WEAK STABILITY OF TRANSONIC CONTACT DISCONTINUITIES IN THREE-DIMENSIONAL STEADY NON-ISENTROPIC COMPRESSIBLE EULER FLOWS *

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Abstract. In this paper, we study the stability of contact discontinuities that separate a C^1 supersonic flow from a static gas, governed by the three dimensional steady non-isentropic compressible Euler equations. The linear stability problem of this transonic contact discontinuity is formulated as a one-phase free boundary value problem for a hyperbolic system with the boundary being characteristics. By calculating the Kreiss-Lopatinskii determinant for this boundary value problem, we conclude that this transonic contact discontinuity is always stable, but only in a weak sense because the Kreiss–Lopatinskii condition fails exactly at the poles of the symbols associated with the linearized hyperbolic operators. Both of planar and non-planar contact discontinuities are studied. We establish the energy estimates of solutions to the linearized problem at a contact discontinuity, by constructing the Kreiss symmetrizers microlocally away from the poles of the symbols, and studying the equations directly at each pole. The non-planar case is studied by using the calculus of paradifferential operators. The failure of the uniform Kreiss-Lopatinskii condition leads to a loss of derivatives of solutions in estimates.

Key words. Non-isentropic Euler equations, steady contact discontinuity, transonic, free boundary, characteristic boundary, weak stability, Kreiss–Lopatinskii condition, energy estimates, loss of derivative

AMS subject classifications. 35L50, 35Q31, 76J20, 76H05

1. Introduction. This work is devoted to investigating the stability of contact discontinuities in three-dimensional steady non-isentropic compressible flows governed by the full Euler equations that separate a C^1 supersonic flow from a static gas (flow with zero velocity, hence subsonic, see Fig. 1.1). Such transonic contact discontinuities occur ubiquitously in supersonic jet flows, cf. [14, §148, p. 387]. They are either vortex sheets or entropy waves or combination of them, since both tangential velocity and entropy may experience jumps across the contact discontinuity fronts.

More specifically, suppose there is a solid convex corner given by $\{(x, y, z) \in \mathbb{R}^3 : x < 0, y \in \mathbb{R}, z < 0\}$ in the space $\mathbb{R}^3$. Set $\mathbb{I} = \{x = 0, y \in \mathbb{R}, z = 0\}$. We can construct the following flow field containing a contact discontinuity front at $\{x > 0, y \in \mathbb{R}, z = 0\}$ that issuing from $\mathbb{I}$: there are uniform supersonic flows with velocity $(\underline{u}, \underline{v}, 0)$, pressure $\underline{p}$, density $\underline{\rho}_+$, in the region $\{x \in \mathbb{R}, y \in \mathbb{R}, z > 0\}$ (supersonic means $\underline{u} > \underline{c}_+$, for the sonic speed $\underline{c}_+$ to be specified later), and the gas in $\{x > 0, y \in \mathbb{R}, z < 0\}$ is static (velocity is zero), with pressure $\underline{p}$, density $\underline{\rho}_-$. Now if there are suitable small perturbations of the upcoming supersonic flow near the edge $\mathbb{I}$, we are wondering whether such flow pattern would still exist in a neighborhood of the origin O.

For the two-dimensional case (i.e., the flow does not depend on y), Chen, Kukreja and Yuan [5, 6] have shown that such transonic contact discontinuities are stable and

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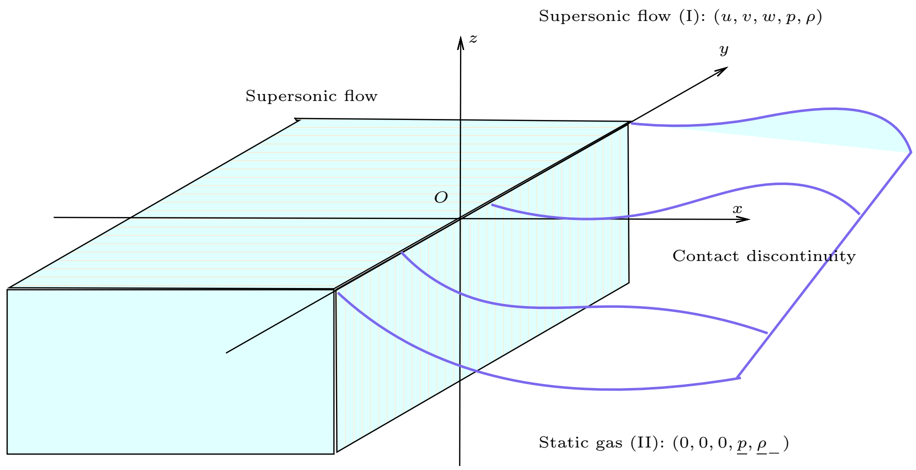

FIG. 1.1. *A contact discontinuity emerged from the edge (y-axis) that separates the gas with velocity zero below from a supersonic flow above.*

well-posed in the class of BV functions by using the front tracking method. The aim of this paper is to study the linear stability of such contact discontinuities for the genuinely multi-dimensional case, by obtaining some fundamental energy estimates of solutions to the related linear problems. This is the crucial step for studying the nonlinear stability of this transonic contact discontinuity. We remark that for the time-dependent problem, it is known that the contact discontinuity is violently unstable for the three-dimensional Euler equations (see [20, pp. 222–225] or [13] and references therein). For the two dimensional case, Coulombel and Secchi showed in their celebrated works [11, 12] that for the isentropic Euler equations, certain rather strong supersonic contact discontinuities are weakly stable. The linear stability of a planar contact discontinuity for the unsteady non-isentropic Euler equations in two space variables was studied by Morando and Trebeschi [19]. Recently, Wang and Yu [22] also studied the linear stability of contact discontinuity for the three dimensional steady Euler equations, but the flow on both sides of the contact discontinuity are supposed to be supersonic. See also [1, 3, 7, 8, 10, 15, 16, 17, 21, 23, 24] and references therein for other related works on the stability of elementary waves in multidimensional hyperbolic conservation laws. Comparing to studies on transonic shocks in steady flows (see, for example, [9]), the significant difference here is that the free boundary is characteristic, and we will mainly study multidimensional hyperbolic equations rather than elliptic-hyperbolic composite systems.

In contrast to the aforementioned works, the problem we considered here is actually a one-phase free boundary problem. At first glace since there involve both supersonic and subsonic flows, the steady Euler equations are then of composite-mixed elliptic-hyperbolic type with a free boundary (i.e., the contact discontinuity). But a merit of our problem is that the gas will stay static in the elliptic region; it will not be influenced by the supersonic flow. So we actually have a one-phase free boundary problem for a hyperbolic system. It seems that this work is the first one dealing with such one-phase hyperbolic free boundary problems.

Like previous works, since the boundary is characteristics, the boundary matrix in the linearized differential operators is singular, which leads to the reduced system for the non-characteristic components of unknowns has poles. Even though in the steady supersonic flows, the three dimensional Euler system is hyperbolic as the supersonic direction of flows is time-like, however we notice that one of crucial differences from the two-dimensional unsteady Euler equations studied in [11], is that there is a velocity component along the "time" direction for the three dimensional steady Euler system. Thus, the tangential velocity field of the three-dimensional transonic steady contact discontinuity has dimension two while it is only one-dimensional along the contact discontinuity front in the two-dimensional unsteady Euler flows (cf. [11]), so the analysis of stability of the contact discontinuities in three dimensional steady Euler system is more subtle. For example, in the following discussion we find a new phenomenon not appeared in [11], namely the associated Lopatinskii determinant for the reduced system of our problem vanishes exactly at the poles of the symbols. This requires one to control the solutions near the poles in a quite delicate way to close the estimates (see section 4.8).

In [22], the authors found that to have the weakly linear stability of a steady supersonic contact discontinuity in three dimensional isentropic Euler systems requires that the tangential velocity field on a space-like plane must be supersonic as well on both sides of the contact discontinuity front. It is quite surprising to discover in this paper that the steady transonic contact discontinuities in three dimensional Euler systems are always weakly stable, even for the non-isentropic flows! This demonstrates the observation that the transonic contact discontinuities are stronger, so it is likely to be more robust.

To state our results of weakly linear stability, we first formulate the nonlinear problem in section 2, where we also review some important properties of the steady Euler system and the notion of contact discontinuities. In section 3 we study the linear stability of a planar transonic contact discontinuity. We will see the Kreiss–Lopatinskii condition holds for the corresponding linearized constant coefficient problem but the uniform Kreiss–Lopatinskii condition always fails exactly at poles of the reduced problem for the associated non-characteristic components of unknowns. By constructing Kreiss' symmetrizers, we obtain the first main result of this paper, Theorem 3.7, a basic L^2 estimate for the linearized problem of transonic contact discontinuity at a planar discontinuity. This estimate exhibits a loss of one derivative, which shows that this transonic contact discontinuity is weakly stable. Then with the argument of this special while crucial case, in section 4, we use para-differential calculus and microlocalization techniques to derive energy estimate of solutions to the linearized problem at a non-planar transonic contact discontinuity, that is Theorem 4.1, the second main result of this paper. In particular, to estimate near the poles needs certain new idea and techniques, since where the uniform Kreiss–Lopatinskii condition fails. We show in detail how to treat this case in section 4.8. The definitions and basic properties of para-differential calculus we used here can be found in appendices of [2] or [10, 11].

2. Formulation of nonlinear problems. In this section, after review some basic facts of the three-dimensional steady non-isentropic Euler equations, we formulate the nonlinear problem on stability of a transonic contact discontinuity.

2.1. Three-dimensional steady non-isentropic Euler equations. The motion of steady non-isentropic flow of perfect gas without exterior force is governed by the following three-dimensional (steady) full compressible Euler system expressing conservation of mass, momentum and energy ([14, (7.09.2) in p.15, (8.02.2) in p.16, and (9.01) in p.17]):

$$(\rho u)_x + (\rho v)_y + (\rho w)_z = 0, \tag{2.1}$$
$$(\rho u^2 + p)_x + (\rho uv)_y + (\rho uw)_z = 0, \tag{2.2}$$
$$(\rho vu)_x + (\rho v^2 + p)_y + (\rho vw)_z = 0, \tag{2.3}$$
$$(\rho wu)_x + (\rho wv)_y + (\rho w^2 + p)_z = 0, \tag{2.4}$$
$$((\rho e + \frac{1}{2}\rho q^2 + p)u)_x + ((\rho e + \frac{1}{2}\rho q^2 + p)v)_y + ((\rho e + \frac{1}{2}\rho q^2 + p)w)_z = 0. \tag{2.5}$$

(In this work subscript means partial derivatives.) Here $q = \sqrt{u^2+v^2+w^2}$ is the speed of the flow, and e is the specific internal energy of the gas.

We consider specifically the polytropic gas. The state function is (cf. [14, (3.03) in p.6]) $p = A(S)\rho^\Gamma$, with S the entropy, $A(S) = (\Gamma-1)\exp((S-S_0)/c_\nu)$, c_ν a positive constant, S_0 a given number, $\Gamma = \frac{R}{c_\nu} + 1$ the adiabatic exponent (cf. (4.10) in [14, p.9]), and R a positive constant. Then the temperature of the gas is given by $T = \frac{p}{R\rho}$; the internal energy $e = c_\nu T = \frac{c_\nu}{R}\frac{p}{\rho} = \frac{1}{\Gamma-1}\frac{p}{\rho}$; and the local sonic speed is given by $c = \sqrt{\Gamma p/\rho}$. For these relations, see [14, p.7].

Since we consider piecewise C^1 solutions, we may use the symmetric form of (2.1)–(2.5) whenever the solution is C^1 and $\rho > 0$ (see [2, p.395]):

$$A_1(U)\partial_x U + A_2(U)\partial_y U + A_3(U)\partial_z U = 0, \tag{2.6}$$

where $U = (u, v, w, p, S)^T$, and

$$A_1 = \begin{pmatrix} \rho u & 0 & 0 & 1 & 0 \\ 0 & \rho u & 0 & 0 & 0 \\ 0 & 0 & \rho u & 0 & 0 \\ 1 & 0 & 0 & \frac{u}{\rho c^2} & 0 \\ 0 & 0 & 0 & 0 & u \end{pmatrix}, \quad A_2 = \begin{pmatrix} \rho v & 0 & 0 & 0 & 0 \\ 0 & \rho v & 0 & 1 & 0 \\ 0 & 0 & \rho v & 0 & 0 \\ 0 & 1 & 0 & \frac{v}{\rho c^2} & 0 \\ 0 & 0 & 0 & 0 & v \end{pmatrix},$$

$$A_3 = \begin{pmatrix} \rho w & 0 & 0 & 0 & 0 \\ 0 & \rho w & 0 & 0 & 0 \\ 0 & 0 & \rho w & 1 & 0 \\ 0 & 0 & 1 & \frac{w}{\rho c^2} & 0 \\ 0 & 0 & 0 & 0 & w \end{pmatrix}.$$

For $u > c$, we can show the system (2.6) is symmetric hyperbolic with respect to x. Let ξ, $\eta \in \mathbb{R}$. We solve the eigenvalue λ by the characteristic equation

$$\det(\lambda A_1 - (\xi A_2 + \eta A_3)) = 0. \tag{2.7}$$

A direct calculation yields that there are two simple solutions λ_1, λ_3, and one solution λ_2 with multiplicity three (cf. [9, p.538]) given as follows:

$$\lambda_1 = \lambda_R - \lambda_I, \quad \lambda_2 = \frac{v\xi + w\eta}{u}, \quad \lambda_3 = \lambda_R + \lambda_I, \tag{2.8}$$

where

$$\lambda_R = \frac{u}{u^2-c^2}(v\xi + w\eta), \quad \lambda_I = \frac{c\sqrt{(v\xi+w\eta)^2 + (u^2-c^2)(\xi^2+\eta^2)}}{u^2-c^2}.$$

One sees immediately that for $\lambda_{1,3}$ to be real, i.e., $(v\xi + w\eta)^2 \geq c^2 - u^2$ whenever $\xi^2+\eta^2 = 1$, there must hold $|u| \geq c$. Since A_1 should be nonsingular, we need $|u| > c$. Also, if $|u| > c$, as can be checked directly, $\lambda_1, \lambda_2, \lambda_3$ are distinct. For $q < c$, the system would be elliptic-hyperbolic composite-mixed type as studied in [9].

Also, suppose $U = (u, v, w, p, S)^T$ satisfies $|u| > c$ and $\rho > 0$, then the eigenvalues λ_1, λ_3 are genuinely nonlinear:

$$\nabla_U \lambda(U; \xi, \eta) \bullet r(U; \xi, \eta) \neq 0 \quad \text{for all} \;\; \xi^2 + \eta^2 = 1,$$

and the eigenvalue λ_2 is linearly degenerate:

$$\nabla_U \lambda(U; \xi, \eta) \bullet r(U; \xi, \eta) \equiv 0 \quad \text{for all} \;\; \xi^2 + \eta^2 = 1.$$

Here r is the corresponding right eigenvector associated with the eigenvalue λ. These claims can also be proved by direct calculations.

2.2. Jump conditions and contact discontinuity. Let $\mathscr{D} : \{\psi(x, y, z) = 0\}$ be a C^1 surface in $\mathbb{R}^3$ across which the piecewise C^1 solution U of the Euler system (2.1)–(2.5) has a jump. With integration by parts, we get the Rankine–Hugoniot jump conditions (cf. [15, (1.8.2) in p.21]):

$$[\rho u]\psi_x + [\rho v]\psi_y + [\rho w]\psi_z = 0, \tag{2.9}$$
$$[\rho u^2 + p]\psi_x + [\rho uv]\psi_y + [\rho uw]\psi_z = 0, \tag{2.10}$$
$$[\rho vu]\psi_x + [\rho v^2 + p]\psi_y + [\rho vw]\psi_z = 0, \tag{2.11}$$
$$[\rho wu]\psi_x + [\rho wv]\psi_y + [\rho w^2 + p]\psi_z = 0, \tag{2.12}$$
$$[\rho u(\frac{1}{2}q^2 + \frac{c^2}{\Gamma - 1})]\psi_x + [\rho v(\frac{1}{2}q^2 + \frac{c^2}{\Gamma - 1})]\psi_y + [\rho w(\frac{1}{2}q^2 + \frac{c^2}{\Gamma - 1})]\psi_z = 0. \tag{2.13}$$

Here as usual $[\cdot]$ stands for the jump of a quantity across $\mathscr{D}$. Set $m = \rho u\psi_x + \rho v\psi_y + \rho w\psi_z$, which is the mass transfer flux across $\mathscr{D}$. Let $m^\pm$ be the mass transfer flux measured at the two sides of $\mathscr{D}$. Equation (2.9) means $m^+ - m^- = 0$. We now specify two special cases:

- $m^+ = m^- = 0$ on $\mathscr{D}$. In this case, $\mathscr{D}$ is called a *contact discontinuity (front).*
- $m^+ = m^- \neq 0$ on $\mathscr{D}$. In this case, $\mathscr{D}$ is called a *shock-front.*

For a contact discontinuity, the Rankine–Hugoniot conditions (2.9)–(2.13) can be significantly simplified as

$$u^\pm\psi_x + v^\pm\psi_y + w^\pm\psi_z = 0, \tag{2.14}$$
$$p^+ - p^- = 0. \tag{2.15}$$

Obviously, the contact discontinuity is a characteristic surface associated with λ_2.

2.3. A free boundary problem. Now we can formulate the transonic contact discontinuity as a free boundary problem, with the characteristic free boundary being the contact discontinuity front. Suppose its equation is

$$z = \psi(x, y) \quad \text{for} \;\; x \geq 0, \; y \in \mathbb{R}, \; \text{with} \;\; \psi(0, y) = 0.$$

Set $\Omega_\psi := \{(x, y, z) \in \mathbb{R}^3 : x > 0, \; z > \psi(x, y), \; y \in \mathbb{R}\}$. Then the unknowns U and ψ should solve:

$$\begin{cases} (2.6), & \text{in} \;\; \Omega_\psi, \\ U = U_0, & \text{on} \;\; \{x = 0, \; z > 0, \; y \in \mathbb{R}\}, \\ p = \underline{p}, & \text{on} \;\; \{z = \psi(x, y)\}, \\ u\psi_x + v\psi_y - w = 0, & \text{on} \;\; \{z = \psi(x, y)\}. \end{cases} \tag{2.16}$$

The last two conditions follows from (2.14) (2.15) applied to the upper unknown supersonic flow $U^+ = U$ and the lower given static gas $U^- = (0,0,0,\underline{p},\underline{\rho_-})$. We suppose the initial data U_0 is a small perturbation of the reference state $\underline{U}_+ = (\underline{u},\underline{v},0,\underline{p},\underline{S}_+)$ with $\underline{u} > \underline{c} := \underline{c}_+$ and expect ψ to be a small perturbation of $z = 0$ for $x > 0$ small.

2.4. Nonlinear problem with a fixed boundary. The problem (2.16) has a free boundary since $\psi(x,y)$ is also unknown. As in [11, 16], to fix the free boundary, we use a change of independent variables $(x,y,z) \mapsto (x',y',z')$ given by

$$x = x', \quad y = y', \quad z = \Psi(x',y',z'),$$

which transforms Ω_ψ to $\mathbb{R}^+_{x'} \times \mathbb{R}_{y'} \times \mathbb{R}^+_{z'}$, with Ψ being an unknown satisfying

$$\Psi(x',y',0) = \psi(x',y'), \quad \partial_{z'}\Psi \geq \kappa$$

for a positive constant κ to make sure the change-of-variables is invertible.

Inspiring by the eikonal equation of ψ, the last line given in (2.16), we require the function $\Psi(x',y',z')$ to be determined by the following problem

$$\begin{cases} \frac{\partial \Psi}{\partial x'} u + \frac{\partial \Psi}{\partial y'} v - w = 0, & \text{in } \{x' > 0,\ z' > 0\}, \\ \Psi(0,y',z') = z'. \end{cases} \tag{2.17}$$

Denote by $\tilde{U}(x',y',z') = U(x',y',\Psi(x',y',z'))$. Then, from (2.6), $\tilde{U}(x',y',z')$ should satisfy, in $\{z' > 0\}$, that

$$A_1(U)\partial_{x'}U + A_2(U)\partial_{y'}U + \tilde{A}_3(U,\mathrm{d}\Psi)\partial_{z'}U = 0, \tag{2.18}$$

where the tildes of U had been dropped for simplicity, and

$$\tilde{A}_3(U,\mathrm{d}\Psi) := \frac{1}{\partial_{z'}\Psi}\Big(A_3(U) - A_2(U)\partial_{y'}\Psi - A_1(U)\partial_{x'}\Psi\Big). \tag{2.19}$$

We then get a fixed boundary problem:

$$\begin{cases} (2.18), & \text{in } \{x' > 0,\ y' \in \mathbb{R},\ z' > 0\}, \\ U = U_0, & \text{on } \{x' = 0,\ y' \in \mathbb{R},\ z' > 0\}, \\ p = \underline{p}, & \text{on } \{x' > 0,\ y' \in \mathbb{R},\ z' = 0\}, \end{cases} \tag{2.20}$$

which is coupled to (2.17), while $\psi(x',y')$ satisfies a transport equation

$$\begin{cases} u\partial_{x'}\psi + v\partial_{y'}\psi = w, & \text{in } \{x' > 0,\ y' \in \mathbb{R}\}, \\ \psi(0,y') = 0, & \text{on } \{x' = 0,\ y' \in \mathbb{R}\}. \end{cases} \tag{2.21}$$

In the sequel, for simplicity of writing, we replace (x',y',z') by (x,y,z). Problem (2.20) and (2.21) is the nonlinear problem we need to study.

3. Constant coefficient linearized problem. In this section, we linearize problem (2.20) and (2.21) at a planar contact discontinuity. We first investigate the related Kreiss–Lopatinskii condition, and then use the information to obtain energy estimates by construction of Kreiss' symmetrizers in the frequency space.

3.1. The constant coefficient problem. Let $\underline{U} = (\underline{u}, \underline{v}, 0, \underline{p}, \underline{S_+})$, $\underline{\psi} \equiv 0$, $\underline{\Psi} = z$ be the constant reference state (recall we assume $\underline{u} > \underline{c_+} > 0$), and $\dot{U}$, $\dot{\psi}$, $\dot{\Psi}$ be their corresponding perturbations. From (2.20), we get the following constant coefficient linearized problem:

$$\begin{cases} A_1(\underline{U})\partial_x \dot{U} + A_2(\underline{U})\partial_y \dot{U} + A_3(\underline{U})\partial_z \dot{U} = f, & \text{in } \{x \in \mathbb{R}^+,\ y \in \mathbb{R},\ z > 0\}, \\ \dot{p} = g, & \text{on } \{x \in \mathbb{R}^+,\ y \in \mathbb{R},\ z = 0\}, \\ \dot{U}|_{x \le 0} = 0. \end{cases} \tag{3.1}$$

We also find, from (2.21), the linearized equation of $\dot{\psi}$:

$$\begin{cases} \underline{u}\partial_x \dot{\psi} + \underline{v}\partial_y \dot{\psi} - \dot{w} = h, & \text{in } \{x \in \mathbb{R}^+,\ y \in \mathbb{R}\}, \\ \dot{\psi}|_{x \le 0} = 0. \end{cases} \tag{3.2}$$

So $\dot{U}$ and $\dot{\psi}$ are actually decoupled in (3.1) and (3.2).

3.2. The Kreiss–Lopatinskii condition. In this section, we are going to see whether the Kreiss–Lopatinskii condition holds for the boundary value problem (3.1), which is a crucial point to have the well-posedness of this problem.

For simplicity of notations, we shall drop the underlines of the background state $\underline{U}$ in the problem (3.1) for the following calculations. So c in the rest of this section actually means $\underline{c_+}$.

