Riccati foliations and Double Riccati foliations

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1. Definition of Foliation

- X: algebaic surface,
  T_X: tangent bundle of X.
  $\mathcal{L}^{-1} \subseteq T_X$: maximal sub-line bundle.
- Foliation $\mathcal{F}$ is a section

$$s \in H^0(X, T_X \otimes \mathcal{L}).$$

- Open covering $X = \cup_\alpha U_\alpha$,

$$s|_{U_\alpha} = A(x_\alpha, y_\alpha)\frac{\partial}{\partial x_\alpha} + B(x_\alpha, y_\alpha)\frac{\partial}{\partial y_\alpha}, \quad (x_\alpha, y_\alpha) \in U_\alpha.$$

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- $T_{\mathcal{F}} := \mathcal{L}^{-1}$ tangent bundle of $\mathcal{F}$,
  $K_{\mathcal{F}} := \mathcal{L}$ canonical bundle of $\mathcal{F}$.
- Exact sequence

$$0 \to T_{\mathcal{F}} \xrightarrow{\cdot s} T_X \to \mathcal{I}_{Z(s)} \otimes N_{\mathcal{F}} \to 0,$$

  $N_{\mathcal{F}}$ line bundle,
  $\mathcal{I}_{Z(s)}$ ideal sheaf of $Z(s)$ (zero set of s).
- Canonical bundle

$$\omega_X := \wedge^2 \Omega_X = K_{\mathcal{F}} \otimes N_{\mathcal{F}}^{-1}$$

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2. Example (1): fibration

- Fibration $f : X \to C$,
  C smooth curve, f holomorphic and surjective.
- Fiber $F_t = f^{-1}(t)$, $t \in C$.
- Foliation $\mathcal{F}$ generated by f:

$$\omega = \frac{1}{\mu(f)}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) \quad \text{(local eq.)},$$

$\mu(f) = \gcd(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$.

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- Canonical bundle of $\mathcal{F}$:

$$K_{\mathcal{F}} = \omega_{X/C}(-D(f)),$$

where

- $\omega_{X/C} := \omega_X \otimes f^*\Omega_C^{-1}$ (relative canonical bundle)
- $D(f) := \sum_{t\in C}(F_t - F_{t,\mathrm{red}})$ (zero divisor of df).
- Conormal bundle of $\mathcal{F}$:

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- Ruled surface $\varphi : X \to B$.
  Riccati Foliation $\mathcal{F}$ w.r.t. $\varphi \overset{def}{\Longleftrightarrow}$ general fiber F of φ transverse to $\mathcal{F}$.
- $\mathcal{F}$ Riccati foliation $\Longleftrightarrow$ $K_{\mathcal{F}} F = 0$.
- Local equation:

$$\omega = \left(g_0(x)y^2 + g_1(x)y + g_2(x)\right) dx - dy, \quad x \in B, \; y \in F.$$

- Canonical bundle of $\mathcal{F}$: $K_{\mathcal{F}} = rF$,
  degree of $\mathcal{F}$: $r := \deg \mathcal{F}$.

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2. Example (3): Double Riccati foliations

- Double cover $\pi : Y \to X$.
- Riccati foliation $\mathcal{F}$ w.r.t. $\varphi : X \to B$.
- Double Riccati foliation $\pi^*\mathcal{F}$: $\omega_Y := \pi^*\omega_X$, where

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3. $\mathcal{F}$-invariant curve

- $\mathcal{F}$ foliation: $\{(U_\alpha, \omega_\alpha)\}$ (or $\{(U_\alpha, s_\alpha)\}$),

$$s_\alpha = A_\alpha \frac{\partial}{\partial x_\alpha} + B_\alpha \frac{\partial}{\partial y_\alpha}$$

  or

$$\omega_\alpha = B_\alpha dx_\alpha - A_\alpha dy_\alpha.$$

- $C \subseteq X$ curve defined by $f_\alpha = 0$ on U_α.
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C is $\mathcal{F}$-invariant iff

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$$f_\alpha \mid s(f_\alpha)$$
- iff f_α is the solution of ODE
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Example

Let $\mathcal{F}$ be a foliation generated by a fibration$f : X \to C$. Then $C \subseteq X$ is $\mathcal{F}$-invariant iff C lies in the fibers of f.

