (X, P) is a rational singular point. We have seen that $Z_1 = Z_1^{(0)}$ is the fundamental cycle of the canonical resolution, thus $p_a(Z_1) = 0$. Because $Z_1 \geq Z_j^{(i)} > 0$ for $j \leq \ell_i$, we have $p_a(Z_j^{(i)}) = 0$ for all $j \leq \ell_i$. Since $\ell_i \geq 1$ and $d_i \leq d$, we obtain

$$\begin{aligned} w_i &= p_a(Z_1^{(i)}) + \cdots + p_a(Z_{\ell_i}^{(i)}) + d_i - \ell_i \\ &= d_i - \ell_i \leq d - 1. \end{aligned}$$

Conversely, suppose $w_i \leq d - 1$ for any i. If $d = 2$, then (2) is true by Theorem 3.3. In what follows, we assume $d = 3$.

CLAIM 1. $\ell_0 \leq 2$.

Suppose that $\ell_0 = 3$, i.e., $Z_3 \neq 0$. Since $w_0 \leq 2$, by Corollary 5.1, we have $p_a(Z_3) = 0$. Let p_1 be a singular point in E_0 $(\subseteq Y_1)$ satisfying Theorem 2.4 (I) and $p_2 = E_2 \cap E_1$ $(\subseteq Y_2)$. From (22),

$$4 \geq w_1 + w_2 = p_a(Z_1) + p_a(Z_2) + 3.$$

Hence $p_a(Z_1) = 1$, $w_1 = w_2 = 2$, Z_2 and Z_3 are (-1)-cycles. By the proof of Corollary 6.1, the cover is totally ramified over E_2. By Corollary 5.4, there is another singular point p_3 in E_2 $(\subseteq Y_2)$. Since $w_1 = 2$, $1 = p_a(Z_1') + p_a(Z_2')$, hence $p_a(Z_1') = 1$ and $p_a(Z_2') = 0$. By (23), we have $w_3 = 3$, a contradiction.

CLAIM 2. *If $w_0 = 2$, then $\ell_0 = 1$, hence $\overline{\pi}^*\mathcal{E}_0 = Z_1$ and $p_a(Z_1) = 0$, i.e., (X, P) is a rational triple point.*

Suppose that $\ell_0 = 2$, i.e., $Z_2 \neq 0$. Then $1 = w_0 - 1 = p_a(Z_1) + p_a(Z_2)$, hence $p_a(Z_1) = 1$, $p_a(Z_2) = 0$. By Corollary 6.1, $\overline{E}_0$ is either a (-3)-curve in $\overline{D}_2$ or a (-2)-curve in $\overline{D}_1$. Hence, if the cover is totally ramified over $\overline{E}_1$, then there are exactly two proximate singular points p_1 and p_2 on E_0. Otherwise, there is a unique proximate singular point on $\overline{E}_1$.

From

$$p_a(\overline{\pi}^*\mathcal{E}_0) = p_a(\overline{\pi}^*\overline{E}_0) + \sum_{p_i \to p} w_i,$$

we have

$$w_0 = \begin{cases} -1 + \sum\limits_{p_i \to p} (w_i - 1), & \overline{\pi} \text{ is totally ramified over } \overline{E}_0, \\ -\frac{1}{2} + \sum\limits_{p_i \to p} (w_i - \frac{1}{2}), & \overline{\pi} \text{ is simple ramified over } \overline{E}_0, \end{cases}$$

where $p_i \to p$ runs over all proximate singular points. By the above discussion, one has

$$w_0 = \begin{cases} w_1 + w_2 - 3, & \overline{\pi} \text{ is totally ramified over } \overline{E}_0, \\ w_1 - 1, & \overline{\pi} \text{ is simple ramified over } \overline{E}_0. \end{cases}$$

Since $w_1, w_2 \leq 2$, the above equality implies $w_0 \leq 1$, a contradiction.

Up to now, we complete the proof. □

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