First, let us introduce certain transformations in order to make the boundary matrix $A_3(\underline{U})$ in (3.1) to be a diagonal one. For this, let us set

$$P = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & c & c & \\ & & \Gamma p & -\Gamma p & \\ & & & & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{\Gamma p} & & & & \\ & \frac{1}{\Gamma p} & & & \\ & & \frac{c}{\Gamma p} & 1 & \\ & & \frac{c}{\Gamma p} & -1 & \\ & & & & 1 \end{pmatrix},$$

and $\dot{U} = PV$. Then, obviously one has

$$V_1 = \dot{u},\ V_2 = \dot{v},\ V_3 = \frac{\dot{w}}{2c} + \frac{\dot{p}}{2\Gamma p},\ V_4 = \frac{\dot{w}}{2c} - \frac{\dot{p}}{2\Gamma p},\ V_5 = \dot{S},$$

$$B_3 := QA_3(\underline{U})P = \mathrm{diag}\{0, 0, 2c, -2c, 0\},$$

and

$$B_1 := QA_1(\underline{U})P = \begin{pmatrix} \frac{u}{c^2} & 0 & 1 & -1 & 0 \\ 0 & \frac{u}{c^2} & 0 & 0 & 0 \\ 1 & 0 & 2u & 0 & 0 \\ -1 & 0 & 0 & 2u & 0 \\ 0 & 0 & 0 & 0 & u \end{pmatrix},$$

$$B_2 := QA_2(\underline{U})P = \begin{pmatrix} \frac{v}{c^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{v}{c^2} & 1 & -1 & 0 \\ 0 & 1 & 2v & 0 & 0 \\ 0 & -1 & 0 & 2v & 0 \\ 0 & 0 & 0 & 0 & v \end{pmatrix}.$$

So (3.1) becomes

$$\begin{cases} B_1\partial_x V + B_2\partial_y V + B_3\partial_z V = Qf, & \text{in } \{z>0\}, \\ V_3 - V_4 = g/(\Gamma p), & \text{on } \{z=0\}. \end{cases} \tag{3.3}$$

Next, we need to change the problem (3.3) into a dynamical system for the non-characteristic components of the unknown vector V, henceforth define the associated Lopatinskii determinant. In this way, we shall see that the coefficient matrix in the reduced system has poles.

Denote by $\hat{V}(z,\tau,\eta)$ the Laplace transform with respect to x (with dual variable $\tau=\gamma+i\delta$, $\gamma\geq 0$, $\delta\in\mathbb{R}$) and Fourier transform with respect to y (with dual variable η) of V. From (3.3) it follows that $\hat{V}(z,\tau,\eta)$ satisfies the following problem

$$\begin{cases} (\tau B_1 + i\eta B_2)\hat{V} + B_3\partial_z\hat{V} = \widehat{Qf}, \\ (0,0,1,-1,0)\hat{V}|_{z=0} = \widehat{g/(\Gamma p)}. \end{cases} \tag{3.4}$$

We find

$$\tau B_1 + i\eta B_2 = \begin{pmatrix} \frac{u\tau+iv\eta}{c^2} & 0 & \tau & -\tau & 0 \\ 0 & \frac{u\tau+iv\eta}{c^2} & i\eta & -i\eta & 0 \\ \tau & i\eta & 2(u\tau+iv\eta) & 0 & 0 \\ -\tau & -i\eta & 0 & 2(u\tau+iv\eta) & 0 \\ 0 & 0 & 0 & 0 & u\tau+iv\eta \end{pmatrix}.$$

Therefore, recall $c^2=\Gamma p/\rho$, equations (3.4) may be written line by line as

$$\begin{aligned} &\rho(u\tau+iv\eta)\hat{V}_1 + \Gamma p\tau(\hat{V}_3-\hat{V}_4) = \widehat{Qf}_1, \\ &\rho(u\tau+iv\eta)\hat{V}_2 + i\Gamma p\eta(\hat{V}_3-\hat{V}_4) = \widehat{Qf}_2, \\ &(u\tau+iv\eta)\hat{V}_5 = \widehat{Qf}_5, \\ &\frac{\mathrm{d}\hat{V}_3}{\mathrm{d}z} + \frac{1}{2c}(\tau\hat{V}_1 + i\eta\hat{V}_2) + \frac{u\tau+iv\eta}{c}\hat{V}_3 = \widehat{Qf}_4, \\ &\frac{\mathrm{d}\hat{V}_4}{\mathrm{d}z} + \frac{1}{2c}(\tau\hat{V}_1 + i\eta\hat{V}_2) - \frac{u\tau+iv\eta}{c}\hat{V}_4 = \widehat{Qf}_5. \end{aligned}$$

Note $\tau\hat{V}_1 + i\eta\hat{V}_2 = c^2\frac{\eta^2-\tau^2}{u\tau+iv\eta}(\hat{V}_3-\hat{V}_4)$, hence the last two equations can be written as

$$\frac{\mathrm{d}\hat{V}^{\mathrm{nc}}}{\mathrm{d}z} = \mathcal{B}(\tau,\eta)\hat{V}^{\mathrm{nc}} + (\widehat{Qf}_4, \widehat{Qf}_5)^T, \tag{3.5}$$

where $\hat{V}^{\mathrm{nc}} = (\hat{V}_3,\hat{V}_4)^T$ are the non-characteristic unknown variables, and

$$\mathcal{B}(\tau,\eta) = \begin{pmatrix} -\left(\frac{c}{2}\frac{\eta^2-\tau^2}{u\tau+iv\eta} + \frac{u\tau+iv\eta}{c}\right) & \frac{c}{2}\frac{\eta^2-\tau^2}{u\tau+iv\eta} \\ -\frac{c}{2}\frac{\eta^2-\tau^2}{u\tau+iv\eta} & \frac{c}{2}\frac{\eta^2-\tau^2}{u\tau+iv\eta} + \frac{u\tau+iv\eta}{c} \end{pmatrix} = \begin{pmatrix} -a & b \\ -b & a \end{pmatrix}. \tag{3.6}$$

The eigenvalues λ of $\mathcal{B}(\tau,\eta)$ are given by

$$\lambda^2 = (a+b)(a-b) = \eta^2 - \tau^2 + \frac{1}{c^2}(u\tau+iv\eta)^2.$$

We need to find the eigenspace $E_-(\tau,\eta)$ associated with the eigenvalue λ_- whose real part is negative when $\mathrm{Re}\,\tau > 0$. For convenience of presentation, we introduce the sets

$$\begin{aligned}\Xi &= \{(\tau,\eta) \in (\mathbb{C}\times\mathbb{R}) \setminus \{(0,0)\} : \ \mathrm{Re}\,\tau \geq 0\},\\ \Sigma &= \{(\tau,\eta) \in \Xi : \ |\tau|^2 + \eta^2 = 1\}.\end{aligned} \tag{3.7}$$

We note λ is homogeneous degree one with respect to (τ,η) in Ξ. By a simple computation, we also see that the eigenvector associated to λ_- can be given as

$$\mathbf{e}_-(\tau,\eta) = (\lambda_-(u\tau + iv\eta) - \frac{c}{2}(\eta^2 - \tau^2) - \frac{(u\tau + iv\eta)^2}{c}, -\frac{c}{2}(\eta^2 - \tau^2))^T, \quad (\tau,\eta) \in \Sigma,$$

and then extended to Ξ with homogenous degree zero. $\mathbf{e}_-(\tau,\eta)$ is a base of $E_-(\tau,\eta)$. So the Lopatinskii determinant for the problem (3.1) is given by

$$\Delta(\tau,\eta) = (1,-1)\mathbf{e}_- = (u\tau + iv\eta)(\lambda_- - \frac{1}{c}(u\tau + iv\eta)), \quad (\tau,\eta) \in \Sigma, \tag{3.8}$$

and is homogenous degree zero in Ξ. For the definitions of Kreiss–Lopatinskii condition and Lopatinskii determinant, see [2, p.108, p.130].

The factor $\lambda - \frac{1}{c}(u\tau + iv\eta)$ vanishes at $\tau = \pm|\eta|$, while $\mathrm{Re}\,\lambda_- < 0$ whenever $\mathrm{Re}\,\tau > 0$. So $\lambda_- - \frac{1}{c}(u\tau + iv\eta)$ can never be zero for $\mathrm{Re}\,\tau \geq 0$. We note the point (τ,η) where $u\tau + iv\eta = 0$ is a pole of the matrix $\mathcal{B}$. Thus, we conclude

Proposition 3.1. *The Lopatinskii determinant $\Delta(\tau,\eta)$ for the problem* (3.1) *vanishes only at the poles of the matrix $\mathcal{B}(\tau,\eta)$.*

This is a new feature comparing to that of [11, 19].

3.3. Estimate of solutions in frequency space. Now we start to derive energy estimate of the solution to the constant coefficient problem (3.1). This mainly relies on the construction of the Kreiss' symmetrizers of the following system

$$\begin{cases}\frac{\mathrm{d}\hat{V}^{\mathrm{nc}}}{\mathrm{d}z} = \mathcal{B}(\tau,\eta)\hat{V}^{\mathrm{nc}}, \quad \text{in } \{z > 0\},\\ \beta\hat{V}^{\mathrm{nc}} = \hat{h}, \quad \text{on } \{z = 0\}, \quad \text{with } \beta = (1,-1),\end{cases} \tag{3.9}$$

for the frequency away from the poles, and a careful analysis of the problem (3.4) near the poles. We note (3.9) is reduced from (3.5) and the boundary condition in (3.4).

Definition 3.2 (Kreiss' Symmetrizers). *For any $(\tau_0,\eta_0) \in \Sigma$, if there is a neighborhood $\mathcal{V}$ and two C^∞ mappings $T : \mathcal{V} \to GL_2(\mathbb{C})$ and $r : \mathcal{V} \to \mathbb{M}_{2\times 2}(\mathbb{C})$ so that*

i) For all $(\tau,\eta) \in \mathcal{V}$, the matrix $r(\tau,\eta)$ is Hermitian, and homogeneous of degree zero with respect to (τ,η);

ii) There exist positive constants k and C so that the following inequalities hold:

$$\mathrm{Re}\Big(r(\tau,\eta)T(\tau,\eta)\mathcal{B}(\tau,\eta)T(\tau,\eta)^{-1}\Big) \geq k\gamma I_2, \qquad \forall (\tau,\eta) \in \mathcal{V}, \tag{3.10}$$

$$r(\tau,\eta) + C\Big(\beta(\tau,\eta)T(\tau,\eta)^{-1}\Big)^*\beta(\tau,\eta)T(\tau,\eta)^{-1} \geq I_2. \tag{3.11}$$

Here I_2 is the 2×2 identity matrix, $\mathrm{Re}\,M := (M + M^*)/2$, and $A \geq B$ mean the matrix $A - B$ is positive-definite.

Then the matrix $r(\tau,\eta)$ is called a local Kreiss' symmetrizer near (τ_0,η_0).

3.3.1. Construction of local symmetrizers. First, let us use Kreiss' idea to construct the symmetrizers of the dynamic system (3.5) for frequencies away from the poles. This will be done for two subcases:

a) Frequency interior points: $\{(\tau,\eta)\in\Sigma:\quad \mathrm{Re}\,\tau>0\}$;

b) Frequency boundary points where Kreiss–Lopatinskii condition holds: $\{(\tau,\eta)\in\Sigma:\quad \mathrm{Re}\,\tau=0\quad\text{and}\quad \Delta(\tau,\eta)\neq 0\}$. We know that these are those points $\{(i\delta,\eta)\in\Sigma:\quad u\delta+v\eta\neq 0\}$.

Case a): Frequency interior point. Suppose (τ_0,η_0) is a frequency interior point ($\mathrm{Re}\,\tau_0>0$), then it has a neighborhood $\mathcal{V}$ that is still contained in the interior of Σ. We know in $\mathcal{V}$ that, since the two eigenvalues of $\mathcal{B}$ must be distinct, the matrix $\mathcal{B}$ is always diagonalizable. Actually, we have the eigenvectors

$$e_-(\tau,\eta)=(\lambda_--a,-b)^T,\quad e_+(\tau,\eta)=(b,\lambda_++a)^T,\qquad (\tau,\eta)\in\mathcal{V},$$

and then $\mathcal{B}(\tau,\eta)(e_-(\tau,\eta),e_+(\tau,\eta))=(e_-(\tau,\eta),e_+(\tau,\eta))\,\mathrm{diag}(\lambda_-,\lambda_+)$. Here and in the following λ_+ is the eigenvalue with positive real part for $\mathrm{Re}\,\tau>0$. So it is natural to define

$$T(\tau,\eta)=(e_-(\tau,\eta),e_+(\tau,\eta))^{-1}=\frac{1}{(\lambda_--a)(\lambda_++a)+b^2}\begin{pmatrix}\lambda_++a & -b\\ b & \lambda_--a\end{pmatrix}.$$

One may check that the denominator $(\lambda_--a)(\lambda_++a)+b^2=2(b^2-a^2)=2\lambda_\pm^2\neq 0$ in $\mathcal{V}$. So $T(\tau,\eta):\mathcal{V}\to GL_2(\mathbb{C})$ is well-defined and smooth. Since

$$T(\tau,\eta)\mathcal{B}(\tau,\eta)T(\tau,\eta)^{-1}=\begin{pmatrix}\lambda_- & 0\\ 0 & \lambda_+\end{pmatrix},$$

we can set

$$r(\tau,\eta)=\begin{pmatrix}-1 & 0\\ 0 & K\end{pmatrix}$$

for some constant $K\geq 1$ to be chosen. It is easily seen that

$$\mathrm{Re}\left(r(\tau,\eta)T(\tau,\eta)\mathcal{B}(\tau,\eta)T(\tau,\eta)^{-1}\right)=\begin{pmatrix}-\mathrm{Re}\,\lambda_- & 0\\ 0 & K\mathrm{Re}\,\lambda_+\end{pmatrix}.$$

Because $\mathcal{V}$ is contained in the interior of Σ, there must be a positive constant $k>0$ so that $\pm\mathrm{Re}\,\lambda_\pm>k$ in $\mathcal{V}$. Therefore, we get

$$\mathrm{Re}\left(r(\tau,\eta)T(\tau,\eta)\mathcal{B}(\tau,\eta)T(\tau,\eta)^{-1}\right)\geq kI_2\geq k\gamma I_2,\qquad \text{in }\mathcal{V},$$

because $0<\gamma\leq 1$.

Let $E_-(\tau,\eta)$ (resp. $E_+(\tau,\eta)$) be the stable (resp. unstable) subspace of $\mathcal{B}(\tau,\eta)$. As the Kreiss–Lopatinskii condition holds in the neighborhood $\mathcal{V}$ of (τ_0,η_0), we have $\ker\beta\cap E_-(\tau,\eta)=\{0\}$ for (τ,η) in $\mathcal{V}$. Since $\dim E_-(\tau,\eta)=1=\dim\ker\beta$, we know $\ker\beta\oplus E_-(\tau,\eta)=\mathbb{C}^2$ and therefore $\beta:\ E_-(\tau,\eta)\to\mathscr{R}(\beta)=\mathbb{C}$ is one-to-one. Then under the change of unknowns $\hat{W}=T\hat{V}^{\mathrm{nc}}$, we explicitly get

$$E_-(\tau,\eta)=\begin{pmatrix}z_-\\ 0\end{pmatrix},\quad E_+(\tau,\eta)=\begin{pmatrix}0\\ z_+\end{pmatrix},\quad z_\pm\in\mathbb{C},$$

and β is replaced by βT^{-1}. Now consider the mapping $\mathbb{C}^2\to\mathbb{C}\oplus\mathbb{C}$ given by

$$Z=\begin{pmatrix}z_-\\ z_+\end{pmatrix}\mapsto(\beta T^{-1}\begin{pmatrix}z_-\\ z_+\end{pmatrix},z_+)^T.$$

This is one-to-one, since for $z_+ = 0$, using the fact that $E_-(\tau,\eta)$ is mapped one-to-one to $\beta T^{-1}E_-(\tau,\eta)$, then $\beta T^{-1}\begin{pmatrix} z_- \\ 0 \end{pmatrix} = 0$ implies $z_- = 0$. Therefore the above mapping is invertible, and there is a constant $C_0 > 0$ independent of $(\tau,\eta) \in \mathcal{V}$ so that $|z_-|^2 \le |Z|^2 \le C_0(|\beta T^{-1}Z|^2 + |z_+|^2)$. Thus we have

$$\begin{aligned} Z^T r(\tau,\eta)Z + 2C_0|\beta T(\tau,\eta)^{-1}Z|^2 &\ge -|z_-|^2 + K|z_+|^2 + 2|z_-|^2 - 2C_0|z_+|^2 \\ &\ge |z_-|^2 + |z_+|^2 = |Z|^2, \quad \forall Z \in \mathbb{C}^2 \end{aligned}$$

by choosing $K = 2C_0 + 1$. The above inequality is equivalent to

$$r(\tau,\eta) + 2C_0\Big(\beta(\tau,\eta)T(\tau,\eta)^{-1}\Big)^*\beta(\tau,\eta)T(\tau,\eta)^{-1} \ge I_2.$$

Therefore we have proved

LEMMA 3.3. *At each frequency interior point (τ_0,η_0), there is a local Kreiss' symmetrizer.*

Case b): Frequency boundary points away from the poles. For points $\{(\tau,\eta) \in \Sigma : \tau = i\delta,\ u\delta + v\eta \ne 0\}$, we have

$$\lambda^2 = \eta^2 + \delta^2 - \frac{1}{c^2}(u\delta + v\eta)^2.$$

The Cauchy–Schwartz inequality implies $(u\delta + v\eta)^2 \le (u^2 + v^2)(\delta^2 + \eta^2)$, so recall $u > c$, we infer the right-hand side may change sign. We discuss this problem for the following three different cases of eigenvalues λ.

Subcase b1). For points $\{(\tau,\eta) \in \Sigma : \tau = i\delta, u\delta + v\eta \ne 0\}$ satisfying $c^2(\eta^2 + \delta^2) - (u\delta+v\eta)^2 > 0$, then $\lambda_\pm = \pm\sqrt{\eta^2 + \delta^2 - \frac{1}{c^2}(u\delta + v\eta)^2}$, hence $\mathcal{B}(\tau,\eta)$ is diagonalizable. One can discuss this case in a way totally similar to that given in Case a), since all $E_\pm(\tau,\eta), e_\pm(\tau,\eta)$ and $\lambda_\pm$ can be continuously extended to this case.

Subcase b2). For the frequency boundary point satisfying $c^2(\eta^2+\delta^2)-(u\delta+v\eta)^2 < 0$, then $\lambda = \pm i\sqrt{\frac{1}{c^2}(u\delta + v\eta)^2 - (\eta^2 + \delta^2)}$, hence $\mathcal{B}(\tau,\eta)$ is still diagonalizable.

We need to decide which of the root should be λ_-. Since $\mathrm{Re}\,\lambda_- < 0$ for $\gamma > 0$ and $\mathrm{Re}\,\lambda_- = 0$ at $\gamma = 0$, we have $\frac{\partial \mathrm{Re}\,\lambda_-}{\partial \mathrm{Re}\,\tau}|_{\gamma=0} < 0$. By the Cauchy–Riemann equation,

$$\begin{aligned} \frac{\partial \mathrm{Re}\,\lambda_-}{\partial \mathrm{Re}\,\tau}\bigg|_{\gamma=0} = \frac{\partial \mathrm{Im}\,\lambda_-}{\partial \mathrm{Im}\,\tau}\bigg|_{\gamma=0} &= \pm\frac{\partial}{\partial\delta}\sqrt{\frac{1}{c^2}(u\delta + v\eta)^2 - (\eta^2 + \delta^2)} \\ &= \pm\frac{1}{c^2}\frac{1}{\sqrt{\frac{1}{c^2}(u\delta + v\eta)^2 - (\eta^2 + \delta^2)}}\big((u^2 - c^2)\delta + uv\eta\big). \end{aligned} \tag{3.12}$$

First, we note that in this Subcase b2), $(u^2 - c^2)\delta + uv\eta \ne 0$. Otherwise, we have $(u\delta + v\eta) = \frac{c^2}{u}\delta$, hence $(u\delta + v\eta)^2/c^2 - (\eta^2 + \delta^2) = ((c^2 - u^2)/u^2)\delta^2 - \eta^2 < 0$ by using $c < u$, which is a contradiction to the assumption of this case!

Thus, we get that

♢ When $(u^2 - c^2)\delta + uv\eta > 0$, we have

$$\lambda_\pm = \pm i\sqrt{\frac{1}{c^2}(u\delta + v\eta)^2 - (\eta^2 + \delta^2)};$$

◇ When $(u^2-c^2)\delta+uv\eta<0$, we have

$$\lambda_\pm=\mp i\sqrt{\frac{1}{c^2}(u\delta+v\eta)^2-(\eta^2+\delta^2)}.$$

Now, we construct a symmetrizer in a neighborhood $\mathcal{V}$ of a point $(\tau_0=i\delta_0,\eta_0)$ of the Subcase b2). We can still use the transform $T(\tau,\eta)$, and $r(\tau,\eta)=\mathrm{diag}(-1,K)$ with $K\geq 1$ to be chosen as before, and obtain that, for $(\tau,\eta)\in\mathcal{V}$ that

$$\mathrm{Re}\left(r(\tau,\eta)T(\tau,\eta)\mathcal{B}(\tau,\eta)T(\tau,\eta)^{-1}\right)=\begin{pmatrix}-\mathrm{Re}\,\lambda_- & 0\\ 0 & K\mathrm{Re}\,\lambda_+\end{pmatrix}.$$

By (3.12), for $\mathcal{V}$ small enough, we have $-\mathrm{Re}\,\lambda_-\geq k\gamma$, and similarly $\mathrm{Re}\,\lambda_+\geq k\gamma$, for some $k>0$ depending only on (τ_0,η_0). This justifies (3.10). The verification of (3.11) is then the same as for frequency interior points, since the stable subspace $E_-(\tau,\eta)$ can be continuously extended to $\mathcal{V}$.

Subcase b3). When $c^2(\eta^2+\delta^2)-(u\delta+v\eta)^2=0$, then $\lambda_\pm=0$, hence $\mathcal{B}(\tau,\eta)$ is not diagonalizable. These (τ,η) are usually called *glancing points*. We shall discuss this case in a way similar to that given in [4, pp. 452–460].

For a point $(i\delta_0,\eta_0)$ of this case, as

$$b=b(i\delta_0,\eta_0)=\frac{c}{2i}\cdot\frac{\eta_0^2+\delta_0^2}{u\delta_0+v\eta_0}=\frac{u\delta_0+v\eta_0}{2ic}\neq 0,$$

there is a small neighborhood $\mathcal{V}$ of $(i\delta_0,\eta_0)$ in Σ, such that $b(\tau,\eta)=\frac{c}{2}\frac{\eta^2-\tau^2}{u\tau+iv\eta}\neq 0$ in $\mathcal{V}$. Define

$$T(\tau,\eta)=\frac{1}{ib}\begin{pmatrix}0 & i\\ -b & -b\end{pmatrix},\quad\text{so}\quad T(\tau,\eta)^{-1}=\begin{pmatrix}-b & -i\\ b & 0\end{pmatrix}.$$

They are smooth in $\mathcal{V}$. Then

$$T(\tau,\eta)\mathcal{B}(\tau,\eta)T(\tau,\eta)^{-1}=\mathrm{a}(\tau,\eta):=\begin{pmatrix}a+b & i\\ 2ib(a+b) & -(a+b)\end{pmatrix}.\tag{3.13}$$

As in [11, p.965], at the point $(i\delta_0,\eta_0)$, we check that (note $(a+b)(i\delta_0,\eta_0)=0$)

$$\begin{aligned}\vartheta&:=\frac{\partial}{\partial\gamma}(2ib(a+b))=\frac{u\delta_0+v\eta_0}{c}\frac{\partial}{\partial\gamma}(a+b)\\&=\frac{u\delta_0+v\eta_0}{c}\frac{\partial}{\partial\gamma}\left(\frac{u\gamma+i(u\delta+v\eta)}{c}+c\frac{\eta^2+\delta^2-\gamma^2-2i\gamma\delta}{u\gamma+i(u\delta+v\eta)}\right)\Bigg|_{(\gamma=0,\delta=\delta_0,\eta=\eta_0)}\\&=2\frac{u\delta_0+v\eta_0}{c}\left(\frac{u}{c}-\frac{c\delta_0}{u\delta_0+v\eta_0}\right)=\frac{2}{c^2}\Big((u^2-c^2)\delta_0+uv\eta_0\Big).\end{aligned}\tag{3.14}$$

We claim $\vartheta\in\mathbb{R}\setminus\{0\}$. Otherwise, it holds $\delta_0=-\frac{uv}{u^2-c^2}\eta_0$. Substituting this into $c^2(\eta^2+\delta^2)-(u\delta+v\eta)^2=0$, we find $\frac{c^2}{c^2-u^2}(u^2+v^2-c^2)\eta_0=0$. Since $u>c$, we conclude $\eta_0=0$ and hence $\delta_0=0$, but $(0,0)$ is not a member of Σ.