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3. $\mathcal{F}$-invariant curve

$s :=$ the number of irreducible compact $\mathcal{F}$-invariant curves.

- Question 1: When does $s = \infty$?
- $\mathcal{F}$ generated by a fibration (i.e. $\mathcal{F}$ is algebraic) $\Longrightarrow s = \infty$.

Theorem (Jouanolou, 1978)

If

$$s \geq h^0(X, K_{\mathcal{F}}) + h^{1,1}(X) - h^{1,0}(X) + 2,$$

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- Question 2: How to determine all $\mathcal{F}$-invariant curves when $s < \infty$?

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4. Invariants of $\mathcal{F}$

- Kodaira dimension of $\mathcal{F}$:

$$Kod(\mathcal{F}) := \limsup_{n\to+\infty} \frac{\log h^0(nK_{\mathcal{F}})}{\log n}.$$

- Numerical Kodaira dimension of $\mathcal{F}$: $\nu(\mathcal{F})$.
- (Brunella) $Kod(\mathcal{F}) \neq \nu(\mathcal{F})$ iff $Kod(\mathcal{F}) = -\infty$ and $\nu(\mathcal{F}) = 1$.
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- Chern number of $\mathcal{F}$ (Tan 2015):

$$c_1^2(\mathcal{F}) \geq 0, c_2(\mathcal{F}) \geq 0, \chi(\mathcal{F}) \geq 0,$$

$$c_1^2(\mathcal{F}) + c_2(\mathcal{F}) = 12\chi(\mathcal{F}).$$

Example (Tan 2015)

Let $\mathcal{F}$ be a foliation generated by a fibration $f : X \to C$. Then

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- $\mathcal{F}$ Riccati foliation w.r.t. a Hirzebruch surface $\varphi : X \to B$.

Theorem (Lu, Tan)

(1) $c_1^2(\mathcal{F}) = c_2(\mathcal{F}) = \chi(\mathcal{F}) = 0$.
(2) $kod(\mathcal{F}) \leq 1$.
(3) $p_n(\mathcal{F}) = \max\left\{ n \deg \mathcal{F} - \sum_p \left\lceil \frac{n}{n_p} \right\rceil + 1, 0 \right\}$
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2. Classification

- Classification of $\mathcal{F}$ with $kod(\omega) = -\infty$ (Lu, Tan)

Theorem

Up to a birational map, we have

1. $\omega = dy$;
2. $\omega = \lambda y dx - x dy$ ($\lambda \in \mathbb{Q}^+$ and $\lambda \leq \frac{1}{2}$);
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4. $\omega = (y^2 + (8x-4)y - 5x)dx - 12x(x-1)dy$;
5. $\omega = (y^2 + (18x-12)y - 7x)dx - 24x(x-1)dy$;
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3. Algebraic Riccati foliation

- When is $\mathcal{F}$ an algebraic Riccati foliation ?

Theorem (Gong, Lu, Tan)

$\mathcal{F}$ is algebraic iff it occurs in one of the following cases (up to a birational map):

$$\begin{array}{ll} (O) & \omega = dy; \\ (A_n) & \omega = \varphi' y dx - n\varphi dy; \\ (D_{n+2}) & \omega = \varphi' \left(y^2 + n(\varphi - 1)y - \varphi\right) dx - 2n\varphi(\varphi - 1)dy; \\ (E_6) & \omega = \varphi' \left(y^2 + 4(2\varphi - 1)y - 5\varphi\right) dx - 12\varphi(\varphi - 1)dy; \\ (E_7) & \omega = \varphi' \left(y^2 + 6(3\varphi - 2)y - 7\varphi\right) dx - 24\varphi(\varphi - 1)dy; \\ (E_8) & \omega = \varphi'(y^2 + 10(4\varphi - 3)y - 11\varphi)dx - 60\varphi(\varphi - 1)dy. \end{array}$$

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3. Algebraic Riccati foliation

- Algebraic foliation of type A_n

Corollary (Gong, Lu, Tan)

$\mathcal{F}$ is an algebraic foliation of type A_n
$\Longleftrightarrow$ it has two $\mathcal{F}$-invariant section of φ
$\Longleftrightarrow$ it is from a fibration $f : X \to \mathbb{P}^1$ with two singular fibers.

- A fibration $f : X \to C$ is Riccatian of type A_n (D_n, E_k,...)
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Type	Riccati foliations	Families	Singular fibers
A_1	$(3x^2+1)ydx - 2(x^3+x+c)dy$	$y^2 = t(x^3+x+c)$	$\mathrm{I}_0^*, \mathrm{I}_0^*$
A_2	$(2x-1)ydx - 3x(x-1)dy$	$y^3 = tx(x-1)$	$\mathrm{IV}, \mathrm{IV}^*$
A_3	$(2x-1)ydx - 4x(x-1)dy$	$y^4 = tx(x-1)$	$\mathrm{III}, \mathrm{III}^*$
A_5	$(3x-2)ydx - 6x(x-1)dy$	$y^6 = tx^2(x-1)$	$\mathrm{II}, \mathrm{II}^*$
D_4	$(y^2-xy-1)dx - (3x^2-4)dy$	$g_1{}^3 = t(3x^2-4)g_2{}^3$	$\mathrm{IV}, \mathrm{IV}^*, 2\mathrm{I}_0$

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$$g_1 := (x^2-1)y^4 - 2xy^3 + 6y^2 - 6xy + 3, \quad g_2 := y^4 - 4xy^3 + 6y^2 - 3.$$

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4.Discriminant

- $\omega = (g_0y^2 + g_1y + g_2)dx - dy$, $g_i \in \mathbb{C}(x)$.
- Take

$$g(x) = \begin{cases} \frac{1}{2}(g_1 + \frac{g_0'}{g_0}), & g_0 \neq 0, \\ \frac{1}{2}g_1, & g_0 = 0. \end{cases}$$

- Discriminant of $\mathcal{F}$:

$$\Delta(\omega) = g'(x) - g(x)^2 - g_0(x)g_2(x).$$

- $\Delta(\omega) \in H^0(S^2\Omega_{\mathbb{P}^1}(\log T))$ where

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- Invariance of Discriminant

Theorem (Gong,Lu,Tan)

$\Delta(\omega) = \Delta(\tilde{\omega})$ iff $\widetilde{\mathcal{F}}$ can becomes $\mathcal{F}$ by flipping maps and choosing suitable coordinates.

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- If all singularities of $\mathcal{F}$ have non-zero eigenvalue, then

$$\Delta(\omega) = \sum_p \frac{1 - {\lambda_p}^2}{4(x-p)^2} + \sum_p \frac{\mu_p}{x-p}$$

where

- $p \in \mathbb{P}^2$ runs over all points whose inverse image F_p is $\mathcal{F}$-invariant.
- $\pm\lambda_p$ is the eigenvalue of the singularities lying on F_p.
- $\sum_p \mu_p = 0$.

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- Double cover $\pi : X \to Y$ with branch locus R
  Riccati foliation $\mathcal{G}$ on Y w.r.t. φ
  Double Riccati foliation $\mathcal{F} = \pi^*\mathcal{G}$
- Formulae for Chern numbers.

Theorem (Hong, Lu, Tan)

$$\chi(\mathcal{F}) = \frac{1}{12}\sum_{p\in R} s_2(p) + \frac{1}{4}(g+1)\deg\mathcal{G},$$

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where $p \in R$ runs over the nodes and the tangent points of R to $\mathcal{G}$, $s_1(p)$ and $s_2(p)$ are local invariants of the branch locus with respect to $\mathcal{G}$, and $\nu(\mathrm{I}_a)$ is the number of fibers of type I_a.

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Thank you!