Now we define

$$\begin{aligned}r(\tau,\eta)&=\begin{pmatrix}0 & \vartheta^{-1}\\ \vartheta^{-1} & e_2\end{pmatrix}+\begin{pmatrix}f(\tau,\eta) & 0\\ 0 & 0\end{pmatrix}-i\gamma\begin{pmatrix}0 & -g\\ g & 0\end{pmatrix}\\&:=E+F(\tau,\eta)-i\gamma G,\end{aligned}\tag{3.15}$$

where e_2 and g are real numbers to be determined, and $f(\tau,\eta)$ is a real-valued C^∞ function that vanishes at $(\tau_0 = i\delta_0, \eta_0)$ [11]. Then

$$r(\tau_0,\eta_0) = \begin{pmatrix} 0 & \vartheta^{-1} \\ \vartheta^{-1} & e_2 \end{pmatrix}.$$

We first verify (3.11) at (τ_0, η_0):

$$r(\tau_0,\eta_0) + C(\beta T(\tau_0,\eta_0)^{-1})^*(\beta T(\tau_0,\eta_0)^{-1}) = \begin{pmatrix} 4C|b|^2 & 2iC\bar{b} + \vartheta^{-1} \\ -2iCb + \vartheta^{-1} & C + e_2 \end{pmatrix}.$$

Choosing $C > 0,\ e_2 > 0$ quite large, the quadratic form satisfies (at (τ_0, η_0))

$$4C|b|^2|z_1|^2 + 2\mathrm{Re}\left((\vartheta^{-1} + 2iC\bar{b})z_1\bar{z}_2\right) + (C+e_2)|z_2|^2 \geq 2(|z_1|^2 + |z_2|^2).$$

So by shrinking $\mathcal{V}$ we may get, for $(\tau,\eta) \in \mathcal{V}$, that

$$r(\tau,\eta) + C(\beta T(\tau,\eta)^{-1})^*(\beta T(\tau,\eta)^{-1}) \geq I_2$$

as desired.

Next we choose $f(\tau,\eta)$ and g to guarantee (3.10). Note $\mathbb{a}(\tau_0,\eta_0) = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} = iN$, we have

$$\mathbb{a}(\tau,\eta) = iN + (\mathbb{a}(i\delta,\eta) - \mathbb{a}(i\delta_0,\eta_0)) + (\mathbb{a}(\tau,\eta) - \mathbb{a}(i\delta,\eta)).$$

By Taylor's formula, it holds

$$\mathbb{a}(\tau,\eta) - \mathbb{a}(i\delta,\eta) = \gamma\frac{\partial}{\partial\gamma}\mathbb{a}(i\delta,\eta) + \gamma^2 M(\tau,\eta),$$

with M a continuous matrix-valued function. We also compute that

$$\begin{aligned}\mathbb{a}(i\delta,\eta) - \mathbb{a}(i\delta_0,\eta_0) &= i\left(\frac{u\delta + v\eta}{c} - c\frac{\eta^2+\delta^2}{u\delta+v\eta}\right)\begin{pmatrix} 1 & 0 \\ c\frac{\eta^2+\delta^2}{u\delta+v\eta} & -1 \end{pmatrix} \\ &:= \begin{pmatrix} ib_1(i\delta,\eta) & 0 \\ ib_2(i\delta,\eta) & ib_3(i\delta,\eta) \end{pmatrix} := i\mathbb{b}(i\delta,\eta).\end{aligned}$$

Now choose

$$f(\tau,\eta) = \vartheta^{-1}(b_1(i\delta,\eta) - b_3(i\delta,\eta)) + e_2 b_2(i\delta,\eta),$$

which is obviously smooth in $\mathcal{V}$ and vanishes at (τ_0, η_0). One then checks that, with such a choice of f,

$$(E + F(\tau,\eta))(N + \mathbb{b}(i\delta,\eta)) = \begin{pmatrix} fb_1 + \vartheta^{-1}b_2 & f + \vartheta^{-1}b_3 \\ \vartheta^{-1}b_1 + e_2 b_2 & \vartheta^{-1} + e_2 b_3 \end{pmatrix}$$

is real and symmetric for all $(\tau,\eta) \in \mathcal{V}$. Consequently, for $(\tau,\eta) \in \mathcal{V}$,

$$\begin{aligned}\mathrm{Re}\,(r(\tau,\eta)\mathbb{a}(\tau,\eta)) &= \mathrm{Re}\left((E+F)(\gamma\frac{\partial\mathbb{a}}{\partial\gamma} + \gamma^2 M) + \gamma G(N + \mathbb{b}) - i\gamma G(\gamma\frac{\partial\mathbb{a}}{\partial\gamma} + \gamma^2 M)\right) \\ &= \gamma\mathrm{Re}\left(E\frac{\partial\mathbb{a}}{\partial\gamma} + GN\right) + \gamma L(\tau,\eta),\end{aligned}$$

where $L(\tau,\eta)$ is smooth and $L(\tau_0,\eta_0)=0$. Direct calculation yields

$$\mathrm{Re}\left(E\frac{\partial \mathbb{a}}{\partial \gamma}(\tau_0,\eta_0)+GN\right)=\begin{pmatrix} 1 & * \\ * & * \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix},$$

where $*$ represents quantities determined by ϑ^{-1}, e_2 and (τ_0,η_0). So by choosing g large enough, the above matrix is bounded below by $\frac{1}{2}I_2$; shrink $\mathcal{V}$ to make $L(\tau,\eta)$ small, we further could have $\mathrm{Re}\,(r(\tau,\eta)\mathbb{a}(\tau,\eta))\geq \frac{1}{4}\gamma I_2$ as desired. So we proved that

Lemma 3.4. *For each frequency boundary point that is not a pole, there is a local Kreiss' symmetrizer.*

3.3.2. Localization. Now, we are going to estimate the solution to the linearized problem (3.1) in each neighborhood of different frequencies.

Since Σ (cf. (3.7)) is compact, there are a finite number of such neighborhoods $\mathcal{V}_1,\cdots,\mathcal{V}_J$ that cover Σ. So there is an associated partition of unity $\{\chi_j^2\}_{j=1}^J$; that is, χ_j are nonnegative real-valued C^∞ functions in Σ, $\mathrm{supp}\,\chi_j\subset\mathcal{V}_j$, $\cup_{j=1}^J\mathrm{supp}\,\chi_j=\Sigma$, and $\sum_{j=1}^J\chi_j(\tau,\eta)^2\equiv 1$ for all $(\tau,\eta)\in\Sigma$.

For a given $\mathcal{V}_j$, there are two possibilities: either it contains only frequency points of cases a) or b) studied in the previous subsection; or it contains a pole of the coefficient matrix of (3.5), for which the estimates need to be done separately below.

3.3.3. Estimate at frequency points away from the poles. First we assume that $\mathrm{supp}\,\chi_j$ does not contain a point $(\tau,\eta)\in\Sigma$ so that $u\tau+iv\eta=0$. In this situation we have constructed a local symmetrizer $r_j(\tau,\eta)$ and a smooth mapping $T_j(\tau,\eta)$ in the previous subsection. We first extend r_j and T_j to be defined in the whole Σ by setting them to be zero outside of $\mathrm{supp}\,\chi_j$, then extend $\chi_j(\tau,\eta)$, $r_j(\tau,\eta)$ and $T_j(\tau,\eta)$ to Ξ (see (3.7)) so that they are homogeneous of degree zero.

Now define

$$W_j(z;\tau,\eta)=\chi_j(\tau,\eta)T_j(\tau,\eta)\hat{V}^{\mathrm{nc}}(z;\tau,\eta). \tag{3.16}$$

From (3.9), we know that W_j satisfies

$$\begin{cases}\dfrac{\mathrm{d}W_j(z;\tau,\eta)}{\mathrm{d}z}=T_j(\tau,\eta)\mathcal{B}(\tau,\eta)T_j(\tau,\eta)^{-1}W_j(z;\tau,\eta), \quad \text{in} \quad \{z>0\},\\ \beta T_j(\tau,\eta)^{-1}W_j(0;\tau,\eta)=\chi_j(\tau,\eta)\hat{h}(\tau,\eta).\end{cases} \tag{3.17}$$

So by using (3.10), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}z}(W_j(z;\tau,\eta)^*r_j(\tau,\eta)W_j(z;\tau,\eta))\\
&=W_j(z;\tau,\eta)^*r_j(\tau,\eta)\left(\frac{\mathrm{d}}{\mathrm{d}z}W_j(z;\tau,\eta)\right)+\left(\frac{\mathrm{d}}{\mathrm{d}z}W_j(z;\tau,\eta)\right)^*r_j(\tau,\eta)W_j(z;\tau,\eta)\\
&=W_j^*(r_jT\mathcal{B}T_j^{-1})W_j+(T_j\mathcal{B}T_j^{-1}W_j)^*r_jW_j\\
&=W_j^*(r_jT_j\mathcal{B}T_j^{-1})W_j+(r_jT_j\mathcal{B}T_j^{-1}W_j)^*W_j\\
&=W_j^*(r_jT_j\mathcal{B}T_j^{-1})W_j+W_j^*(r_jT_j\mathcal{B}T_j^{-1})^*W_j\\
&=2W_j^*\mathrm{Re}\,(r_jT_j\mathcal{B}T_j^{-1})W_j\geq 2k\gamma|W_j(z;\tau,\eta)|^2.
\end{aligned}$$

Integrating the above both sides for $z \in [0,\infty)$, we get

$$\begin{aligned}2k\gamma\int_0^\infty |W_j(z;\tau,\eta)|^2\,\mathrm{d}z &\le -W_j(0;\tau,\eta)^* r_j(\tau,\eta)W_j(0;\tau,\eta)\\ &\le -|W_j(0;\tau,\eta)|^2 + C|\beta T_j(\tau,\eta)^{-1}W_j(0;\tau,\eta)|^2\\ &\le -|W_j(0;\tau,\eta)|^2 + C|\chi_j(\tau,\eta)\hat{h}(\tau,\eta)|^2\end{aligned}$$

via (3.11) and boundary condition in (3.17). Since $T_j(\tau,\eta)$ is invertible in $\operatorname{supp}\chi_j$, it follows that

$$\begin{aligned}&\gamma|\chi_j(\tau,\eta)|^2\int_0^\infty |\hat{V}^{\mathrm{nc}}(z;\tau,\eta)|^2\,\mathrm{d}z + |\chi_j(\tau,\eta)|^2|\hat{V}^{\mathrm{nc}}(0;\tau,\eta)|^2\\ &\le C_j|\chi_j(\tau,\eta)|^2|\hat{h}(\tau,\eta)|^2\end{aligned}\tag{3.18}$$

for a positive constant C_j independent of $(\tau,\eta)\in\operatorname{supp}\chi_j$.

3.3.4. Estimate near the poles. Consider the point $(i\delta_0,\eta_0)\in\Sigma$ so that $u\delta_0+v\eta_0=0$, which is a pole of the matrix $\mathcal{B}(\tau,\eta)$. Let $\mathcal{V}$ be a small neighborhood of the point $(i\delta_0,\eta_0)$ in Σ. We define

$$T(\tau,\eta)=\begin{pmatrix} m(\lambda_- - a) & mb\\ -mb & m(\lambda_- - a)\end{pmatrix}^{-1},$$

with a,b defined in (3.6), and $m=u\tau+iv\eta$. Since $(m(\lambda_- - a))^2+(mb)^2|_{(i\delta_0,\eta_0)} = \frac{c^2}{2}(\eta_0^2+\delta_0^2)^2\neq 0$, the $T(\tau,\eta)$ given above makes sense. One checks in $\mathcal{V}$ that

$$T(\tau,\eta)\mathcal{B}(\tau,\eta)T(\tau,\eta)^{-1}=\begin{pmatrix}\lambda_-(\tau,\eta) & 2b(\tau,\eta)\\ 0 & \lambda_+(\tau,\eta)\end{pmatrix}.\tag{3.19}$$

We also note that $\lambda_\pm(i\delta_0,\eta_0)=\pm\sqrt{\eta_0^2+\delta_0^2}$, and $\lambda_\pm$ is continuous in $\mathcal{V}$, so there is a positive constant k so that

$$\operatorname{Re}\lambda_-(\tau,\eta)<-k,\quad \operatorname{Re}\lambda_+(\tau,\eta)>k,\quad \forall(\tau,\eta)\in\mathcal{V}.\tag{3.20}$$

As above, denote by $\chi_j(\tau,\eta)$ the cut-off function supported in $\mathcal{V}$, we can still extend $\chi_j(\tau,\eta), T_j(\tau,\eta)$ to be defined in Ξ so that they are homogenous of degree zero. Define $W_j(z;\tau,\eta)$ as in (3.16), then it solves, in $\{z\ge 0\}$ that,

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}W_j^1(z;\tau,\eta)\\ W_j^2(z;\tau,\eta)\end{pmatrix}=\begin{pmatrix}\lambda_-(\tau,\eta) & 2b(\tau,\eta)\\ 0 & \lambda_+(\tau,\eta)\end{pmatrix}\begin{pmatrix}W_j^1(z;\tau,\eta)\\ W_j^2(z;\tau,\eta)\end{pmatrix}.\tag{3.21}$$

By (3.20), to make sure $W_j\in H_z^2([0,\infty))$, we should have

$$W_j^2(z;\tau,\eta)\equiv 0 \qquad \forall z\in[0,\infty),\quad \forall(\tau,\eta)\in\mathcal{V}.$$

Hence although $b(\tau,\eta)$ has a pole $(-i(v/u)\eta_0,\eta_0)$ in $\mathcal{V}$, the first equation in (3.21) reads

$$\frac{\mathrm{d}}{\mathrm{d}z}W_j^1(z;\tau,\eta)=\lambda_-(\tau,\eta)W_j^1(z;\tau,\eta).$$

Therefore, from (3.20), note $\lambda_-(\tau,\eta)$ is extended to Ξ to be homogeneous degree one, then $\operatorname{Re}\lambda_-(\tau,\eta)<-k\sqrt{|\tau|^2+\eta^2}$ and it implies

$$\frac{\mathrm{d}}{\mathrm{d}z}|W_j^1(z;\tau,\eta)|^2 = 2W_j^1(z;\tau,\eta)^*\operatorname{Re}\lambda_-(\tau,\eta)W_j^1(z;\tau,\eta)\le -2k\sqrt{|\tau|^2+\eta^2}|W_j^1(z;\tau,\eta)|^2.$$

As $W_j^1(z;\tau,\eta)\in H_z^2([0,\infty))$, we obtain

$$2k\sqrt{|\tau|^2+\eta^2}\int_0^\infty |W_j^1(z;\tau,\eta)|^2\,\mathrm{d}z \le |W_j^1(0;\tau,\eta)|^2. \tag{3.22}$$

The boundary condition on $W_j^1(z=0;\tau,\eta)$ reads

$$m(\tau,\eta)(\lambda_-(\tau,\eta)-(a(\tau,\eta)-b(\tau,\eta)))W_j^1(z=0;\tau,\eta)=\chi_j(\tau,\eta)\hat{h}(\tau,\eta), \tag{3.23}$$

and recall $m(\tau,\eta)(\lambda_-(\tau,\eta)-(a(\tau,\eta)-b(\tau,\eta)))$ is the Lopatinskii determinant $\Delta(\tau,\eta)$ given by (3.8). We can easily verify that there is a positive constant C_j so that

$$|\Delta(\tau,\eta)|\ge \gamma/C_j \qquad \forall(\tau,\eta)\in\mathcal{V}.$$

Note although it looks that $\Delta(\tau,\eta)$ is homogeneous degree two for $(\tau,\eta)\in\Xi$, however, we actually extend T_j with homogeneous degree zero, so actually $\Delta(\tau,\eta)$ is extended to $(\tau,\eta)\in\Xi$ homogeneous degree zero. Therefore

$$|\Delta(\tau,\eta)|\ge \gamma/(C_j\sqrt{|\tau|^2+\eta^2}) \qquad \forall(\tau,\eta)\in\{(t\tau',t\eta'):t\in\mathbb{R}^+,\ (\tau',\eta')\in\mathcal{V}\}.$$

We remark this estimate is the reason for the principle that *the order of vanishing of Lopatinskii determinant is the same as the order of loss of derivatives in the energy estimate*. From (3.23), it follows that

$$|W_j^1(0;\tau,\eta)|\le \frac{C_j\sqrt{|\tau|^2+\eta^2}}{\gamma}|\chi_j(\tau,\eta)\hat{h}(\tau,\eta)|,$$

as well as

$$2k\sqrt{|\tau|^2+\eta^2}\int_0^\infty |W_j^1(z;\tau,\eta)|^2\,\mathrm{d}z+|W_j^1(0;\tau,\eta)|^2\le \frac{C_j(|\tau|^2+\eta^2)}{\gamma^2}|\chi_j(\tau,\eta)\hat{h}(\tau,\eta)|^2$$

from (3.22). Since $W_j^2\equiv 0$, and $\sqrt{|\tau|^2+\eta^2}\ge\gamma$, this implies

$$\gamma\int_0^\infty |W_j(z;\tau,\eta)|^2\,\mathrm{d}z+|W_j(0;\tau,\eta)|^2\le \frac{C_j}{\gamma^2}|\chi_j(\tau,\eta)\hat{h}(\tau,\eta)|^2(|\tau|^2+\eta^2).$$

Remember $T_j(\tau,\eta)$ is also invertible in $\{(t\tau',t\eta'):t\in\mathbb{R}^+,\ (\tau',\eta')\in\mathcal{V}\}$, we find

$$\begin{aligned}&\gamma|\chi_j(\tau,\eta)|^2\int_0^\infty |\hat{V}^{\mathrm{nc}}(z;\tau,\eta)|^2\,\mathrm{d}z+|\chi_j(\tau,\eta)|^2|\hat{V}^{\mathrm{nc}}(0;\tau,\eta)|^2\\ &\le \frac{C_j}{\gamma^2}|\chi_j(\tau,\eta)|^2|\hat{h}(\tau,\eta)|^2(|\tau|^2+\eta^2)\end{aligned} \tag{3.24}$$

for a positive constant C_j independent of $(\tau,\eta)\in\mathcal{V}$.

3.3.5. Conclusion. Considering (3.18) and (3.24), note $(|\tau|^2+\eta^2)/\gamma^2\ge 1$, we see (3.24) actually holds for all $j=1,\cdots,J$. Then summing them up for j from 1 to J, and as $\sum_{j=1}^J|\chi_j|^2\equiv 1$, one obtains

PROPOSITION 3.5. *For the solution of the problem* (3.9) *that vanishes as $z\to\infty$, one has the estimate*

$$\gamma\int_0^\infty |\hat{V}^{\mathrm{nc}}(z;\tau,\eta)|^2\,\mathrm{d}z+|\hat{V}^{\mathrm{nc}}(0;\tau,\eta)|^2\le \frac{C}{\gamma^2}|\hat{h}(\tau,\eta)|^2(|\tau|^2+\eta^2) \tag{3.25}$$

with a positive constant C independent of $(\tau,\eta)\in\Xi$.

3.4. Energy estimate for the constant coefficient problem. We continue to establish energy estimate of the solutions to the problem (3.1). It is reduced equivalently to the form (3.3) by introducing explicitly the characteristic variables $V^{\rm c} = (V_1, V_2, V_5)$ and non-characteristic variables $V^{\rm nc} = (V_3, V_4)$. In the following we further introduce some reductions of this problem, which simplifies greatly the derivation of energy estimate.

3.4.1. Function spaces. First, we introduce several definitions and notations of function spaces and norms, which will be used in the following estimates.

We define $H_{s,\gamma}(\mathbb{R}^2)$ with index $s \in \mathbb{R}$ and parameter $\gamma \geq 1$ to be the Hilbert space consists of those Sobolev functions $u \in H^s(\mathbb{R}^2)$ so that $\|u\|_{s,\gamma} < \infty$, where

$$\|u\|_{s,\gamma} = \left(\int_{\mathbb{R}^2} |\hat{u}(\delta,\eta)|^2 (\gamma^2 + \delta^2 + \eta^2)^s \, \mathrm{d}\delta \mathrm{d}\eta\right)^{\frac{1}{2}}.$$

Note $\|u\|_{0,\gamma} = \|u\|_{L^2(\mathbb{R}^2)}$.

We then define a weighted Sobolev space

$$H^*_{s,\gamma}(\mathbb{R}^2) := \{u \in \mathscr{D}'(\mathbb{R}^2) : \ e^{-\gamma x} u(x,y) \in H_{s,\gamma}(\mathbb{R}^2)\}$$

with the norm $\|u\|_{H^*_{s,\gamma}} := \|e^{-\gamma x} u(x,y)\|_{s,\gamma}$. Finally, we write the norm of a function $f(x,y,z) \in L^2(\mathbb{R}^+_z; H^*_{s,\gamma}(\mathbb{R}^2_{x,y}))$ to be

$$\|f\|_{L^2(H^*_{s,\gamma})} = \left(\int_{\mathbb{R}^+} \|f(x,y,z)\|^2_{H^*_{s,\gamma}(\mathbb{R}^2_{x,y})} \, \mathrm{d}z\right)^{\frac{1}{2}}.$$

3.4.2. Introducing of weight and elimination of interior source term. For $\gamma \geq 1$ a parameter, let $\tilde{V} = \exp(-\gamma x) V$. Then the problem (3.3) becomes

$$\begin{cases} \gamma B_1 \tilde{V} + B_1 \partial_x \tilde{V} + B_2 \partial_y \tilde{V} + B_3 \partial_z \tilde{V} = e^{-\gamma x} Q f, \\ \beta \tilde{V}^{\rm nc} = e^{-\gamma x} g/(\Gamma p), \qquad \text{on } \{z=0\}; \quad \text{here } \ \beta = (1,-1). \end{cases} \tag{3.26}$$

Now consider an auxiliary problem for unknown $\tilde{V}_1$:

$$\begin{cases} \gamma B_1 \tilde{V}_1 + B_1 \partial_x \tilde{V}_1 + B_2 \partial_y \tilde{V}_1 + B_3 \partial_z \tilde{V}_1 = e^{-\gamma x} Q f, \quad z > 0, \\ M_1 \tilde{V}_1 = 0, \quad z = 0; \quad \text{here } \ M_1 = (0,0,1,0,0). \end{cases} \tag{3.27}$$

It is easy to check that this boundary value problem is maximal dissipative (for the definition, see [2, p.86]). It follows from standard result that there exists a solution $\tilde{V}_1$ (cf. [2, 18]).

Now set $\tilde{V}_2 = \tilde{V} - \tilde{V}_1$, which solves

$$\begin{cases} \gamma B_1 \tilde{V}_2 + B_1 \partial_x \tilde{V}_2 + B_2 \partial_y \tilde{V}_2 + B_3 \partial_z \tilde{V}_2 = 0, \quad \text{in } \ \{z > 0\}, \\ \beta \tilde{V}_2^{\rm nc} = h := e^{-\gamma x} \frac{g}{\Gamma p} - \beta \tilde{V}_1^{\rm nc}, \quad \text{on } \ \{z = 0\}. \end{cases} \tag{3.28}$$

3.4.3. Estimate of $\tilde{V}_2$ by $\left(\tilde{V}_2^{\rm nc}\right)|_{z=0}$. Multiplying $\tilde{V}_2$ to the equations in (3.28), and integrating on $(x,y,z) \in \Omega = \mathbb{R}^2 \times \mathbb{R}^+$, we find

$$\int_\Omega \Big((B_1 \partial_x \tilde{V}_2, \tilde{V}_2) + (B_2 \partial_y \tilde{V}_2, \tilde{V}_2) + (B_3 \partial_z \tilde{V}_2, \tilde{V}_2) \Big) \, \mathrm{d}x\mathrm{d}y\mathrm{d}z + \gamma \int_\Omega (B_1 \tilde{V}_2, \tilde{V}_2) \, \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0.$$

As all B_k $(k=1,2,3)$ are symmetric and $\tilde{V}_2$ is real, integration by parts for the above identity yields

$$\gamma\int_\Omega (B_1\tilde{V}_2,\tilde{V}_2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z=\frac{1}{2}\int_{\mathbb{R}^2}(B_3\tilde{V}_2,\tilde{V}_2)|_{z=0}\,\mathrm{d}x\,\mathrm{d}y=c\int_{\mathbb{R}^2}((\tilde{V}_2^3)^2-(\tilde{V}_2^4)^2)|_{z=0}\,\mathrm{d}x\,\mathrm{d}y.$$

Recall B_1 is positive-definite, we find a positive constant C so that

$$\gamma\int_0^\infty \left\|\tilde{V}_2(\cdot,z)\right\|^2_{L^2(\mathbb{R}^2)}\,\mathrm{d}z\le C\int_{\mathbb{R}^2}\left|(\tilde{V}_2^{\mathrm{nc}})_{|z=0}\right|^2\,\mathrm{d}x\,\mathrm{d}y. \tag{3.29}$$

3.4.4. Estimate of $\left(\tilde{V}_2^{\mathrm{nc}}\right)|_{z=0}$. Now to estimate $\int_{\mathbb{R}^2}|\tilde{V}_2^{\mathrm{nc}}(x,y,0)|^2\,\mathrm{d}x\,\mathrm{d}y$, we apply the Fourier transform with respect to $(x,y)\in\mathbb{R}^2$ to (3.28). Let $\hat{V}=\mathscr{F}_{(x,y)\to(\delta,\eta)}\tilde{V}_2$, and write $\tau=\gamma+i\delta$, we find

$$\begin{cases}B_3\partial_z\hat{V}+(\tau B_1+i\eta B_2)\hat{V}=0,\\ \beta\hat{V}^{\mathrm{nc}}=\hat{h}.\end{cases}$$

This is exactly (3.4) (with $f=0$). Then as computation shown there, we get the ODE (3.9) for $\hat{V}^{\mathrm{nc}}$. So we can use (3.25) now. Integrating it with respect to $(\delta,\eta)\in\mathbb{R}^2$, we have

$$\int_{\mathbb{R}^2}|\hat{V}^{\mathrm{nc}}(\gamma+i\delta,\eta,0)|^2\,\mathrm{d}\delta\mathrm{d}\eta\le\frac{C}{\gamma^2}\int_{\mathbb{R}^2}|\hat{h}(\gamma+i\delta,\eta)|^2|\gamma^2+\delta^2+\eta^2|\,\mathrm{d}\delta\mathrm{d}\eta.$$

Note here $\gamma\ge 1$ is a parameter. Using Plancherel's theorem, we find that

$$\int_{\mathbb{R}^2}|(\tilde{V}_2^{\mathrm{nc}})_{|z=0}|^2\,\mathrm{d}x\mathrm{d}y\le\frac{C}{\gamma^2}\left\|h\right\|^2_{1,\gamma}. \tag{3.30}$$

3.4.5. Conclusion. From (3.29) and (3.30) we directly have the estimate for $\tilde{V}_2$:

$$\gamma\int_0^\infty\left\|\tilde{V}_2(\cdot,z)\right\|^2_{L^2(\mathbb{R}^2)}\,\mathrm{d}z+\int_{\mathbb{R}^2}|(\tilde{V}_2^{\mathrm{nc}})_{|z=0}|^2\,\mathrm{d}x\mathrm{d}y\le\frac{C}{\gamma^2}\left\|h\right\|^2_{1,\gamma}. \tag{3.31}$$

Recall that for $\tilde{V}_1$, it holds [2, p.96]

$$\gamma\int_0^\infty\left\|\tilde{V}_1(\cdot,z)\right\|^2_{L^2(\mathbb{R}^2)}\,\mathrm{d}z+\int_{\mathbb{R}^2}|(\tilde{V}_1^{\mathrm{nc}})_{|z=0}|^2\,\mathrm{d}x\mathrm{d}y\le\frac{C}{\gamma}\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}|e^{-\gamma x}Qf|^2\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}z. \tag{3.32}$$

Since $\|h\|^2_{1,\gamma}\le C(\|e^{-\gamma x}g\|^2_{1,\gamma}+\left\|\tilde{V}_1^{\mathrm{nc}}\right\|^2_{1,\gamma})$, in order to close the estimates (3.31) and (3.32), we still need an estimate of $\left\|\tilde{V}_1^{\mathrm{nc}}\right\|_{1,\gamma}$.

From the problem (3.27), it can be proved that we actually have (refer to [2, p.227, Proposition 9.1]):

$$\left\|\tilde{V}_1^{\mathrm{nc}}\right\|^2_{1,\gamma}\le\frac{C}{\gamma}\int_{\mathbb{R}^+}\left\|e^{-\gamma x}Qf(\cdot,z)\right\|^2_{H_{1,\gamma}(\mathbb{R}^2_{(x,y)})}\,\mathrm{d}z.$$

So combining (3.31) and (3.32), we find

$$\begin{aligned}\gamma\int_0^\infty\left\|\tilde{V}(\cdot,z)\right\|^2_{L^2(\mathbb{R}^2)}\,\mathrm{d}z+\int_{\mathbb{R}^2}|(\tilde{V}^{\mathrm{nc}})_{|z=0}|^2\,\mathrm{d}x\mathrm{d}y\le\frac{C}{\gamma}\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}|e^{-\gamma x}Qf|^2\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}z\\ +\frac{C}{\gamma^3}\int_{\mathbb{R}^+}\left\|e^{-\gamma x}Qf(\cdot,z)\right\|^2_{H_{1,\gamma}(\mathbb{R}^2_{(x,y)})}\,\mathrm{d}z+\frac{C}{\gamma^2}\left\|e^{-\gamma x}g\right\|^2_{H_{1,\gamma}(\mathbb{R}^2_{(x,y)})}.\end{aligned} \tag{3.33}$$

Since $\gamma \geq 1$, using definition of the norm $\|\cdot\|_{s,\gamma}$, one verifies that

$$\|u\|_{s,\gamma} \leq \gamma^{s-r} \|u\|_{r,\gamma} \quad \text{for} \quad s < r. \tag{3.34}$$

Applying this to $e^{-\gamma \cdot} Qf(\cdot, z)$ with $s = 0$ and $r = 1$, we find the second term on the right-hand side of (3.33) can control the first term there. So finally, note Q is a constant matrix, we have

$$\begin{aligned} &\gamma \int_0^\infty \left\|\tilde{V}(\cdot, z)\right\|^2_{L^2(\mathbb{R}^2)} \,\mathrm{d}z + \int_{\mathbb{R}^2} |(\tilde{V}^{\mathrm{nc}})_{|z=0}|^2 \,\mathrm{d}x\mathrm{d}y \\ &\leq C\left(\frac{1}{\gamma^3}\int_{\mathbb{R}^+} \left\|e^{-\gamma x} f(\cdot, z)\right\|^2_{H_{1,\gamma}(\mathbb{R}^2_{(x,y)})} \,\mathrm{d}z + \frac{C}{\gamma^2}\left\|e^{-\gamma x} g\right\|^2_{H_{1,\gamma}(\mathbb{R}^2_{(x,y)})}\right). \end{aligned} \tag{3.35}$$

Returning to the linear problem (3.3), we conclude

Proposition 3.6. *For the solution of the problem* (3.3)*, we have the estimate*

$$\gamma \|V\|^2_{L^2(H^*_{0,\gamma})} + \|V^{\mathrm{nc}}(\cdot, 0)\|^2_{H^*_{0,\gamma}} \leq C\left(\frac{1}{\gamma^3}\|f\|^2_{L^2(H^*_{1,\gamma})} + \frac{1}{\gamma^2}\|g\|^2_{H^*_{1,\gamma}}\right). \tag{3.36}$$

Here $\gamma \geq 1$ and C is a constant independent of γ and (f, g).

3.5. Estimate of the free boundary. We turn to study the initial value problem of transport equation (3.2) for the perturbation of the contact discontinuity front:

$$\begin{cases} u\partial_x \dot{\psi} + v\partial_y \dot{\psi} = 2c(1,1) V^{\mathrm{nc}}_{|z=0}, & \text{in } \{x > 0,\ y \in \mathbb{R}\}, \\ \dot{\psi}|_{x=0} = 0. \end{cases} \tag{3.37}$$

Let $\tilde{\psi} = \exp(-\gamma x)\dot{\psi}$. Then we have $u(\gamma + \partial_x)\tilde{\psi} + v\partial_y \tilde{\psi} = 2c(1,1)\tilde{V}^{\mathrm{nc}}_{|z=0}$. After taking the Fourier transform in x and y, with dual variables δ and η as before, it follows

$$(u\tau + iv\eta)\hat{\psi} = 2c(1,1)\hat{V}^{\mathrm{nc}}(0; \tau, \eta),$$

with $\tau = \gamma + i\delta$, and $\hat{\psi}$ the Fourier transform of $\tilde{\psi}$. Since $|u\tau + iv\eta| \geq C\gamma$, by (3.37) we obtain that

$$|\hat{\psi}|^2 \leq \frac{C}{\gamma^2} |\hat{V}^{\mathrm{nc}}(0; \tau, \eta)|^2.$$

Integrating this over $(\delta, \eta) \in \mathbb{R}^2$, Plancherel's theorem and (3.25) yield

$$\left\|\tilde{\psi}\right\|_{0,\gamma} \leq \frac{C}{\gamma} \left\|\tilde{V}^{\mathrm{nc}}|_{z=0}\right\|_{0,\gamma} \leq \frac{C}{\gamma^2}\left(\left\|e^{-\gamma x} g\right\|_{1,\gamma} + \left\|(\tilde{V}^{\mathrm{nc}}_1)_{|z=0}\right\|_{1,\gamma}\right).$$

Recall that (cf. [2, (9.1.15) in p.227])

$$\left\|(\tilde{V}^{\mathrm{nc}}_1)_{|z=0}\right\|_{1,\gamma} \leq \frac{C}{\sqrt{\gamma}} \left\|e^{-\gamma x} Qf\right\|_{L^2_z(H_{1,\gamma}(\mathbb{R}^2_{(x,y)}))},$$

we get

$$\left\|\dot{\psi}\right\|_{H^*_{0,\gamma}} \leq \frac{C}{\gamma^2}\left(\|g\|_{H^*_{1,\gamma}} + \frac{1}{\sqrt{\gamma}}\|f\|_{L^2(\mathbb{R}^+_z; H^*_{1,\gamma}(\mathbb{R}^2_{(x,y)}))}\right). \tag{3.38}$$

Hence for the free boundary, there is still a loss of one derivative.

Finally, we summarize all the obtained estimates as the following theorem, which is the first main result of this paper.

THEOREM 3.7. *There is a constant $C > 0$ so that for all $\gamma \geq 1$ and $(\dot{U}, \dot{\psi}) \in H^2(\Omega) \times H^2(\mathbb{R}^2)$, it holds*

$$\gamma \left\|\dot{U}\right\|^2_{L^2(H^*_{0,\gamma})} + \|(\dot{w}, \dot{p})|_{z=0}\|^2_{H^*_{0,\gamma}} + \gamma^2 \left\|\dot{\psi}\right\|^2_{H^*_{0,\gamma}} \leq C\left(\frac{1}{\gamma^3}\|f\|^2_{L^2(H^*_{1,\gamma})} + \frac{1}{\gamma^2}\|g\|^2_{H^*_{1,\gamma}}\right). \tag{3.39}$$

Here $f := A_1(\underline{U})\partial_x\dot{U} + A_2(\underline{U})\partial_y\dot{U} + A_3(\underline{U})\partial_z\dot{U}$, $g := \dot{p}(x, y, 0)$, and $\dot{\psi}$ satisfies $\underline{u}\partial_x\dot{\psi} + \underline{v}\partial_y\dot{\psi} = \dot{w}$ on $\{z = 0\}$.

Proof. If $\dot{U}, \dot{\psi}$ are smooth functions with compact support, the above inequality follows directly from (3.38) and (3.36) (recall $\dot{U} = PV$). It also holds for $(\dot{U}, \dot{\psi}) \in H^2(\Omega) \times H^2(\mathbb{R}^2)$ just by standard approximation. □

4. The variable coefficient linear problem. Guided by the analysis of constant coefficient case developed in section 3, from now on we study the linear problem for the general case. This variable coefficient linear problem is derived by linearizing the nonlinear problem (2.20) and (2.21) around a non-planar transonic contact discontinuity. This analysis of linear problem is a crucial step towards the study of the nonlinear problem. We first derive the linearized problem, then state the main estimate in Theorem 4.1. The rest of this paper is devoted to reduction of the estimates and finally proving it by using para-differential calculus.

4.1. Linearization of the nonlinear problem. We need linearize the nonlinear problem

$$L(U, \nabla\Psi)U = A_1(U)\partial_x U + A_2(U)\partial_y U + \tilde{A}_3(U, \nabla\Psi)\partial_z U = 0, \quad z > 0, \tag{4.1}$$

$$p = \underline{p}, \quad z = 0, \tag{4.2}$$

where $\tilde{A}_3(U, \nabla\Psi) = \frac{1}{\partial_z\Psi}\Big(A_3(U) - A_2(U)\partial_y\Psi - A_1(U)\partial_x\Psi\Big)$, $U = (u, v, w, p, S)^T$, and Ψ should satisfy

$$u\partial_x\Psi + v\partial_y\Psi - w = 0, \quad z \geq 0, \tag{4.3}$$

$$\partial_z\Psi \geq \kappa_0 > 0, \quad z \geq 0 \tag{4.4}$$

for a fixed constant κ_0.

Let U and Ψ be a (non-planar) background state satisfying (4.3)(4.4) in the whole domain $\{z \geq 0\}$, and denote by V and Φ their small perturbations respectively. By a direct computation (cf. [1]), we get the following linearized equation of (4.1) at (U, Ψ):

$$A_1(U)\partial_x V + A_2(U)\partial_y V + \tilde{A}_3(U, \nabla\Psi)\partial_z V + (\mathrm{d}_U A_1(U)\cdot V)\partial_x U + (\mathrm{d}_U A_2(U)\cdot V)\partial_y U$$
$$+\Big(\mathrm{d}_U\tilde{A}_3(U, \nabla\Psi)\cdot V + \mathrm{d}_{\nabla\Psi}\tilde{A}_3(U, \nabla\Psi)\cdot\nabla\Phi\Big)\partial_z U = f.$$

As Alinhac discovered in [1], to remove the mixture of the first derivatives of V and Φ in the above equation, by introducing the good unknowns:

$$\dot{U} = V - \frac{\Phi}{\partial_z\Psi}\partial_z U, \tag{4.5}$$

the above equation for V can be rewritten as the following one for $\dot{U}$,

$$L(U,\nabla\Psi)\dot{U}+C(U,\nabla U,\nabla\Psi)\cdot\dot{U}+\frac{\Phi}{\partial_z\Psi}\Big(\partial_z(L(U,\nabla\Psi)U)\Big)=f. \tag{4.6}$$

where

$$C(U,\nabla U,\nabla\Psi)\cdot\dot{U}=(\mathrm{d}_U A_1(U)\cdot\dot{U})\partial_x U+(\mathrm{d}_U A_2(U)\cdot\dot{U})\partial_y U+(\mathrm{d}_U\tilde{A}_3(U,\nabla\Psi)\cdot\dot{U})\partial_z U.$$

In terms of the good unknowns, the linearized boundary condition for $\dot{U}$ is given by

$$\dot{p}+\frac{\Phi}{\partial_z\Psi}\partial_z p=g_1,\quad \text{on}\ \ \{z=0\}. \tag{4.7}$$

By a simple computation, we know that the linearized equation of the transport equation (4.3) is given by

$$u\partial_x\Phi+v\partial_y\Phi+V_1\partial_x\Psi+V_2\partial_y\Psi-V_3=h_1$$

which can be rewritten as the following one for the good unknowns

$$u\partial_x\Phi+v\partial_y\Phi-\dot{w}+\dot{u}\partial_x\Psi+\dot{v}\partial_y\Psi-\frac{\Phi}{\partial_z\Psi}(\partial_z w-\partial_z u\partial_x\Psi-\partial_z v\partial_y\Psi)=h_1. \tag{4.8}$$

4.2. The effective linear problem. Since the zero-th order term of Φ appears in a quadratic form together with U in the equation (4.6), as usual [11], to study the linear stability of the transonic contact discontinuity it suffices to consider the following effective linear equation with the zero-th order term of Φ being emerged into the source term f:

$$L'\dot{U}=L(U,\nabla\Psi)\dot{U}+C(U,\nabla U,\nabla\Psi)\cdot\dot{U}=f,\quad \text{in}\ \ \{z>0\}. \tag{4.9}$$

A simple calculation yields that

$$\tilde{A}_3(U,\nabla\Psi)=\frac{1}{\partial_z\Psi}\begin{pmatrix} * & 0 & 0 & -\partial_x\Psi & 0\\ 0 & * & 0 & -\partial_y\Psi & 0\\ 0 & 0 & * & 1 & 0\\ -\partial_x\Psi & -\partial_y\Psi & 1 & */(\rho^2c^2) & 0\\ 0 & 0 & 0 & 0 & */\rho \end{pmatrix}, \tag{4.10}$$

where $*=\rho(w-u\partial_x\Psi-v\partial_y\Psi)$. Under the assumption that the (non-planar) background state (U,Ψ) satisfies the eikonal equation (4.3), the element $*$ in (4.10) is identically zero for $z\geq 0$. So we can only expect control of $\dot{U}_4=\dot{p}$ and $-\partial_x\Psi\dot{U}_1-\partial_y\Psi\dot{U}_2+\dot{U}_3$ on $\{z=0\}$. Therefore we introduce as in [11], with $\psi(x,y)=\Psi(x,y,0)$, that

$$\mathbb{P}(\psi)\dot{U}|_{z=0}=\begin{pmatrix} -\partial_x\psi\dot{U}_1-\partial_y\psi\dot{U}_2+\dot{U}_3\\ \dot{U}_4\end{pmatrix}\Bigg|_{z=0}, \tag{4.11}$$

which is the non-characteristic part of the unknown $\dot{U}$ when restricted on the boundary. We see the linearized boundary conditions (4.7) and (4.8) only involve these non-characteristic part.

As above, since the zero-th order term of Φ in the boundary condition (4.7) is of a quadratic form with U, one can shift this zero-th order term of Φ into the source term g_1 in the boundary condition to have

$$\dot{p} = g_1, \quad \text{on } \{z = 0\}. \tag{4.12}$$

Denote by $b = \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix}$, $\nabla\phi = (\partial_x\phi, \partial_y\phi)^T$, with $\phi(x, y) = \Phi(x, y, 0)$, and

$$M' = \begin{pmatrix} \partial_x\psi & \partial_y\psi & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} \partial_x\psi & \partial_y\psi & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, from (4.12) and (4.8), the boundary conditions can be formulated as:

$$B'(\dot{U}, \phi) := b\nabla\phi + \frac{1}{\partial_z\Psi}(M'\partial_z U)\phi + M\dot{U} = (h_1, g_1)^T, \quad \text{on } \{z = 0\}, \tag{4.13}$$

In the following we study the estimates of the solution to the variable coefficient linear problem (4.9) and (4.13).

4.3. Basic L^2 estimate of linear problem.

Assumptions. For the linear problem (4.9) and (4.13), the unknowns are a vector $\dot{U}$ and a function ϕ. The coefficients involve a vector-valued function U and a function Ψ defined in $\{z \geq 0\}$. By definition of good unknown (4.5), $\dot{U}$ actually contains a new unknown Φ, the perturbed front, whose restriction on $\{z = 0\}$ is ϕ. Also, $\psi = \Psi|_{z=0}$. In addition, we require that Ψ and U to satisfy

$$u\partial_x\Psi + v\partial_y\Psi = w \quad \text{in } \{z \geq 0\}, \tag{4.14}$$

$$\partial_z\Psi \geq \kappa_0 \tag{4.15}$$

for a fixed positive constant κ_0.

Apart from these, we also need the following regularity and smallness conditions on U and Ψ:

$$\begin{aligned} &U \in W^{2,\infty}(\Omega), \quad \nabla\Psi \in W^{2,\infty}(\Omega), \\ &\|U - \underline{U}\|_{W^{2,\infty}(\Omega)} + \|\nabla\Psi - (0, 0, 1)\|_{W^{2,\infty}(\Omega)} \leq K_0 \end{aligned} \tag{4.16}$$

for a suitable constant $K_0 > 0$. Finally, we also require both $U - \underline{U}$ and $\nabla(\Psi - z)$ have compact support in (y, z). Recall here that $\underline{U} = (\underline{u}, \underline{v}, 0, \underline{p}, \underline{S}_+)$ and $\underline{\Psi} = z$ represent the planar reference state for which $\underline{u} > \underline{c}_+$.

Main result on L^2 estimate. Under the above assumptions, we have the following theorem, which is the second main result of this paper.

THEOREM 4.1. *There exist constant C_1 and $\gamma_1 \geq 1$ that depend only on K_0 and κ_0 so that for all $\gamma \geq \gamma_1$ and all $(\dot{U}, \phi) \in H^*_{2,\gamma}(\Omega) \times H^*_{2,\gamma}(\mathbb{R}^2)$, the following estimate holds:*

$$\begin{aligned} &\gamma \left\|\dot{U}\right\|^2_{L^2(H^*_{0,\gamma})} + \left\|\mathbb{P}(\psi)\dot{U}|_{z=0}\right\|^2_{H^*_{0,\gamma}} + \gamma^2 \|\phi\|^2_{H^*_{0,\gamma}} \\ &\leq C_1 \left(\frac{1}{\gamma^3}\left\|L'\dot{U}\right\|^2_{L^2(H^*_{1,\gamma})} + \frac{1}{\gamma^2}\left\|B'(\dot{U}, \phi)\right\|^2_{H^*_{1,\gamma}}\right). \end{aligned} \tag{4.17}$$

We note that unlike the two-phase hyperbolic free boundary problem studied in [11], where the perturbed front ϕ satisfies an elliptic equation and hence it gets one more regularity, in our case ϕ only satisfies a transport equation on $\{z = 0\}$ and its regularity is the same as that of the trace of the non-characteristic components of the state $\dot{U}$. The rest of this section is devoted to proving this theorem.

4.4. Some preliminary transformations.

4.4.1. Reduction of estimate. Set $\dot{V} = e^{-\gamma x}\dot{U}$. Then, from (4.9), $\dot{V}$ solves

$$A_1(U)\partial_x \dot{V} + A_2(U)\partial_y \dot{V} + \tilde{A}_3(U, \nabla\Psi)\partial_z \dot{V} + \gamma A_1(U)\dot{V} + C(U, \nabla U, \nabla\Psi)\cdot \dot{V} = e^{-\gamma x} f. \tag{4.18}$$

Since this equation is symmetric hyperbolic with respect to x, taking inner product with $\dot{V}$ and integrating over $\{z \geq 0\}$, it follows that

$$\begin{aligned}
&\int_{\{z\geq 0\}} \Big(\partial_x(A_1(U)\dot{V}, \dot{V}) + \partial_y(A_2(U)\dot{V}, \dot{V}) + \partial_z(\tilde{A}_3(U, \nabla\Psi)\dot{V}, \dot{V})\Big)\, \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&+ \int_{\{z\geq 0\}} \Big(\{\gamma A_1(U) - [\partial_x A_1(U) + \partial_y A_2(U) + \partial_z A_3(U, \nabla\Psi)]\}\dot{V}, \dot{V}\Big)\, \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&+ \int_{\{z\geq 0\}} (C(U, \nabla U, \nabla\Psi)\cdot \dot{V}, \dot{V})\, \mathrm{d}x\mathrm{d}y\mathrm{d}z = 2\int_{\{z\geq 0\}} (e^{-\gamma x} f, \dot{V})\, \mathrm{d}x\mathrm{d}y\mathrm{d}z.
\end{aligned}$$

Note that $A_1(\underline{U})$ is positive-definite, so there is a constant $c(K_0)$ such that $A_1(U) \geq c(K_0)I_5$. We also observe

$$\begin{aligned}
&\|\partial_x A_1(U) + \partial_y A_2(U) + \partial_z A_3(U, \nabla\Psi)\|_{W^{1,\infty}} \leq C(K_0, \kappa_0), \\
&|(C(U, \nabla U, \nabla\Psi)\cdot \dot{V}, \dot{V})| \leq C(K_0, \kappa_0)|\dot{V}|^2;
\end{aligned}$$

hence, with the help of Young's inequality,

$$\begin{aligned}
(\gamma c - C)\int_{\{z\geq 0\}} |\dot{V}|^2\, \mathrm{d}x\mathrm{d}y\mathrm{d}z \leq \frac{C_\varepsilon}{\gamma}\int_{\{z\geq 0\}} |e^{-\gamma x} f|^2\, \mathrm{d}x\mathrm{d}y\mathrm{d}z + \varepsilon\gamma \int_{\{z\geq 0\}} |\dot{V}|^2\, \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
+ \int_{\mathbb{R}^2} (\tilde{A}_3(U, \nabla\Psi)\dot{V}, \dot{V})|_{z=0}\, \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

A direct computation shows that

$$(\tilde{A}_3(U, \nabla\Psi)\dot{V}, \dot{V})|_{z=0} = 2\dot{V}_4(-\partial_x\Psi \dot{V}_1 - \partial_y\Psi \dot{V}_2 + \dot{V}_3)|_{z=0} \leq |\mathbb{P}(\psi)\dot{V}_{|z=0}|^2.$$

Plugging this relation into the above inequality, and choosing a proper small ε, we get the following result:

Lemma 4.2. *There are constants $C > 0$ and $\gamma_0 > 1$ so that for any $\gamma \geq \gamma_0$, it holds*

$$\gamma \left\|\dot{U}\right\|^2_{L^2(H^*_{0,\gamma})} \leq C\left(\frac{1}{\gamma}\left\|L'\dot{U}\right\|^2_{L^2(H^*_{0,\gamma})} + \left\|\mathbb{P}(\psi)(\dot{U})\right\|^2_{H^*_{0,\gamma}}\right). \tag{4.19}$$

So we only need to obtain estimate of $\left\|\mathbb{P}(\psi)(\dot{U})\right\|_{H^*_{0,\gamma}}$ and $\|\phi\|_{H^*_{0,\gamma}}$ below.

4.4.2. Diagonalization of boundary matrix in interior equation. The next step is to transform the linearized interior equation so that the coefficient matrix of ∂_z is diagonal.

Set $\Theta(U) = \mathrm{diag}\{\rho u, \rho u, \rho u, \frac{u}{\rho c^2}, u\}$. We solve the eigenvalues of $\tilde{A}_3$ with respect to Θ, that is, numbers λ so that

$$\det\Big(\lambda\Theta(U) - \tilde{A}_3(U, \nabla\Psi)\Big) = 0.$$

A direct calculation yields

$$\lambda_{1,2,3}=0,\quad \lambda_4=-\frac{c}{u}\frac{\sqrt{1+|\partial_x\Psi|^2+|\partial_y\Psi|^2}}{\partial_z\Psi}<0,\quad \lambda_5=\frac{c}{u}\frac{\sqrt{1+|\partial_x\Psi|^2+|\partial_y\Psi|^2}}{\partial_z\Psi}>0$$

with associated right eigenvectors being

$$\begin{aligned}&r_1=(0,0,0,0,1)^T,\quad r_2=(1,0,\partial_x\Psi,0,0)^T,\quad r_3=(0,1,\partial_y\Psi,0,0)^T,\\ &r_{4,5}=(-\partial_x\Psi,-\partial_y\Psi,1,\rho u\lambda_{4,5}\partial_z\Psi,0)^T.\end{aligned}$$

So by taking $T(U,\nabla\Psi)=(r_1,r_2,r_3,r_4,r_5)$, we have

$$T(U,\nabla\Psi)^{-1}\Theta(U)^{-1}\tilde{A}_3(U,\nabla\Psi)T(U,\nabla\Psi)=\mathrm{diag}\{0,0,0,\lambda_4,\lambda_5\}.$$

Set $W=T(U,\nabla\Psi)^{-1}\dot{V}$. Then from (4.18), W satisfies

$$\begin{aligned}&T^{-1}\Theta^{-1}A_1T\partial_xW+T^{-1}\Theta^{-1}A_2T\partial_yW+T^{-1}\Theta^{-1}\tilde{A}_3T\partial_zW\\ &+T^{-1}\Theta^{-1}\Big[A_1\partial_xT+A_2\partial_yT+\tilde{A}_3\partial_zT+\gamma A_1T\Big]W+T^{-1}\Theta^{-1}C\cdot(TW)\\ =\ &e^{-\gamma x}T^{-1}\Theta^{-1}f.\end{aligned}$$

Now introduce $A_0(U,\nabla\Psi)=\mathrm{diag}\{1,1,1,\lambda_4^{-1},\lambda_5^{-1}\}$, then by multiplying A_0 from left to the above equation, we get

$$L^\gamma W:=\gamma\mathbf{A}_1W+\mathbf{A}_1\partial_xW+\mathbf{A}_2\partial_yW+\mathcal{I}_5\partial_zW+\mathbf{C}W=e^{-\gamma x}F,\tag{4.20}$$

with

$$\begin{aligned}&\mathbf{A}_1:=A_0T^{-1}\Theta^{-1}A_1T(U,\nabla\Psi),\quad \mathbf{A}_2:=A_0T^{-1}\Theta^{-1}A_2T(U,\nabla\Psi),\\ &\mathbf{C}:=A_0\Big[T^{-1}\Theta^{-1}\Big(A_1\partial_xT+A_2\partial_yT+\tilde{A}_3\partial_zT\Big)T+T^{-1}\Theta^{-1}CT\Big](U,\nabla\Psi),\\ &\mathcal{I}_5:=\mathrm{diag}\{0,0,0,1,1\},\quad F=A_0T^{-1}(U,\nabla\Psi)\Theta^{-1}f.\end{aligned}$$

Here, with some abuse of notations, we write $T^{-1}\Theta^{-1}C\cdot(TW)$ as $T^{-1}\Theta^{-1}CTW$. It is easy to know that

$$\mathbf{A}_j\in W^{2,\infty}(\Omega),\ j=1,2;\quad \mathbf{C}\in W^{1,\infty}(\Omega).$$

For later reference, by some tedious computations, we find that

$$T^{-1}(U,\nabla\Psi)=\begin{pmatrix}0&0&0&0&1\\ \frac{1+|\partial_y\Psi|^2}{\langle\Psi\rangle^2}&\frac{-\partial_x\Psi\partial_y\Psi}{\langle\Psi\rangle^2}&\frac{\partial_x\Psi}{\langle\Psi\rangle^2}&0&0\\ \frac{-\partial_x\Psi\partial_y\Psi}{\langle\Psi\rangle^2}&\frac{1+|\partial_x\Psi|^2}{\langle\Psi\rangle^2}&\frac{\partial_y\Psi}{\langle\Psi\rangle^2}&0&0\\ -\frac12\frac{\partial_x\Psi}{\langle\Psi\rangle^2}&-\frac12\frac{\partial_y\Psi}{\langle\Psi\rangle^2}&\frac12\frac{1}{\langle\Psi\rangle^2}&-\frac12\frac{1}{\rho c}\frac{1}{\langle\Psi\rangle}&0\\ -\frac12\frac{\partial_x\Psi}{\langle\Psi\rangle^2}&-\frac12\frac{\partial_y\Psi}{\langle\Psi\rangle^2}&\frac12\frac{1}{\langle\Psi\rangle^2}&\frac12\frac{1}{\rho c}\frac{1}{\langle\Psi\rangle}&0\end{pmatrix}.$$

Here we have used the notation

$$\langle\Psi\rangle=\sqrt{1+|\partial_x\Psi|^2+|\partial_y\Psi|^2}.$$

Since $W = T^{-1}\dot{V}$, we see $W_1 = \dot{V}_5$ and all $W_{2,3,4,5}$ have the dimension of velocity. We also have

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{c}{u}\frac{1+|\partial_y\Psi|^2}{\langle\Psi\rangle} & \frac{c}{u}\frac{1+|\partial_y\Psi|^2}{\langle\Psi\rangle} \\ 0 & 0 & 1 & \frac{c}{u}\frac{\partial_x\Psi\partial_y\Psi}{\langle\Psi\rangle} & -\frac{c}{u}\frac{\partial_x\Psi\partial_y\Psi}{\langle\Psi\rangle} \\ 0 & \frac{1}{2}\frac{\partial_z\Psi}{\langle\Psi\rangle^2} & 0 & -\frac{\partial_z\Psi}{\langle\Psi\rangle}\left(\frac{u}{c}+\frac{\partial_x\Psi}{\langle\Psi\rangle}\right) & 0 \\ 0 & \frac{1}{2}\frac{\partial_z\Psi}{\langle\Psi\rangle^2} & 0 & 0 & \frac{\partial_z\Psi}{\langle\Psi\rangle}\left(\frac{u}{c}-\frac{\partial_x\Psi}{\langle\Psi\rangle}\right) \end{pmatrix}, \tag{4.21}$$

$$\mathbf{A}_2 = \begin{pmatrix} \frac{v}{u} & 0 & 0 & 0 & 0 \\ 0 & \frac{v}{u} & 0 & \frac{c}{u}\frac{\partial_x\Psi\partial_y\Psi}{\langle\Psi\rangle} & -\frac{c}{u}\frac{\partial_x\Psi\partial_y\Psi}{\langle\Psi\rangle} \\ 0 & 0 & \frac{v}{u} & -\frac{c}{u}\frac{1+|\partial_x\Psi|^2}{\langle\Psi\rangle} & \frac{c}{u}\frac{1+|\partial_x\Psi|^2}{\langle\Psi\rangle} \\ 0 & 0 & \frac{1}{2}\frac{\partial_z\Psi}{\langle\Psi\rangle^2} & -\frac{\partial_z\Psi}{\langle\Psi\rangle}\left(\frac{v}{c}+\frac{\partial_y\Psi}{\langle\Psi\rangle}\right) & 0 \\ 0 & 0 & \frac{1}{2}\frac{\partial_z\Psi}{\langle\Psi\rangle^2} & 0 & \frac{\partial_z\Psi}{\langle\Psi\rangle}\left(\frac{v}{c}-\frac{\partial_y\Psi}{\langle\Psi\rangle}\right) \end{pmatrix}. \tag{4.22}$$

Therefore, for $\tau \in \mathbb{C}$ and $\eta \in \mathbb{R}$, we could compute that

$$\tau\mathbf{A}_1 + i\eta\mathbf{A}_2 = \begin{pmatrix} \omega & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & -\alpha_1 & \alpha_1 \\ 0 & 0 & \omega & \beta_1 & -\beta_1 \\ 0 & \mu\tau & i\eta\mu & -a(\omega+\theta) & 0 \\ 0 & \mu\tau & i\eta\mu & 0 & a(\omega-\theta) \end{pmatrix}. \tag{4.23}$$

For simplicity, we have introduced here

$$\omega = \tau + i\eta\frac{v}{u}, \quad \theta = \frac{c}{u}\frac{1}{\langle\Psi\rangle}(\tau\partial_x\Psi + i\eta\partial_y\Psi), \quad \mu = \frac{1}{2}\frac{\partial_z\Psi}{\langle\Psi\rangle^2}, \quad a = \frac{u}{c}\frac{\partial_z\Psi}{\langle\Psi\rangle}, \tag{4.24}$$
$$\alpha_1 = \frac{c}{u}\frac{1}{\langle\Psi\rangle}[(1+|\partial_y\Psi|^2)\tau - i\eta\partial_x\Psi\partial_y\Psi], \quad \beta_1 = \frac{c}{u}\frac{1}{\langle\Psi\rangle}[\tau\partial_x\Psi\partial_y\Psi - i\eta(1+|\partial_x\Psi|^2)].$$

4.4.3. Boundary conditions. Next we consider the boundary conditions (4.13) in terms of the new unknown W. Recall $\dot{V} = e^{-\gamma x}\dot{U}$, and now set $\tilde{\Phi} = e^{-\gamma x}\Phi$, $\varphi = e^{-\gamma x}\phi$, then (4.13), in terms of $\dot{V}$, becomes

$$\begin{pmatrix} \dot{V}_4 \\ -\dot{V}_3 + \dot{V}_1\partial_x\psi + \dot{V}_2\partial_y\psi + \gamma u\varphi + u\partial_x\varphi + v\partial_y\varphi - \frac{\varphi}{\partial_z\Psi}(\partial_z w - \partial_z u\partial_x\psi - \partial_z v\partial_y\psi) \end{pmatrix}$$
$$= e^{-\gamma x}g, \quad g := (g_1, h_1)^T. \tag{4.25}$$

In the following we will consider estimate of $\left\|\mathbb{P}(\psi)\dot{U}|_{z=0}\right\|_{H^*_{0,\gamma}}$ and $\|\phi\|_{H^*_{0,\gamma}}$ by utilizing problem (4.20) and (4.25). In terms of the unknowns listed in this problem, we just need to control $\|\varphi\|_{L^2}$ and $\left\|\mathbb{P}(\psi)\dot{V}|_{z=0}\right\|_{L^2}$. Direct calculation shows that

$$\dot{V}_1 = W_2 - \partial_x\Psi(W_4+W_5), \quad \dot{V}_2 = W_3 - \partial_y\Psi(W_4+W_5),$$
$$\dot{V}_3 = \partial_x\Psi W_2 + \partial_y\Psi W_3 + W_4 + W_5, \; \dot{V}_4 = \rho c\langle\Psi\rangle(-W_4+W_5), \; \dot{V}_5 = W_1.$$

So

$$\mathbb{P}\dot{V} = \begin{pmatrix} \dot{V}_3 - \dot{V}_1\partial_x\psi - \dot{V}_2\partial_y\psi \\ \dot{V}_4 \end{pmatrix} = \begin{pmatrix} \langle\Psi\rangle^2(W_4+W_5) \\ \rho c\langle\Psi\rangle(-W_4+W_5) \end{pmatrix},$$

and it follows that

$$\left\|\mathbb{P}\dot{V}|_{z=0}\right\|_{L^2(\mathbb{R}^2)} \le C(K_0)(\left\|(W_4)_{|z=0}\right\|_{L^2} + \left\|(W_5)_{|z=0}\right\|_{L^2}). \tag{4.26}$$

Hence in the following we only need to estimate the traces of vector (W_4, W_5) on $\{z = 0\}$, and $\|\varphi\|_{L^2}$. Meanwhile, the boundary condition (4.25) becomes

$$-W_4 + W_5 = \frac{e^{-\gamma x}}{\rho c\sqrt{1 + |\partial_x\psi|^2 + |\partial_y\psi|^2}} g_1, \quad z = 0, \tag{4.27}$$

$$u\partial_x\varphi + v\partial_y\varphi + \varphi\Big(\gamma u - \frac{1}{\partial_z\Psi}(\partial_z w - \partial_z u\partial_x\psi - \partial_z v\partial_y\psi)\Big) \\ -(1 + |\partial_x\psi|^2 + |\partial_y\psi|^2)(W_4 + W_5) = e^{-\gamma x}h_1, \quad z = 0. \tag{4.28}$$

4.4.4. Estimate of $\|\varphi\|_{L^2}$. As the equation (4.28) is a linear transport equation for φ, by a classical way, it is easy to have

$$\gamma^2\|\varphi\|^2_{L^2} \le C(K_0,\kappa_0)\Big(\|W_4\|^2_{L^2} + \|W_5\|^2_{L^2} + \left\|e^{-\gamma x}h_1\right\|^2_{L^2}\Big) \\ \le C(K_0,\kappa_0)\Big(\|(W_4, W_5)|_{z=0}\|^2_{L^2} + \frac{1}{\gamma^2}\|h_1\|^2_{H^*_{1,\gamma}}\Big). \tag{4.29}$$

Summarizing the above analysis ((4.19), (4.26) and (4.29)), we see to prove Theorem 4.1, one only needs to prove for problem (4.20) and (4.27) the following result:

THEOREM 4.3. *There exist constants C_1 and $\gamma_1 \ge 1$ that depend only on K_0 and κ_0 so that for all $\gamma \ge \gamma_1$ and all $(W,\varphi) \in H_{2,\gamma}(\Omega) \times H_{2,\gamma}(\mathbb{R}^2)$, the following estimate holds:*

$$\|(W_4, W_5)|_{z=0}\|^2_{L^2} \le C_1\left(\frac{1}{\gamma^3}\left\|e^{-\gamma x}f\right\|^2_{L^2(H_{1,\gamma})} + \frac{1}{\gamma^2}\left\|e^{-\gamma x}g_1\right\|^2_{1,\gamma}\right). \tag{4.30}$$

4.5. Paralinearization. Because the problem (4.20) (4.27) has non-smooth coefficients, we shall use the para-differential calculus to study it. In this step we replace the differential operators by para-differential operators T^γ_α with suitable symbols α and a parameter γ, and estimate the error. One can refer to the appendices of [2] or [10, 11] for an introduction of para-differential operators and theorems we used below.

4.5.1. Error of paralinearization. For the boundary condition (4.27), we set $\beta = (1, -1)$ and write it as

$$\beta W^{\mathrm{nc}} = \tilde{G}, \quad \text{on} \ \ \{z = 0\}.$$

Here $\tilde{G} = \frac{e^{-\gamma x}}{\rho c\sqrt{1+|\partial_x\psi|^2+|\partial_y\psi|^2}} g_1$, and we have introduced the non-characteristic unknown

$$W^{\mathrm{nc}} = (W_5, W_4)^T.$$

Since β is constant, the above boundary condition may be written directly as

$$T^\gamma_\beta W^{\mathrm{nc}} = \tilde{G} \tag{4.31}$$

and there is no any error.

We now turn to the equation (4.20). As done in [11], we replace each term on the left-hand side of (4.20) by corresponding para-differential operator, and estimate the error first for fixed z which is considered as a parameter in the symbol, and then integrate with respect to $z>0$. For example, by Theorem B.9 in [11] we have

$$\begin{aligned}&\left\|\gamma \mathbf{A}_1 W - T^{\gamma}_{\gamma\mathbf{A}_1} W\right\|^2_{L^2(H_{1,\gamma})} = \int_0^\infty \gamma^2 \left\|\mathbf{A}_1 W(\cdot,z) - T^{\gamma}_{\mathbf{A}_1} W(\cdot,z)\right\|^2_{1,\gamma}\,\mathrm{d}z\\ &\le C\int_0^\infty \|\mathbf{A}_1(\cdot,z)\|^2_{W^{2,\infty}(\mathbb{R}^2)}\|W(\cdot,z)\|^2_{L^2(\mathbb{R}^2)}\,\mathrm{d}z \le C\|\mathbf{A}_1\|^2_{W^{2,\infty}(\Omega)}\int_\Omega |W|^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &\le C(K_0)\|W\|^2_{L^2(\Omega)}.\end{aligned}$$

Denote by δ and η the dual variables of x and y respectively. Then similarly we have

$$\begin{aligned}\left\|\mathbf{A}_1\partial_x W - T^{\gamma}_{i\delta\mathbf{A}_1} W\right\|^2_{L^2(H_{1,\gamma})} &= \int_0^\infty \left\|\mathbf{A}_1\partial_x W(\cdot,z) - T^{\gamma}_{i\delta\mathbf{A}_1} W(\cdot,z)\right\|^2_{1,\gamma}\,\mathrm{d}z\\ &\le C\int_0^\infty \|\mathbf{A}_1\|^2_{W^{2,\infty}(\mathbb{R}^2)}\|W(\cdot,z)\|^2_{L^2(\mathbb{R}^2)}\,\mathrm{d}z\\ &\le C\|\mathbf{A}_1\|^2_{W^{2,\infty}(\Omega)}\|W\|^2_{L^2(\Omega)} \le C(K_0)\|W\|^2_{L^2(\Omega)},\end{aligned}$$

and $\left\|\mathbf{A}_2\partial_y W - T^{\gamma}_{i\eta\mathbf{A}_2} W\right\|_{L^2(H_{1,\gamma})} \le C(K_0)\|W\|_{L^2(\Omega)}$. Using the third inequality in Theorem C.20 of [2, p.490], we also get

$$\begin{aligned}\|\mathbf{C}W - T^{\gamma}_{\mathbf{C}} W\|^2_{L^2(H_{1,\gamma})} &= \int_0^\infty \|\mathbf{C}W(\cdot,z) - T^{\gamma}_{\mathbf{C}} W(\cdot,z)\|^2_{1,\gamma}\,\mathrm{d}z\\ &\le \frac{C}{\gamma^2}\int_0^\infty \|\mathbf{C}(\cdot,z)\|^2_{W^{1,\infty}(\mathbb{R}^2)}\|W\|^2_{L^2(\mathbb{R}^2)}\,\mathrm{d}z\\ &\le C\|\mathbf{C}\|^2_{W^{1,\infty}(\Omega)}\|W\|^2_{L^2(\Omega)} \le C(K_0,\kappa_0)\|W\|^2_{L^2(\Omega)}.\end{aligned}$$

Therefore the total error of para-linearization of (4.20) is (recall $\tau=\gamma+i\delta$.)

$$\left\|L^\gamma W - (T^{\gamma}_{\tau\mathbf{A}_1+i\eta\mathbf{A}_2+\mathbf{C}} W + \mathcal{I}_5\partial_z W)\right\|_{L^2(H_{1,\gamma})} \le C(K_0,\kappa_0)\|W\|_{L^2(\Omega)}. \quad (4.32)$$

4.5.2. The boundary value problem of para-differential equations. We now consider the following boundary value problem of para-linearized system:

$$\begin{cases}\mathcal{I}_5\partial_z W + T^{\gamma}_{\tau\mathbf{A}_1+i\eta\mathbf{A}_2+\mathbf{C}} W = \tilde{F}, & z>0,\\ T^{\gamma}_{\beta} W^{\mathrm{nc}} = \tilde{G}, & z=0,\end{cases} \quad (4.33)$$

where $\tilde{F} = e^{-\gamma x}A_0 T^{-1}\Theta^{-1} f$.

THEOREM 4.4. *Assume there are constants C_0,γ_0 depending only on K_0 and κ_0 so that the solution W to the problem (4.33) satisfies the estimate*

$$\|W^{\mathrm{nc}}|_{z=0}\|^2_{L^2(\mathbb{R}^2)} \le C_0\left(\frac{1}{\gamma^3}\left\|\tilde{F}\right\|^2_{L^2(H_{1,\gamma})} + \frac{1}{\gamma^2}\left\|\tilde{G}\right\|^2_{1,\gamma}\right) \quad (4.34)$$

for all $\gamma>\gamma_0$. Then Theorem 4.3 holds.

Proof. The equation (4.20) may be written equivalently as

$$\begin{aligned}&\mathcal{I}_5\partial_z W + T^{\gamma}_{\tau\mathbf{A}_1+i\eta\mathbf{A}_2+\mathbf{C}} W = -[L^\gamma W - (\mathcal{I}_5\partial_z W + T^{\gamma}_{\tau\mathbf{A}_1+i\eta\mathbf{A}_2+\mathbf{C}} W)] + \tilde{F},\\ &T^{\gamma}_{\beta} W^{\mathrm{nc}} = \tilde{G}.\end{aligned}$$

Using (4.34) and (4.32), we find

$$\begin{aligned}\|W^{\mathrm{nc}}|_{z=0}\|^2_{L^2(\mathbb{R}^2)} &\le C_0\left(\frac{1}{\gamma^3}\left\|L^\gamma W-(\mathcal{I}_5\partial_z W+T^\gamma_{\tau\mathbf{A}_1+i\eta\mathbf{A}_2+\mathbf{C}}W)\right\|^2_{L^2(H_{1,\gamma})}\right.\\ &\qquad\left.+\frac{1}{\gamma^3}\left\|\tilde F\right\|^2_{L^2(H_{1,\gamma})}+\frac{1}{\gamma^2}\left\|\tilde G\right\|^2_{1,\gamma}\right)\\ &\le C_0\left(\frac{1}{\gamma^3}\|W\|^2_{L^2(\Omega)}+\frac{1}{\gamma^3}\left\|\tilde F\right\|^2_{L^2(H_{1,\gamma})}+\frac{1}{\gamma^2}\left\|\tilde G\right\|^2_{1,\gamma}\right). \quad (4.35)\end{aligned}$$

For the first term on the right-hand side, recall $W=T^{-1}\dot V$, then by Lemma 4.2 and estimate (4.26),

$$\|W\|^2_{L^2(\Omega)}\le C(\kappa_0)\left\|\dot V\right\|^2_{L^2(\Omega)}\le C(K_0,\kappa_0)\frac{1}{\gamma}\left(\frac{1}{\gamma^3}\left\|e^{-\gamma x}f\right\|^2_{L^2(H_{1,\gamma})}+\|W^{\mathrm{nc}}|_{z=0}\|^2_{L^2(\mathbb{R}^2)}\right).$$

For the second and third terms on the right-hand side of (4.35), it is straightforward to check that

$$\begin{aligned}\left\|\tilde F\right\|^2_{L^2(H_{1,\gamma})}&=\int_0^\infty\left\|e^{-\gamma x}A_0T^{-1}\Theta^{-1}f(\cdot,z)\right\|^2_{1,\gamma}\,\mathrm{d}z\le C(K_0,\kappa_0)\left\|e^{-\gamma x}f\right\|^2_{L^2(H_{1,\gamma})},\\ \left\|\tilde G\right\|^2_{1,\gamma}&=\left\|\frac{e^{-\gamma x}g_1}{\rho c\sqrt{1+|\partial_x\psi|^2+|\partial_y\psi|^2}}\right\|^2_{1,\gamma}\le C(K_0,\kappa_0)\left\|e^{-\gamma x}g_1\right\|^2_{1,\gamma}.\end{aligned}$$

Substituting the above three inequalities into (4.35), we find that by taking γ_0 further larger (depending only on κ_0 and K_0), there holds

$$\|W^{\mathrm{nc}}|_{z=0}\|^2_{L^2}\le C(K_0,\kappa_0)\left(\frac{1}{\gamma^3}\left\|e^{-\gamma x}f\right\|^2_{L^2(H_{1,\gamma})}+\frac{1}{\gamma^2}\left\|e^{-\gamma x}g_1\right\|^2_{1,\gamma}\right).$$

This is exactly (4.30) claimed in Theorem 4.3. □

4.6. Microlocalization. From now on we focus on the problem (4.33) and our aim is to derive estimate (4.34).

Since $\mathcal{I}_5=\mathrm{diag}(0,0,0,1,1)$, the first three equations in (4.33) do not involve $\partial_z W$. The main idea is to solve W_1,W_2,W_3 from the first three equations and then substitute them into the last two equations, and get a para-differential problem for $W^{\mathrm{nc}}=(W_5,W_4)^T$ of the form

$$\begin{cases}\partial_z W^{\mathrm{nc}}=T^\gamma_{\mathbb{A}}W^{\mathrm{nc}}+T^\gamma_{\mathbb{E}}W^{\mathrm{nc}}+\text{source term}, & z>0,\\ T^\gamma_\beta W^{\mathrm{nc}}|_{z=0}=\text{source term}, & z=0.\end{cases}\quad (4.36)$$

Here $\mathbb{A}$ is a 2×2 matrix symbol of order one and $\mathbb{E}$ is a 2×2 matrix symbol of order zero. We first illustrate in a formal way how to determine $\mathbb{A}$.

Recall the symbol $\tau\mathbf{A}_1+i\eta\mathbf{A}_2$ is given by (4.23). Denote by $\mathbf{C}=(\mathbf{c}_1^T,\cdots,\mathbf{c}_5^T)^T$ for the matrix $\mathbf{C}$ appeared in (4.20), with each $\mathbf{c}_j$ being a row. The second and third equations in (4.33) read

$$\begin{aligned}T^\gamma_\omega W_2+T^\gamma_{-\alpha_1}W_4+T^\gamma_{\alpha_1}W_5&=-T^\gamma_{\mathbf{c}_2}W+\tilde F_2,\\ T^\gamma_\omega W_3+T^\gamma_{\beta_1}W_4+T^\gamma_{-\beta_1}W_5&=-T^\gamma_{\mathbf{c}_3}W+\tilde F_3.\end{aligned}$$

Now acting $T^{\gamma}_{\omega^{-1}}$ on both sides of these equations, and using symbolic calculus, we find

$$\begin{cases} W_2 = T^{\gamma}_{\alpha_1\omega^{-1}} W_4 + T^{\gamma}_{-\alpha_1\omega^{-1}} W_5 + R_{-1}W + T^{\gamma}_{\omega^{-1}}\tilde{F}_2, \\ W_3 = T^{\gamma}_{-\beta_1\omega^{-1}} W_4 + T^{\gamma}_{\beta_1\omega^{-1}} W_5 + R_{-1}W + T^{\gamma}_{\omega^{-1}}\tilde{F}_3, \end{cases} \tag{4.37}$$

where R_{-1} is some operator of order -1. Recall by (4.23), the forth and fifth equations in (4.33) are

$$\partial_z W_4 + T^{\gamma}_{\mu\tau} W_2 + T^{\gamma}_{i\eta\mu} W_3 + T^{\gamma}_{-a(\omega+\theta)} W_4 + T^{\gamma}_{\mathbf{c}_4} W = \tilde{F}_4,$$
$$\partial_z W_5 + T^{\gamma}_{\mu\tau} W_2 + T^{\gamma}_{i\eta\mu} W_3 + T^{\gamma}_{a(\omega-\theta)} W_5 + T^{\gamma}_{\mathbf{c}_5} W = \tilde{F}_5.$$

By using the expressions (4.37) of W_2 and W_3, and symbolic calculus of composition of operators, we get that the above two equations can be written as the matrix form (4.36), with

$$\mathbb{A} = \begin{pmatrix} \alpha - a(\omega-\theta) & -\alpha \\ \alpha & -\alpha + a(\omega+\theta) \end{pmatrix}, \tag{4.38}$$

where

$$\alpha := \mu\omega^{-1}(\tau\alpha_1 - i\eta\beta_1) = \frac{c}{2}\frac{\partial_z\Psi}{\langle\Psi\rangle^3}\frac{(\tau^2-\eta^2) + (\tau\partial_y\Psi - i\eta\partial_x\Psi)^2}{u\tau + iv\eta}.$$

For the planar case $\Psi = z$ and $U = \underline{U}$, this matrix $\mathbb{A}$ is reduced exactly to $\mathcal{B}$ appeared in (3.6).

We note that the symbol $\mathbb{A}$ depends on (x, y, z, τ, η), with the corresponding operator acting on functions of (x, y), and $z > 0$ is a parameter. Here $\tau = \gamma + i\delta$, and $(\delta, \eta) \in \mathbb{R}^2$ are dual variables of (x, y), and $\gamma \geq 1$ is a parameter. It is obvious that

$$\begin{aligned} \Upsilon_{\mathbf{p}} &= \{(x, y, z, \tau, \eta) \in \Omega \times \Xi : u(x, y, z)\tau + i\eta v(x, y, z) = 0\} \\ &= \{(x, y, z, \tau = \gamma + i\delta, \eta) \in \Omega \times \Xi : \gamma = 0,\ \delta = -\frac{v(x, y, z)}{u(x, y, z)}\eta\} \end{aligned}$$

is the set of poles of $\mathbb{A}$; recall here $\Xi := \{(\tau, \eta) \in (\mathbb{C}\times\mathbb{R}) \setminus \{(0, 0)\} : \mathrm{Re}\,\tau \geq 0\}$. So the above calculation should be taken away from $\Upsilon_{\mathbf{p}}$.

The eigenvalues λ of $\mathbb{A}$ are the roots to the equation

$$(\lambda - a\theta)^2 = (a\omega)^2 - 2\alpha a\omega.$$

We observe that there is no singularity for λ as a function of (τ, η) and it is homogeneous of degree one. The eigenvalue with positive (resp. negative) real part when $\mathrm{Re}\,\tau > 0$ is denoted by λ_+ (resp. λ_-). The stable subspace of $\mathbb{A}$ is given by $E_-(x, y, z, \tau, \eta) = \mathrm{span}\{\mathbf{e}_-\}$ with $\mathbf{e}_- = (u\tau + iv\eta)(\lambda_- + \alpha - aw - a\theta, \alpha)^T$, so the space E_- can be extended continuously to $\mathrm{Re}\,\tau \geq 0$. The Lopatinskii determinant is then given by $\Delta(x, y, z, \tau, \eta) = (u\tau + iv\eta)(\lambda_- - (aw + a\theta))$ and one may check that the latter factor never vanishes. So Δ vanishes only at poles of the matrix $\mathbb{A}$. All these can be checked in the same fashion as for the constant coefficient case, under the assumption that the perturbation $U - \underline{U}$ and $\nabla(\Psi - z)$ has compact support with respect to (y, z), and

$$\|U - \underline{U}\|_{L^\infty} \leq \nu, \quad \|\nabla\Psi - (0, 0, 1)\|_{L^\infty} \leq \nu$$

for ν small.

To deal with different situations of points in $\Omega\times\Xi$, that is, those points belong to the set of poles $\Upsilon_{\mathbf{p}}$ where Kreiss–Lopatinskii condition fails at the boundary $\Upsilon_{\mathbf{p}}\cap\{z=0\}$ and those points in $\Omega\times\Xi\setminus\Upsilon_{\mathbf{p}}$ where the Lopatinskii determinant is nonzero and the matrix $\mathbb{A}$ is well-defined, we introduce the following two cut-off functions:

- ♡ $\chi_{\mathbf{p}}$ is a C^∞ function on $\Omega\times\Xi$, homogeneous of degree zero with respect to (τ,η); $\chi_{\mathbf{p}}\equiv 1$ on $\Upsilon_{\mathbf{p}}$, and $\operatorname{supp}\chi_{\mathbf{p}}\subset\mathcal{V}_{\mathbf{p}}$, with $\mathcal{V}_{\mathbf{p}}$ an open subset of $\Omega\times\Xi$ that containing $\Upsilon_{\mathbf{p}}$;
- ♡ $\chi_{\mathbf{u}}=1-\chi_{\mathbf{p}}$. So $\chi_{\mathbf{u}}$ is supported far away from the poles and $\operatorname{supp}\chi_{\mathbf{u}}\cap(\partial\Omega\times\Xi)$ consists only those points where the uniform Kreiss–Lopatinskii condition holds.

In the following two subsections, we will estimate the traces of $T^\gamma_{\chi_{\mathbf{u}}}W^{\mathrm{nc}}$ and $T^\gamma_{\chi_{\mathbf{p}}}W^{\mathrm{nc}}$ on $\{z=0\}$ in two different ways. Then as $\chi_{\mathbf{u}}+\chi_{\mathbf{p}}=1$, we get $W^{\mathrm{nc}}=T^\gamma_{\chi_{\mathbf{u}}}W^{\mathrm{nc}}+T^\gamma_{\chi_{\mathbf{p}}}W^{\mathrm{nc}}$ and finally by taking into account of some errors appearing due to symbolic calculus, we get the desired estimate of $W^{\mathrm{nc}}|_{z=0}$.

4.7. Derivation of energy estimate: frequencies away from poles.

4.7.1. Derivation of the equations. We start again from problem (4.33). Set $W_{\mathbf{u}}=T^\gamma_{\chi_{\mathbf{u}}}W$. Noting that $T^\gamma_{\chi_{\mathbf{u}}}$ is a para-differential operator acting on functions of (x,y) with z being a parameter, we have

$$\begin{aligned}\mathcal{I}_5\partial_z(W_{\mathbf{u}})=\mathcal{I}_5\partial_z(T^\gamma_{\chi_{\mathbf{u}}}W)&=\mathcal{I}_5T^\gamma_{\chi_{\mathbf{u}}}\partial_zW+\mathcal{I}_5T^\gamma_{\partial_z(\chi_{\mathbf{u}})}W\\&=\mathcal{I}_5T^\gamma_{\partial_z(\chi_{\mathbf{u}})}W-T^\gamma_{\chi_{\mathbf{u}}}T^\gamma_{\tau\mathbf{A}_1+i\eta\mathbf{A}_2+\mathbf{C}}W+T^\gamma_{\chi_{\mathbf{u}}}\tilde{F}.\end{aligned}$$

By symbolic calculus, recalling that $\chi_{\mathbf{u}}\in\Gamma^0_k$ ($k\in\mathbb{N}$), $\tau\mathbf{A}_1+i\eta\mathbf{A}_2\in\Gamma^1_2$, and $\mathbf{C}\in\Gamma^0_1$, we find

$$\begin{aligned}&-T^\gamma_{\chi_{\mathbf{u}}}T^\gamma_{\mathbf{C}}W=-T^\gamma_{\mathbf{C}}T^\gamma_{\chi_{\mathbf{u}}}W+R_{-1}W=-T^\gamma_{\mathbf{C}}W_{\mathbf{u}}+R_{-1}W,\\&-T^\gamma_{\chi_{\mathbf{u}}}T^\gamma_{\tau\mathbf{A}_1+i\eta\mathbf{A}_2}W=-T^\gamma_{\tau\mathbf{A}_1+i\eta\mathbf{A}_2}T^\gamma_{\chi_{\mathbf{u}}}W+R_{-1}W+T^\gamma_{-i\{\tau\mathbf{A}_1+i\eta\mathbf{A}_2,\chi_{\mathbf{u}}\}}W,\end{aligned}$$

where R_{-1} is an operator of order -1, and

$$\{a,b\}=\frac{\partial a}{\partial\delta}\frac{\partial b}{\partial x}+\frac{\partial a}{\partial\eta}\frac{\partial b}{\partial y}-\frac{\partial a}{\partial x}\frac{\partial b}{\partial\delta}-\frac{\partial a}{\partial y}\frac{\partial b}{\partial\eta}$$

is the Poisson bracket. Recall $\partial_z\chi_{\mathbf{u}}\in\Gamma^0_k$, so if we set

$$r=i\{\tau\mathbf{A}_1+i\eta\mathbf{A}_2,\chi_{\mathbf{u}}\}-\partial_z\chi_{\mathbf{u}}\mathcal{I}_5\in\Gamma^0_1,$$

which vanishes in a neighborhood of the pole set $\Upsilon_{\mathbf{p}}$ and also outside of $\mathcal{V}_{\mathbf{p}}$, then we have an equation for $W_{\mathbf{u}}$:

$$\mathcal{I}_5\partial_zW_{\mathbf{u}}+T^\gamma_{\tau\mathbf{A}_1+i\eta\mathbf{A}_2}W_{\mathbf{u}}+T^\gamma_{\mathbf{C}}W_{\mathbf{u}}+T^\gamma_rW=R_{-1}W+T^\gamma_{\chi_{\mathbf{u}}}\tilde{F}. \tag{4.39}$$

In the following we continue to use symbolic calculus to decouple the above algebraic para-differential system for the characteristic components and non-characteristic components respectively. We shall denote by α^m a generic symbol of class Γ^m_1, and r any symbol in Γ^0_1 that vanishes in a neighborhood of $\Upsilon_{\mathbf{p}}$ and outside $\mathcal{V}_{\mathbf{p}}$. The notation R_m is also used to denote a generic operator of order m. We also write

$$W_{\mathbf{u}}=(w_1,w_2,w_3,w_4,w_5)^T,\quad W=(W_1,W_2,W_3,W_4,W_5)^T.$$

Now the system (4.39) may be written line by line as

$$T^\gamma_\omega w_1 + \sum_{i=1}^5 T^\gamma_{\alpha_0} w_i + T^\gamma_r W = R_{-1}W + T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_1, \tag{4.40}$$

$$T^\gamma_\omega w_2 + T^\gamma_{\alpha_1}(w_5 - w_4) + \sum_{i=1}^5 T^\gamma_{\alpha_0} w_i + T^\gamma_r W = R_{-1}W + T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_2, \tag{4.41}$$

$$T^\gamma_\omega w_3 + T^\gamma_{-\beta_1}(w_5 - w_4) + \sum_{i=1}^5 T^\gamma_{\alpha_0} w_i + T^\gamma_r W = R_{-1}W + T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_3, \tag{4.42}$$

$$\partial_z w_4 + T^\gamma_{\mu\tau} w_2 + T^\gamma_{i\eta\mu} w_3 + T^\gamma_{-a(\omega+\theta)} w_4 + \sum_{i=1}^5 T^\gamma_{\alpha_0} w_i + T^\gamma_r W = R_{-1}W + T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_4, \tag{4.43}$$

$$\partial_z w_5 + T^\gamma_{\mu\tau} w_2 + T^\gamma_{i\eta\mu} w_3 + T^\gamma_{a(\omega-\theta)} w_5 + \sum_{i=1}^5 T^\gamma_{\alpha_0} w_i + T^\gamma_r W = R_{-1}W + T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_5. \tag{4.44}$$

Recall that the symbols ω etc. had been defined in (4.24).

We now try to solve w_1, w_2, w_3 from (4.40), (4.41) and (4.42), and substitute them into (4.43) and (4.44). To apply a localized Gårding's inequality later, we need to introduce cut-off functions χ_0, χ_1 and χ_2 such that

- χ_0, χ_1 and χ_2 are C^∞ functions, taking values in $[0, 1]$, and homogeneous of degree zero with respect to (τ, η);
- $\chi_0 \equiv 1$ on $\operatorname{supp}\chi_{\mathbf{u}}$, $\chi_1 \equiv 1$ on $\operatorname{supp}\chi_0$, and $\chi_2 \equiv 1$ on $\operatorname{supp}\chi_1$;
- χ_2 (and therefore χ_1, χ_0) vanishes in a small neighborhood of the set of poles $\Upsilon_{\mathbf{p}}$.

From these we see $\chi_2\omega^{-1}$ is well-defined and is a symbol of class Γ^{-1}_2. Now applying $T^\gamma_{\chi_2\omega^{-1}}$ to the equations (4.41)(4.42), by using symbolic calculus, we obtain

$$T^\gamma_{\chi_2} w_2 + T^\gamma_{\chi_2\alpha_1\omega^{-1}}(w_5 - w_4) + \sum_{i=1}^5 T^\gamma_{\alpha^{-1}} w_i + T^\gamma_{\chi_2 r\omega^{-1}} W = R_{-2}W + T^\gamma_{\chi_2\omega^{-1}} T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_2,$$

$$T^\gamma_{\chi_2} w_3 + T^\gamma_{-\chi_2\beta_1\omega^{-1}}(w_5 - w_4) + \sum_{i=1}^5 T^\gamma_{\alpha^{-1}} w_i + T^\gamma_{\chi_2 r\omega^{-1}} W = R_{-2}W + T^\gamma_{\chi_2\omega^{-1}} T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_3.$$

Observing that, as $\chi_2 \equiv 1$ on $\operatorname{supp}\chi_{\mathbf{u}}$, so $\{\chi_2, \chi_{\mathbf{u}}\} \equiv 0$, hence

$$T^\gamma_{\chi_2} W_{\mathbf{u}} = T^\gamma_{\chi_2} T^\gamma_{\chi_{\mathbf{u}}} W = T^\gamma_{\chi_{\mathbf{u}}} W + R_{-2}W = W_{\mathbf{u}} + R_{-2}W,$$

we then solve

$$w_2 = -T^\gamma_{\chi_2\alpha_1\omega^{-1}}(w_5 - w_4) - \sum_{i=1}^5 T^\gamma_{\alpha^{-1}} w_i - T^\gamma_{\chi_2 r\omega^{-1}} W + R_{-2}W + T^\gamma_{\chi_2\omega^{-1}} T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_2, \tag{4.45}$$

$$w_3 = -T^\gamma_{-\chi_2\beta_1\omega^{-1}}(w_5 - w_4) - \sum_{i=1}^5 T^\gamma_{\alpha^{-1}} w_i - T^\gamma_{\chi_2 r\omega^{-1}} W + R_{-2}W + T^\gamma_{\chi_2\omega^{-1}} T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_3. \tag{4.46}$$

Similarly, from (4.40) we have

$$w_1 = -\sum_{i=1}^{5} T^\gamma_{\alpha^{-1}} w_i - T^\gamma_{\chi_2 r\omega^{-1}} W + R_{-2}W + T^\gamma_{\chi_2\omega^{-1}} T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_1. \tag{4.47}$$

From (4.45) and (4.46) we easily have

$$T^\gamma_{\mu\tau} w_2 = -T^\gamma_{\chi_2\mu\tau\alpha_1\omega^{-1}}(w_5 - w_4) + \sum_{i=1}^{5} T^\gamma_{\alpha^0} w_i + T^\gamma_r W + R_{-1}W + T^\gamma_{\mu\tau} T^\gamma_{\chi_2\omega^{-1}} T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_2,$$

$$T^\gamma_{i\mu\eta} w_3 = -T^\gamma_{-i\chi_2\mu\eta\beta_1\omega^{-1}}(w_5 - w_4) + \sum_{i=1}^{5} T^\gamma_{\alpha^0} w_i + T^\gamma_r W + R_{-1}W + T^\gamma_{i\mu\eta} T^\gamma_{\chi_2\omega^{-1}} T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_3.$$

We further note that for any $\alpha^1 \in \Gamma^1_1$, it holds

$$T^\gamma_{\chi_2\alpha^1} W_{\mathbf{u}} = T^\gamma_{\alpha^1} T^\gamma_{\chi_2} W_{\mathbf{u}} + R_0 W_{\mathbf{u}} = T^\gamma_{\alpha^1} W_{\mathbf{u}} + R_0 W_{\mathbf{u}} + R_{-1}W.$$

So from (4.44) we have

$$\begin{aligned}\partial_z w_5 = {} & T^\gamma_{\chi_2\mu(\tau\alpha_1 - i\eta\beta_1)\omega^{-1}}(w_5 - w_4) - T^\gamma_{\chi_2 a(\omega-\theta)} w_5 + \sum_{i=1}^{5} T^\gamma_{\alpha^0} w_i + T^\gamma_r W + R_{-1}W \\ & + T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_5 - T^\gamma_{i\mu\eta} T^\gamma_{\chi_2\omega^{-1}} T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_3 - T^\gamma_{\mu\tau} T^\gamma_{\chi_2\omega^{-1}} T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_2,\end{aligned}$$

and from (4.43),

$$\begin{aligned}\partial_z w_4 = {} & T^\gamma_{\chi_2\mu(\tau\alpha_1 - i\eta\beta_1)\omega^{-1}}(w_5 - w_4) + T^\gamma_{\chi_2 a(\omega+\theta)} w_4 + \sum_{i=1}^{5} T^\gamma_{\alpha^0} w_i + T^\gamma_r W + R_{-1}W \\ & + T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_4 - T^\gamma_{i\mu\eta} T^\gamma_{\chi_2\omega^{-1}} T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_3 - T^\gamma_{\mu\tau} T^\gamma_{\chi_2\omega^{-1}} T^\gamma_{\chi_{\mathbf{u}}} \tilde{F}_2.\end{aligned}$$

If we use (4.45)–(4.47) to replace the zero-th order terms w_i ($i = 1, 2, 3$) on the right-hand sides of the above two equations, then we get the following system

$$\partial_z \begin{pmatrix} w_5 \\ w_4 \end{pmatrix} = T^\gamma_{\chi_2\mathbb{A}} \begin{pmatrix} w_5 \\ w_4 \end{pmatrix} + T^\gamma_{\mathbb{E}} \begin{pmatrix} w_5 \\ w_4 \end{pmatrix} + T^\gamma_r W + R_{-1}W + R_0\tilde{F}. \tag{4.48}$$

Here $\mathbb{A} \in \Gamma^1_2$ is the symbol given by (4.38), and $\mathbb{E} \in \Gamma^0_1$.

Since β is a constant vector, we have directly the boundary condition of $W^{\mathrm{nc}}_{\mathbf{u}} = (w_5, w_4)^T$:

$$\beta(w_5, w_4)^T|_{z=0} = T^\gamma_{\chi_{\mathbf{u}}} \tilde{G}|_{z=0}. \tag{4.49}$$

4.7.2. Energy estimate. We now study the estimates of the non-characteristic components of the unknown to the problem (4.48) (4.49) by the method of Kreiss' symmetrizers. Since we know the Kreiss–Lopatinskii condition holds at the points where $\chi_2 \neq 0$, and $\mathbb{A}$ is also well-defined there, the construction of such symmetrizers is quite standard. The estimate we will prove reads

$$\begin{aligned}&\gamma \left\|T^\gamma_{\chi_{\mathbf{u}}} W\right\|^2_{L^2(H_{1/2,\gamma})} + \left\|T^\gamma_{\chi_{\mathbf{u}}} W^{\mathrm{nc}}(0)\right\|^2_{1/2,\gamma} \\ &\leq \frac{C}{\gamma}\left(\frac{1}{\gamma}\left\|\tilde{F}\right\|^2_{L^2(H_{1,\gamma})} + \|W\|^2_{L^2(L^2)} + \|T^\gamma_r W\|^2_{L^2(H_{1/2,\gamma})} + \left\|\tilde{G}\right\|^2_{1,\gamma} + \|W^{\mathrm{nc}}(0)\|^2_{L^2}\right).\end{aligned} \tag{4.50}$$

The reason why we give a $L^2(H_{1/2,\gamma})$ estimate as above is that this is the only one available near the poles $\Upsilon_{\mathbf{p}}$, to be derived in the next subsection. Recall here that $r \in \Gamma_1^0$ vanishes in a neighborhood of $\Upsilon_{\mathbf{p}}$ and out of $\mathcal{V}_{\mathbf{p}}$, which is an error of microlocalization. The appearance of $\|W^{\rm nc}(0)\|_{L^2}^2$ on the right-hand side of (4.50) is due to an error when applying localized Gårding's inequality.

Proposition 4.5 (Kreiss' symmetrizers). *There exists a mapping $S : \bar{\Omega} \times \Xi \to \mathbb{M}_{2\times 2}(\mathbb{C})$ so that the following properties hold:*

- *$\forall X = (x, y, z, \gamma, \eta) \in \bar{\Omega} \times \Xi$, $S(X)$ is Hermitian and $S \in \Gamma_2^1$;*
- *Set $\lambda^{s,\gamma} = (|\tau|^2 + |\eta|^2)^{\frac{s}{2}}$. For any $X \in \partial\Omega \times \Xi$, there holds*

$$\chi_2^2 S(X) + C\chi_2^2 \lambda^{1,\gamma}(\tau, \eta)\beta^*\beta \geq c\chi_2^2 \lambda^{1,\gamma}(\tau, \eta) I_2; \tag{4.51}$$

- *There exists a finite set of matrix-valued mappings such that*

$$\mathrm{Re}\,(S(X)\chi_2\mathbb{A}(X)) = \sum_l V_l(X)^* \begin{pmatrix} \chi_2\gamma H_l(X) & 0 \\ 0 & \chi_2 E_l(X) \end{pmatrix} V_l(X), \tag{4.52}$$

where V_l (E_l resp.) is homogeneous of degree $1/2$ (1 resp.) with respect to (τ, η), and belong to $\Gamma_2^{\frac{1}{2}}$ (Γ_2^1 resp.); H_l is homogeneous of degree zero with respect to (τ, η), and belongs to Γ_2^0, and the following inequalities hold:

$$\begin{aligned} &\sum_l V_l(X)^* V_l(X) \geq c\lambda^{1,\gamma}(\tau, \eta) I_2, \\ &\chi_2 H_l(X) \geq c\chi_2 I_2, \qquad \chi_2 E_l(X) \geq c\chi_2 \lambda^{1,\gamma}(\tau, \eta) I_2. \end{aligned} \tag{4.53}$$

This result will be proved in section 4.7.3. We adopt the ideas presented in [10] to derive energy estimates using these symmetrizers.

Let $\{\mathbf{S}^\gamma(z)\}$ be given by

$$\mathbf{S}^\gamma(z) = \frac{1}{2}\left((T^\gamma_{S(z)})^* + T^\gamma_{S(z)}\right),$$

with $S(z)$ denoting the above symmetrizer and z a parameter. Since $S \in \Gamma_2^1$, we know $\{\mathbf{S}^\gamma\}$ are uniformly bounded self-adjoint operators from $H_{s,\gamma}(\mathbb{R}^2_{x,y})$ to $H_{s-1,\gamma}(\mathbb{R}^2_{x,y})$. The starting point to derive the energy estimate is to take the scalar product of (4.48) with $\mathbf{S}^\gamma W^{\rm nc}_{\mathbf{u}} = \mathbf{S}^\gamma (w_5, w_4)^T$, and integrating with respect to $(x, y, z) \in \Omega$. Actually, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}(\mathbf{S}^\gamma W^{\rm nc}_{\mathbf{u}}, W^{\rm nc}_{\mathbf{u}}) &= 2\mathrm{Re}\,(\mathbf{S}^\gamma \partial_z W^{\rm nc}_{\mathbf{u}}, W^{\rm nc}_{\mathbf{u}}) + ((\partial_z \mathbf{S}^\gamma) W^{\rm nc}_{\mathbf{u}}, W^{\rm nc}_{\mathbf{u}}) \\ &= 2\mathrm{Re}\,(\mathbf{S}^\gamma T^\gamma_{\mathbb{A}\chi_2} W^{\rm nc}_{\mathbf{u}}, W^{\rm nc}_{\mathbf{u}}) + 2\mathrm{Re}\,(\mathbf{S}^\gamma T^\gamma_{\mathbb{E}} W^{\rm nc}_{\mathbf{u}}, W^{\rm nc}_{\mathbf{u}}) \\ &\quad + 2\mathrm{Re}\,(\mathbf{S}^\gamma T^\gamma_r W, W^{\rm nc}_{\mathbf{u}}) + 2\mathrm{Re}\,(\mathbf{S}^\gamma R_{-1} W, W^{\rm nc}_{\mathbf{u}}) \\ &\quad + 2\mathrm{Re}\,(\mathbf{S}^\gamma R_0 F, W^{\rm nc}_{\mathbf{u}}) + ((\partial_z \mathbf{S}^\gamma) W^{\rm nc}_{\mathbf{u}}, W^{\rm nc}_{\mathbf{u}}), \end{aligned}$$

which is equivalent to the identity

$$A+B := (\mathbf{S}^\gamma(0)W^{\rm nc}_{\mathbf{u}}(0), W^{\rm nc}_{\mathbf{u}}(0))_{L^2(\mathbb{R}^2)} + 2\int_{\mathbb{R}^+}\int_{\mathbb{R}^2} \mathrm{Re}\,(\mathbf{S}^\gamma T^\gamma_{\chi_2\mathbb{A}} W^{\rm nc}_{\mathbf{u}}, W^{\rm nc}_{\mathbf{u}})\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}z$$

$$= \sum_{k=1}^{5} J_k := -\int_{\mathbb{R}^+}\int_{\mathbb{R}^2} 2\mathrm{Re}\,(\mathbf{S}^\gamma T^\gamma_{\mathbb{E}} W^{\rm nc}_{\mathbf{u}}, W^{\rm nc}_{\mathbf{u}})\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}z$$
$$-\int_{\mathbb{R}^+}\int_{\mathbb{R}^2} 2\mathrm{Re}\,(\mathbf{S}^\gamma T^\gamma_r W, W^{\rm nc}_{\mathbf{u}})\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}z - \int_{\mathbb{R}^+}\int_{\mathbb{R}^2} 2\mathrm{Re}\,(\mathbf{S}^\gamma R_{-1} W, W^{\rm nc}_{\mathbf{u}})\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}z$$
$$-\int_{\mathbb{R}^+}\int_{\mathbb{R}^2} ((\partial_z\mathbf{S}^\gamma)W^{\rm nc}_{\mathbf{u}}, W^{\rm nc}_{\mathbf{u}})\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}z - \int_{\mathbb{R}^+}\int_{\mathbb{R}^2} 2\mathrm{Re}\,(\mathbf{S}^\gamma R_0 F, W^{\rm nc}_{\mathbf{u}})\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}z.$$

Upper bound of $A+B$. We now estimate each term on the above right-hand side. Let $\Lambda^{s,\gamma} = T^\gamma_{\lambda^{s,\gamma}}$, we may write $\mathbf{S}^\gamma = \Lambda^{1/2,\gamma}\Lambda^{-1/2,\gamma}\mathbf{S}^\gamma$. Note $\Lambda^{s,\gamma}$ is self-adjoint and $\Lambda^{-1/2,\gamma}\mathbf{S}^\gamma$ is of order $1/2$, we have, by the Cauchy–Schwartz inequality,

$$|J_1| \le C\,\|W^{\rm nc}_{\mathbf{u}}\|^2_{L^2(H_{1/2,\gamma})}, \qquad |J_4| \le C\,\|W^{\rm nc}_{\mathbf{u}}\|^2_{L^2(H_{1/2,\gamma})},$$
$$|J_2| \le C\,\|T^\gamma_r W\|_{L^2(H_{1/2,\gamma})}\,\|W^{\rm nc}_{\mathbf{u}}\|_{L^2(H_{1/2,\gamma})} \le \frac{C_\varepsilon}{\gamma}\,\|T^\gamma_r W\|^2_{L^2(H_{1/2,\gamma})} + \varepsilon\gamma\,\|W^{\rm nc}_{\mathbf{u}}\|^2_{L^2(H_{1/2,\gamma})},$$
$$|J_3| \le C\,\|W\|_{L^2(H_{-1/2,\gamma})}\,\|W^{\rm nc}_{\mathbf{u}}\|_{L^2(H_{1/2,\gamma})} \le \frac{C_\varepsilon}{\gamma^2}\,\|W\|^2_{L^2(L^2)} + \varepsilon\gamma\,\|W^{\rm nc}_{\mathbf{u}}\|^2_{L^2(H_{1/2,\gamma})},$$
$$|J_5| \le C\,\|F\|_{L^2(H_{1/2,\gamma})}\,\|W^{\rm nc}_{\mathbf{u}}\|_{L^2(H_{1/2,\gamma})} \le \frac{C_\varepsilon}{\gamma}\,\left\|\tilde{F}\right\|^2_{L^2(H_{1/2,\gamma})} + \varepsilon\gamma\,\|W^{\rm nc}_{\mathbf{u}}\|^2_{L^2(H_{1/2,\gamma})}.$$

It follows that

$$A+B \le (C+3\varepsilon\gamma)\,\|W^{\rm nc}_{\mathbf{u}}\|^2_{L^2(H_{1/2,\gamma})} + \frac{C_\varepsilon}{\gamma}\,\left\|\tilde{F}\right\|^2_{L^2(H_{1/2,\gamma})} + \frac{C_\varepsilon}{\gamma}\left(\frac{1}{\gamma}\,\|W\|^2_{L^2(L^2)} + \|T^\gamma_r W\|^2_{L^2(H_{1/2,\gamma})}\right). \tag{4.54}$$

We continue to derive a lower bound for the term $A+B$ by means of Gårding's inequalities. We first deal with A.

Lower bound of A. Since $\mathbf{S}^\gamma(0) - T^\gamma_{S(0)} = \frac{1}{2}((T^\gamma_{S(0)})^* - T^\gamma_{S(0)}) = \frac{1}{2}((T^\gamma_{S(0)})^* - T^\gamma_{(S(0))^*})$ is of order 0, we have

$$A = (T^\gamma_{S(0)} W^{\rm nc}_{\mathbf{u}}(0), W^{\rm nc}_{\mathbf{u}}(0))_{L^2} + O(1)\,\|W^{\rm nc}_{\mathbf{u}}(0)\|_{1/2,\gamma}\,\|W^{\rm nc}_{\mathbf{u}}(0)\|_{-1/2,\gamma}.$$

From (4.51), we apply the localized Gårding's inequality [10, Theorem A.4] to obtain, for $W^{\rm nc}_{\mathbf{u}}(0) \in H_{1/2,\gamma}$ that

$$\mathrm{Re}\,(T^\gamma_{S(0)+\lambda^{1,\gamma}\beta^*\beta} T^\gamma_{\chi_1} W^{\rm nc}_{\mathbf{u}}(0), T^\gamma_{\chi_1} W^{\rm nc}_{\mathbf{u}}(0))_{(H_{-1/2,\gamma}, H_{1/2,\gamma})}$$
$$\ge c\,\left\|T^\gamma_{\chi_1} W^{\rm nc}_{\mathbf{u}}(0)\right\|^2_{1/2,\gamma} - C\,\|W^{\rm nc}_{\mathbf{u}}(0)\|^2_{-1/2,\gamma}.$$

By the construction of χ_1, χ_2, we have $T^\gamma_{\chi_1} W^{\rm nc}_{\mathbf{u}} = W^{\rm nc}_{\mathbf{u}} + R_{-2}W^{\rm nc}$, so the right-hand side of the above inequality is larger than or equal to $c\,\|W^{\rm nc}_{\mathbf{u}}(0)\|^2_{1/2,\gamma} - C'\,\|W^{\rm nc}(0)\|^2_{-1/2,\gamma}$, while the left-hand side is smaller than or equal to

$$\mathrm{Re}\,(T^\gamma_{S(0)+\lambda^{1,\gamma}\beta^*\beta} W^{\rm nc}_{\mathbf{u}}(0), W^{\rm nc}_{\mathbf{u}}(0))_{(H_{-1/2,\gamma}, H_{1/2,\gamma})} + C\,\|W^{\rm nc}(0)\|^2_{-1/2,\gamma}.$$

Hence we obtain that

$$\mathrm{Re}\,(T^{\gamma}_{S(0)+\lambda^{1,\gamma}\beta^*\beta}W^{\mathrm{nc}}_{\mathbf{u}}(0),W^{\mathrm{nc}}_{\mathbf{u}}(0))_{(H_{-1/2,\gamma},H_{1/2,\gamma})}\geq c\left\|W^{\mathrm{nc}}_{\mathbf{u}}(0)\right\|^2_{1/2,\gamma}-C\left\|W^{\mathrm{nc}}(0)\right\|^2_{-1/2,\gamma}.$$

Observe that $T^{\gamma}_{\lambda^{1,\gamma}\beta^*\beta}-[\Lambda^{1/2,\gamma}T^{\gamma}_{\beta}]^*[\Lambda^{1/2,\gamma}T^{\gamma}_{\beta}]$ is of order 0, we have

$$\begin{aligned}&\mathrm{Re}\,(T^{\gamma}_{\lambda^{1,\gamma}\beta^*\beta}W^{\mathrm{nc}}_{\mathbf{u}}(0),W^{\mathrm{nc}}_{\mathbf{u}}(0))_{(H_{-1/2,\gamma},H_{1/2,\gamma})}\\ &\leq(\Lambda^{1/2,\gamma}T^{\gamma}_{\beta}W^{\mathrm{nc}}_{\mathbf{u}}(0),\Lambda^{1/2,\gamma}T^{\gamma}_{\beta}W^{\mathrm{nc}}_{\mathbf{u}}(0))_{(L^2,L^2)}+C\left\|W^{\mathrm{nc}}_{\mathbf{u}}(0)\right\|_{-1/2,\gamma}\left\|W^{\mathrm{nc}}_{\mathbf{u}}(0)\right\|_{1/2,\gamma}\\ &=\left\|\tilde{G}\right\|^2_{1/2,\gamma}+C\left\|W^{\mathrm{nc}}_{\mathbf{u}}(0)\right\|_{-1/2,\gamma}\left\|W^{\mathrm{nc}}_{\mathbf{u}}(0)\right\|_{1/2,\gamma}.\end{aligned}$$

Hence we discover that

$$\begin{aligned}A&\geq(T^{\gamma}_{S(0)}W^{\mathrm{nc}}_{\mathbf{u}}(0),W^{\mathrm{nc}}_{\mathbf{u}}(0))_{(H_{-1/2,\gamma},H_{1/2,\gamma})}-C\left\|W^{\mathrm{nc}}_{\mathbf{u}}(0)\right\|_{1/2,\gamma}\left\|W^{\mathrm{nc}}_{\mathbf{u}}(0)\right\|_{-1/2,\gamma}\\ &\geq c\left\|W^{\mathrm{nc}}_{\mathbf{u}}(0)\right\|^2_{1/2,\gamma}-C\left\|W^{\mathrm{nc}}(0)\right\|^2_{-1/2,\gamma}-\left\|\tilde{G}\right\|^2_{1/2,\gamma}-C\left\|W^{\mathrm{nc}}_{\mathbf{u}}(0)\right\|_{-1/2,\gamma}\left\|W^{\mathrm{nc}}_{\mathbf{u}}(0)\right\|_{1/2,\gamma}\\ &\geq c'\left\|W^{\mathrm{nc}}_{\mathbf{u}}(0)\right\|^2_{1/2,\gamma}-\frac{C}{\gamma}\left(\left\|W^{\mathrm{nc}}(0)\right\|^2_{L^2}+\left\|\tilde{G}\right\|^2_{1,\gamma}\right).\end{aligned}\tag{4.55}$$

Lower bound of B. We then consider the term B. Since $\mathrm{Re}\,\mathbf{S}^{\gamma}T^{\gamma}_{\chi_2\mathbb{A}}-T^{\gamma}_{\mathrm{Re}\,(\chi_2S\mathbb{A})}$ is of order one, we have

$$B\geq\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}2(T^{\gamma}_{\mathrm{Re}\,(\chi_2S\mathbb{A})}W^{\mathrm{nc}}_{\mathbf{u}},W^{\mathrm{nc}}_{\mathbf{u}})\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}z-C\left\|W^{\mathrm{nc}}_{\mathbf{u}}\right\|^2_{L^2(H_{1/2,\gamma})}.$$

For $X\in\Omega\times\Xi$, define

$$a_l(X)=\begin{pmatrix}\gamma H_l(X)&0\\0&E_l(X)\end{pmatrix}.\tag{4.56}$$

Recall by (4.53) we have

$$\chi_1^2\chi_2H_l(X)\geq c\chi_1^2\chi_2I_2=c\chi_1^2I_2,\quad\chi_1^2\chi_2E_l(X)\geq c\chi_1^2\chi_2\lambda^{1,\gamma}(\tau,\eta)I_2=c\chi_1^2\lambda^{1,\gamma}(\tau,\eta)I_2.$$

Note the remainder $T^{\gamma}_{V_l^*a_lV_l}-(T^{\gamma}_{V_l})^*T^{\gamma}_{a_l}T^{\gamma}_{V_l}$ is of order 1. Therefore it holds

$$\begin{aligned}&\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}2(T^{\gamma}_{\mathrm{Re}\,(\chi_2S\mathbb{A})}W^{\mathrm{nc}}_{\mathbf{u}},W^{\mathrm{nc}}_{\mathbf{u}})\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}z\\ &=\sum_l\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}2(T^{\gamma}_{V_l^*\chi_2a_lV_l}W^{\mathrm{nc}}_{\mathbf{u}},W^{\mathrm{nc}}_{\mathbf{u}})\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}z\\ &\geq\sum_l\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}2((T^{\gamma}_{V_l})^*T^{\gamma}_{\chi_2a_l}T^{\gamma}_{V_l}W^{\mathrm{nc}}_{\mathbf{u}},W^{\mathrm{nc}}_{\mathbf{u}})\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}z-C\left\|W^{\mathrm{nc}}_{\mathbf{u}}\right\|^2_{L^2(H_{1/2,\gamma})}\\ &=\sum_l\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}2(T^{\gamma}_{\chi_2a_l}W_l,W_l)\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}z-C\left\|W^{\mathrm{nc}}_{\mathbf{u}}\right\|^2_{L^2(H_{1/2,\gamma})},\end{aligned}$$

where for short we have introduced $W_l:=T^{\gamma}_{V_l}W^{\mathrm{nc}}_{\mathbf{u}}$. We may split $W_l=(W_l^1,W_l^2)$ according to the block structure of (4.56). So

$$\begin{aligned}&\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}2(T^{\gamma}_{\chi_2a_l}W_l,W_l)\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}z=I_1+I_2\qquad(4.57)\\ &=:\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}2(T^{\gamma}_{\chi_2\gamma H_l}W_l^1,W_l^1)\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}z+\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}2(T^{\gamma}_{\chi_2E_l}W_l^2,W_l^2)\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}z.\end{aligned}$$

Lower bound of I_1. We first show a lower bound of I_1. By applying the localized Gårding's inequality to $\chi_2 H_l$ (with $m=0$), we obtain

$$\mathrm{Re}\,(T^\gamma_{\chi_2 H_l} T^\gamma_{\chi_0} W_l^1, T^\gamma_{\chi_0} W_l^1)_{(L^2,L^2)} \geq c\left\|T^\gamma_{\chi_0} W_l^1\right\|_0^2 - C\left\|W_l^1\right\|_{-1,\gamma}^2. \tag{4.58}$$

Note that

$$\begin{aligned}
T^\gamma_{\chi_0} W_l^1 &= (T^\gamma_{\chi_0} T^\gamma_{V_l} W_{\mathbf{u}}^{\mathrm{nc}})^1 = (T^\gamma_{V_l} T^\gamma_{\chi_0} W_{\mathbf{u}}^{\mathrm{nc}})^1 + (R_0 W_{\mathbf{u}}^{\mathrm{nc}})^1 \\
&= (T^\gamma_{V_l} W_{\mathbf{u}}^{\mathrm{nc}})^1 + (R_{-1} W^{\mathrm{nc}})^1 + (R_0 W_{\mathbf{u}}^{\mathrm{nc}})^1 \\
&= W_l^1 + (R_{-1} W^{\mathrm{nc}})^1 + (R_0 W_{\mathbf{u}}^{\mathrm{nc}})^1,
\end{aligned} \tag{4.59}$$

so the right-hand side of (4.58)

$$\begin{aligned}
&\geq c\left\|W_l^1\right\|_0^2 - c\left\|R_{-1} W^{\mathrm{nc}}\right\|_0^2 - c\left\|R_0 W_{\mathbf{u}}^{\mathrm{nc}}\right\|_0^2 - \frac{C}{\gamma^2}\left\|W_l^1\right\|_0^2 \\
&\geq (c - \frac{C}{\gamma^2})\left\|W_l^1\right\|_0^2 - \frac{C}{\gamma^2}\left\|W^{\mathrm{nc}}\right\|_0^2 - C\left\|W_{\mathbf{u}}^{\mathrm{nc}}\right\|_0^2,
\end{aligned} \tag{4.60}$$

and the left-hand side of (4.58) is bounded by (recall $H_l \in \Gamma_2^0$)

$$\begin{aligned}
&\mathrm{Re}\,(T^\gamma_{\chi_2 H_l} W_l^1, W_l^1)_{(L^2,L^2)} + \left\|W_l^1\right\|_0 \left\|R_{-1} W^{\mathrm{nc}}\right\|_0 + \left\|W_l^1\right\|_0 \left\|W_{\mathbf{u}}^{\mathrm{nc}}\right\|_0 + \left\|R_{-1} W^{\mathrm{nc}}\right\|_0^2 \\
&\quad + \left\|R_{-1} W^{\mathrm{nc}}\right\|_0 \left\|W_{\mathbf{u}}^{\mathrm{nc}}\right\|_0 + \left\|W_{\mathbf{u}}^{\mathrm{nc}}\right\|_0^2 \\
&\leq \mathrm{Re}\,(T^\gamma_{\chi_2 H_l} W_l^1, W_l^1)_{(L^2,L^2)} + 2\epsilon\left\|W_l^1\right\|_0^2 + C\left\|W_{\mathbf{u}}^{\mathrm{nc}}\right\|_0^2 + C\left\|R_{-1} W^{\mathrm{nc}}\right\|_0^2 \\
&\leq \mathrm{Re}\,(T^\gamma_{\chi_2 H_l} W_l^1, W_l^1)_{(L^2,L^2)} + 2\epsilon\left\|W_l^1\right\|_0^2 + C\left\|W_{\mathbf{u}}^{\mathrm{nc}}\right\|_0^2 + \frac{C}{\gamma^2}\left\|W^{\mathrm{nc}}\right\|_0^2.
\end{aligned} \tag{4.61}$$

Therefore by taking $\varepsilon = c/4$, from (4.58), (4.60) and (4.61), we have

$$\mathrm{Re}\,(T^\gamma_{\chi_2 H_l} W_l^1, W_l^1)_{(L^2,L^2)} \geq (\frac{c}{2} - \frac{C}{\gamma^2})\left\|W_l^1\right\|_0^2 - \frac{C}{\gamma^2}\left\|W^{\mathrm{nc}}\right\|_0^2 - C\left\|W_{\mathbf{u}}^{\mathrm{nc}}\right\|_0^2.$$

Hence we obtain that,

$$\begin{aligned}
I_1 &\geq (c\gamma - \frac{C}{\gamma})\left\|W_l^1\right\|_{L^2(L^2)}^2 - \frac{C}{\gamma}\left\|W^{\mathrm{nc}}\right\|_{L^2(L^2)}^2 - C\gamma\left\|W_{\mathbf{u}}^{\mathrm{nc}}\right\|_{L^2(L^2)}^2 \\
&\geq (c\gamma - \frac{C}{\gamma})\left\|W_l^1\right\|_{L^2(L^2)}^2 - \frac{C}{\gamma}\left\|W^{\mathrm{nc}}\right\|_{L^2(L^2)}^2 - C\left\|W_{\mathbf{u}}^{\mathrm{nc}}\right\|_{L^2(H_{1/2,\gamma})}^2.
\end{aligned}$$

Lower bound of I_2. Next we show a lower bound of I_2 given in (4.57). Applying the localized Gårding's inequality to $\chi_2 E_l$ (with $m = 1/2$), we find that, for large γ, it holds

$$\mathrm{Re}\,(T^\gamma_{\chi_2 E_l} T^\gamma_{\chi_0} W_l^2, T^\gamma_{\chi_0} W_l^2)_{(H^{-\frac12},H^{\frac12})} \geq c\left\|T^\gamma_{\chi_0} W_l^2\right\|_{\frac12,\gamma}^2 - C\left\|W_l^2\right\|_{-\frac12,\gamma}^2.$$

From (4.59) (applied to W_l^2), we have a lower bound of the right-hand side, so

$$\begin{aligned}
&\mathrm{Re}\,(T^\gamma_{\chi_2 E_l} T^\gamma_{\chi_0} W_l^2, T^\gamma_{\chi_0} W_l^2)_{(H^{-\frac12},H^{\frac12})} \\
&\geq c\left\|W_l^2\right\|_{\frac12,\gamma}^2 - C\left\|W^{\mathrm{nc}}\right\|_{-\frac12,\gamma}^2 - C\left\|W_{\mathbf{u}}^{\mathrm{nc}}\right\|_{\frac12,\gamma}^2 - C\left\|W_l^2\right\|_{-\frac12,\gamma}^2.
\end{aligned} \tag{4.62}$$

It is also straightforward to check

$$\begin{aligned}&\mathrm{Re}\,(T^{\gamma}_{\chi_2 E_l} T^{\gamma}_{\chi_0} W_l^2, T^{\gamma}_{\chi_0} W_l^2)_{(H^{-\frac{1}{2}},H^{\frac{1}{2}})}\\ &\le \mathrm{Re}\,(T^{\gamma}_{\chi_2 E_l} W_l^2, W_l^2)_{(L^2,L^2)} + \varepsilon \left\|W_l^2\right\|^2_{\frac{1}{2},\gamma} + C\left\|W^{\mathrm{nc}}\right\|^2_{-\frac{1}{2},\gamma} + C\left\|W^{\mathrm{nc}}_{\mathbf{u}}\right\|^2_{\frac{1}{2},\gamma}. \quad (4.63)\end{aligned}$$

Hence by setting $\varepsilon = c/2$ in (4.63), from (4.62) and (4.63), we get

$$\begin{aligned}\mathrm{Re}\,(T^{\gamma}_{\chi_2 E_l} W_l^2, W_l^2)_{(L^2,L^2)} &\ge \frac{c}{2}\left\|W_l^2\right\|^2_{\frac{1}{2},\gamma} - C\left\|W^{\mathrm{nc}}\right\|^2_{-\frac{1}{2},\gamma} - C\left\|W^{\mathrm{nc}}_{\mathbf{u}}\right\|^2_{\frac{1}{2},\gamma} - C\left\|W_l^2\right\|^2_{-\frac{1}{2},\gamma}\\ &\ge (\frac{c}{2}\gamma - \frac{C}{\gamma})\left\|W_l^2\right\|^2_0 - \frac{C}{\gamma}\left\|W^{\mathrm{nc}}\right\|^2_0 - C\left\|W^{\mathrm{nc}}_{\mathbf{u}}\right\|^2_{1/2,\gamma}.\end{aligned}$$

After integrating with respect to z on $\mathbb{R}^+$, there follows

$$I_2 \ge (\frac{c}{2}\gamma - \frac{C}{\gamma})\left\|W_l^2\right\|^2_{L^2(L^2)} - \frac{C}{\gamma}\left\|W^{\mathrm{nc}}\right\|^2_{L^2(L^2)} - C\left\|W^{\mathrm{nc}}_{\mathbf{u}}\right\|^2_{L^2(H_{1/2,\gamma})}.$$

So up to now, summing up over all l (recall it is finite), we have

$$B \ge (c\gamma - \frac{C}{\gamma})\sum_l \left\|W_l\right\|^2_{L^2(L^2)} - \frac{C}{\gamma}\left\|W^{\mathrm{nc}}\right\|^2_{L^2(L^2)} - C\left\|W^{\mathrm{nc}}_{\mathbf{u}}\right\|^2_{L^2(H_{1/2,\gamma})}.$$

Here we used the fact $\|W_l\| \le \left\|W_l^1\right\| + \left\|W_l^2\right\|$. We need a lower bound of $\sum_l \|W_l\|^2_{L^2(L^2)}$.

Lower bound of $\sum_l \|W_l\|^2_{L^2(L^2)}$. Since the symbol $\sum_l V_l^* V_l$ is elliptic, it follows (by Gårding's inequality, [11, Theorem B.7]) that

$$\begin{aligned}&C_1\left\|W^{\mathrm{nc}}_{\mathbf{u}}\right\|^2_{L^2(H_{1/2,\gamma})} \le \sum_l \int_{\mathbb{R}^+}\int_{\mathbb{R}^2} \mathrm{Re}\,(T^{\gamma}_{V_l^* V_l} W^{\mathrm{nc}}_{\mathbf{u}}, W^{\mathrm{nc}}_{\mathbf{u}})\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &\le C_2\sum_l \|W_l\|^2_{L^2(L^2)} + \int_{\mathbb{R}^+}\int_{\mathbb{R}^2}(R_0 W^{\mathrm{nc}}_{\mathbf{u}}, W^{\mathrm{nc}}_{\mathbf{u}})\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &\le C_2\sum_l \|W_l\|^2_{L^2(L^2)} + C\left\|W^{\mathrm{nc}}_{\mathbf{u}}\right\|^2_{L^2(L^2)} \le C_2\sum_l \|W_l\|^2_{L^2(L^2)} + \frac{C}{\gamma}\left\|W^{\mathrm{nc}}_{\mathbf{u}}\right\|^2_{L^2(H_{1/2,\gamma})}.\end{aligned}$$

Therefore, by taking γ large, we have $\sum_l \|W_l\|^2_{L^2(L^2)} \ge C_0\left\|W^{\mathrm{nc}}_{\mathbf{u}}\right\|^2_{L^2(H_{1/2,\gamma})}$. So we get

$$B \ge C_0(c\gamma - \frac{C}{\gamma})\left\|W^{\mathrm{nc}}_{\mathbf{u}}\right\|^2_{L^2(H_{1/2,\gamma})} - \frac{C}{\gamma}\left\|W^{\mathrm{nc}}\right\|^2_{L^2(L^2)} - C\left\|W^{\mathrm{nc}}_{\mathbf{u}}\right\|^2_{L^2(H_{1/2,\gamma})}. \quad (4.64)$$

Conclusion: estimate for frequencies away from the poles. Combining the lower and upper bounds of $A + B$ we proved in (4.54), (4.55) and (4.64), there comes

$$\begin{aligned}&C_0(c\gamma - \frac{C}{\gamma})\left\|W^{\mathrm{nc}}_{\mathbf{u}}\right\|^2_{L^2(H_{1/2,\gamma})} + C'\left\|W^{\mathrm{nc}}_{\mathbf{u}}(0)\right\|^2_{1/2,\gamma}\\ &\le \frac{C}{\gamma}(\left\|W^{\mathrm{nc}}(0)\right\|^2_0 + \left\|\tilde{G}\right\|^2_{1,\gamma} + \left\|W^{\mathrm{nc}}\right\|^2_{L^2(L^2)}) + C\left\|W^{\mathrm{nc}}_{\mathbf{u}}\right\|^2_{L^2(H_{1/2,\gamma})}\\ &\quad +(C + 3\varepsilon\gamma)\left\|W^{\mathrm{nc}}_{\mathbf{u}}\right\|^2_{L^2(H_{1/2,\gamma})} + \frac{C}{\gamma}\left\|\tilde{F}\right\|^2_{L^2(H_{1/2,\gamma})}\\ &\quad + \frac{C}{\gamma}\left(\frac{1}{\gamma}\|W\|^2_{L^2(L^2)} + \left\|T^{\gamma}_r W\right\|^2_{L^2(H_{1/2,\gamma})}\right).\end{aligned}$$

First taking $\varepsilon = \frac{C_0 c}{6}$, and choosing a sufficiently large γ_0, then for $\gamma \geq \gamma_0$, it follows that

$$\begin{aligned} &\gamma \|W_{\mathbf{u}}^{\mathrm{nc}}\|_{L^2(H_{1/2,\gamma})}^2 + \|W_{\mathbf{u}}^{\mathrm{nc}}(0)\|_{1/2,\gamma}^2 \\ &\leq \frac{C}{\gamma}\left(\|W^{\mathrm{nc}}(0)\|_0^2 + \left\|\tilde{G}\right\|_{1,\gamma}^2 + \frac{1}{\gamma}\left\|\tilde{F}\right\|_{L^2(H_{1,\gamma})}^2 + \|W\|_{L^2(L^2)}^2 + \|T_r^\gamma W\|_{L^2(H_{1/2,\gamma})}^2 \right). \end{aligned}$$

This is exactly (4.50).

4.7.3. Construction of symmetrizers. We now indicate how to construct the symmetrizers claimed in Proposition 4.5.

Recall we have constructed local Kreiss' symmetrizers r_j of constant coefficient problems near a frequency point in section 3.3.1. From the process we infer that there is a pair (r_j, T_j) with the properties (3.11) and (3.10) (with $\mathcal{B}$ replaced by $\mathbb{A}$ now) given in section 3.3.1 in a neighborhood $\mathcal{V}_j$ of $X_j \in (\partial\Omega \times \Sigma) \cap \operatorname{supp}\chi_2$, with $\Omega = \mathbb{R}_{x,y}^2 \times \mathbb{R}_z^+$, and $\Sigma = \{(\tau, \eta) \in \mathbb{C} \times \mathbb{R} : \operatorname{Re}\tau \geq 0, |\tau|^2 + |\eta|^2 = 1\}$. For a neighborhood $\mathcal{V}_j$ of $X_j \in ((\bar{\Omega} \setminus \partial\Omega) \times \Sigma) \cap \operatorname{supp}\chi_2$, the pair (r_j, T_j) satisfies (3.10).

We then define the symmetrizers of order zero on $\mathcal{V}_j$:

$$S_j^0 = T_j^* r_j T_j,$$

and it follows that $\operatorname{Re}(S_j^0\mathbb{A}) = T_j^* \operatorname{Re}(r_j T_j \mathbb{A} T_j^{-1}) T_j$. For cases a) and b1), we see $\operatorname{Re}(r_j T_j \mathbb{A} T_j^{-1})$ is of the diagonal form E_j that is a diagonal 2×2 matrix with order one satisfying $E_j \geq cI_2$; for cases b2) and b3), $\operatorname{Re}(r_j T_j \mathbb{A} T_j^{-1}) = \gamma H_j$ with H_j a 2×2 matrix of order 0 so that $H_j \geq cI_2$. Therefore, for all these cases we may take $V_j = \lambda^{1/2,\gamma} T_j$ so that each term $\operatorname{Re}(\lambda^{1,\gamma} S_j^0 \mathbb{A})$ satisfying the third requirement in the Proposition 4.5. One easily checks that the second requirement there is guaranteed by (3.11).

Since we assume that U, Ψ in the coefficients of the linearized problem are constant outside a compact set of $\bar{\Omega}$ (recall the initial perturbations have compact set with respect to (y, z) and we consider the case that x is only contained in a finite interval later), we just construct symmetrizers in finite (say, N) open set $\mathcal{V}_j$ for $j = 1, \cdots, N$. Then by a partition of unity $1 = \sum_{j=1}^n \psi_j(X)$ with each $\psi_j \in \Gamma_k^0$ $(k \in \mathbb{N})$, we may define the "global" symmetrizer S on $(\bar{\Omega} \times \Sigma) \cap \operatorname{supp}\chi_2$ to be

$$S(X) = \sum_{l=1}^N \psi_l(X) \lambda^{1,\gamma} S_l^0.$$

Note each S_j^0 and r_j, T_j are homogeneous of degree 0 for (τ, η) when extended to $\bar{\Omega} \times \Xi$, while V_j and $\mathbb{A}$ are extended by degrees 1/2 and 1 respectively. Proposition 4.5 is the proved.

We remark the construction of symmetrizers S for more general hyperbolic problems are given in Theorem 5.1 of [2, p.144]. The S we need here is actually $-\Sigma$ given in that theorem.

4.8. Derivation of energy estimate: the case of poles.

4.8.1. Derivation of equations. We now turn to the energy estimate near the poles where the uniform Kreiss–Lopatinskii conditions fails and we also can not use the reduced equations like (4.48). As in [11], the strategy is to consider the original system after some kind of diagonalization and then estimate each unknown directly.

Diagonalization of symbols. To derive the equations, we first carry out some algebraic manipulations to transform the matrix $\tau\mathbf{A}_1+i\eta\mathbf{A}_2$ defined by (4.23) to an almost diagonal form. To simplify the computations, we introduce here some notations. We set

$$\mathcal{A}=\begin{pmatrix}\omega I_2 & M\\ N & \Lambda\end{pmatrix},\quad M=\begin{pmatrix}-\alpha_1 & \alpha_1\\ \beta_1 & -\beta_1\end{pmatrix},\quad N=\begin{pmatrix}\mu\tau & i\eta\mu\\ \mu\tau & i\eta\mu\end{pmatrix},$$

and $\Lambda=\mathrm{diag}\{-a(\omega+\theta),a(\omega-\theta)\}$. We wish to find two matrices L and R defined near the set of poles $\Upsilon_{\mathbf{p}}$ so that

1) $L\,\mathrm{diag}\{0,0,1,1\}R=\mathrm{diag}\{0,0,1,1\}$,

2) $L\mathcal{A}R$ takes the form $\begin{pmatrix}\omega & 0 & 0 & *\\ 0 & \omega & 0 & *\\ * & * & e' & 0\\ 0 & 0 & 0 & e\end{pmatrix}$, and the argument e (resp. e') satisfies $\mathrm{Re}\,e<0$ (resp. $\mathrm{Re}\,e'>0$) near $\Upsilon_{\mathbf{p}}\cap\{|\tau|^2+\eta^2=1\}$.

To this end, we suppose $L=\begin{pmatrix}I_2 & 0\\ L_1 & L_2\end{pmatrix}$ and $R=\begin{pmatrix}I_2 & R_1\\ 0 & R_2\end{pmatrix}$. Then the first requirement above is equivalent to

$$L_2R_2=I_2. \tag{4.65}$$

This also implies that both L and R are invertible. By a direct computation, we have

$$L\mathcal{A}R=\begin{pmatrix}\omega I_2 & \omega R_1+MR_2\\ \omega L_1+L_2N & (\omega L_1+L_2N)R_1+(L_1M+L_2\Lambda)R_2\end{pmatrix}.$$

Now suppose specifically that,

$$L_1=\begin{pmatrix}0 & 0\\ l_3 & l_4\end{pmatrix},\quad L_2=\begin{pmatrix}0 & 1\\ 1 & m_4\end{pmatrix},$$

$$R_1=\begin{pmatrix}r_1 & 0\\ r_3 & 0\end{pmatrix},\quad R_2=\begin{pmatrix}s_1 & 1\\ 1 & 0\end{pmatrix},$$

then we get

$$\omega L_1+L_2N=\begin{pmatrix}\mu\tau & i\eta\mu\\ \underbrace{l_3\omega+(1+m_4)\mu\tau}_{\mathrm{a}_1} & \underbrace{l_4\omega+(1+m_4)i\eta\mu}_{\mathrm{a}_2}\end{pmatrix},$$

$$\omega R_1+MR_2=\begin{pmatrix}\underbrace{r_1\omega+(-s_1+1)\alpha_1}_{\mathbb{b}_1} & -\alpha_1\\ \underbrace{r_3\omega+(s_1-1)\beta_1}_{\mathbb{b}_2} & \beta_1\end{pmatrix}.$$

By the above second requirement, all $\mathrm{a}_1,\mathrm{a}_2,\mathbb{b}_1,\mathbb{b}_2$ should be zero. These imply that

$$m_4=-1-\frac{l_4\omega}{i\eta\mu},\quad l_3=\frac{l_4\tau}{i\eta}, \tag{4.66}$$

$$s_1=1-\frac{r_3\omega}{\beta_1},\quad r_1=-\frac{r_3\alpha_1}{\beta_1}, \tag{4.67}$$

by noting here that η and β_1 are nonzero near the poles. Also, (4.65) means

$$s_1 = -m_4 \quad \text{or} \quad r_3 = -\frac{\beta_1}{i\eta\mu} l_4. \tag{4.68}$$

Using these relations, we further have

$$\begin{aligned}&(\omega L_1 + L_2 N)R_1 + (L_1 M + L_2\Lambda)R_2\\ =& \begin{pmatrix} \underbrace{\mu\tau r_1 + i\eta\mu r_3 + a(\omega-\theta)}_{e'} & 0 \\ \underbrace{(\alpha_1 l_3 - \beta_1 l_4)(1-s_1) + 2am_4\omega}_{\mathbb{c}} & \underbrace{-(\alpha_1 l_3 - \beta_1 l_4) - a(\omega+\theta)}_{e} \end{pmatrix}.\end{aligned}$$

Requirement 2) above also implies $\mathbb{c} = 0$. So it follows the equation

$$m_4 = \frac{1}{2a\mu}\left(\frac{l_4}{i\eta}\right)^2 (\alpha_1\tau - i\eta\beta_1). \tag{4.69}$$

We should find $m_4, s_1, l_3, l_4, r_1, r_3$ so that (4.66)–(4.69) hold, and, near the poles,

$$(\mathrm{Re}\, e)/\sqrt{|\tau|^2+\eta^2} := \mathrm{Re}\left([-\frac{l_4}{i\eta}(\alpha_1\tau - i\eta\beta_1) - a(\omega+\theta)]/\sqrt{|\tau|^2+\eta^2}\right) < -c_0 < 0 \tag{4.70}$$

for some constant c_0.

From (4.66) and (4.69), we see $\zeta = l_4/i\eta$ should solve

$$\zeta^2(\alpha_1\tau - i\eta\beta_1) + 2a\zeta\omega + 2a\mu = 0.$$

For any fixed point (x, y, z, τ, η) so that $\omega = 0$ and the reference state $\Psi = z$, we can solve that $\zeta = \pm\frac{u^2}{c\sqrt{u^2+v^2}}\frac{1}{|\eta|}$. If we choose $l_4 = -i\frac{u^2}{c\sqrt{u^2+v^2}}\mathrm{sign}(\eta)$, then $(\mathrm{Re}\, e)/\sqrt{|\tau|^2+\eta^2} = -1 < 0$. By continuity, we see there is a neighborhood of $\Upsilon_{\mathbf{p}}$ and ν small, so that one could solve l_4 and (4.70) holds. Once l_4 is known, all other parameters are solved. We then obtain

$$L\mathcal{A}R = \begin{pmatrix} \omega & 0 & 0 & -\alpha_1 \\ 0 & \omega & 0 & \beta_1 \\ \mu\tau & i\eta\mu & \underbrace{-e-2a\theta}_{e'} & 0 \\ 0 & 0 & 0 & e \end{pmatrix}.$$

It is easily noted that there is a constant c_0' so that $(\mathrm{Re}\, e')/\sqrt{|\tau|^2+\eta^2} > c_0' > 0$ near the poles set $\Upsilon_{\mathbf{p}}$.

Finally, we introduce $\mathbf{L} = \mathrm{diag}\{1, L\}$, $\mathbf{R} = \mathrm{diag}\{1, R\}$, and

$$\mathbf{A}^d := \mathbf{L}(\tau\mathbf{A}_1 + i\eta\mathbf{A}_2)\mathbf{R} = \mathrm{diag}\{1, L\mathcal{A}R\}.$$

We remark that $\mathbf{L}$ and $\mathbf{R}$ constructed here, as well as their inverses, may be considered as symbols of class Γ_2^0, and $\mathbf{A}^d \in \Gamma_2^1$.

Derivation of equations near poles. Similar to the beginning of section 4.7.1, set $W_{\mathbf{p}} = T^{\gamma}_{\chi_{\mathbf{p}}} W$. Then it solves

$$\mathcal{I}_5 \partial_z W_{\mathbf{p}} + T^{\gamma}_{\tau \mathbf{A}_1 + i\eta \mathbf{A}_2} W_{\mathbf{p}} + T^{\gamma}_{\mathbf{C}} W_{\mathbf{p}} + T^{\gamma}_{r'} W = R_{-1} W + T^{\gamma}_{\chi_{\mathbf{p}}} F, \tag{4.71}$$

with

$$r' = i\{\tau \mathbf{A}_1 + i\eta \mathbf{A}_2, \chi_{\mathbf{p}}\} - \partial_z \chi_{\mathbf{p}} \mathcal{I}_5 \in \Gamma^0_1,$$

which vanishes in a neighborhood of the poles set $\Upsilon_{\mathbf{p}}$ and out of $\mathcal{V}_{\mathbf{p}}$. As before, we also need some cut-off functions ζ_j, $j = 1, 2, 3, 4$ such that

- ζ_j are C^∞ functions, taking values in $[0, 1]$, and homogeneous of degree 0 with respect to (τ, η);
- $\zeta_1|_{\operatorname{supp} \chi_{\mathbf{p}}} \equiv 1$, and for $k = 2, 3, 4$, there hold $\zeta_j|_{\operatorname{supp} \zeta_{j-1}} \equiv 1$;
- ζ_4 (and therefore $\zeta_j, j = 1, 2, 3$) is supported in a neighborhood of the set of poles $\Upsilon_{\mathbf{p}}$.

Then we introduce

$$V = T^{\gamma}_{\zeta_1 \mathbf{R}^{-1}} W_{\mathbf{p}}. \tag{4.72}$$

From (4.72) and the equation (4.71), we find

$$\begin{aligned}
\mathcal{I}_5 \partial_z V &= \mathcal{I}_5 T^{\gamma}_{\partial_z \zeta_1 \mathbf{R}^{-1}} W_{\mathbf{p}} + \mathcal{I}_5 T^{\gamma}_{\zeta_1 \partial_z \mathbf{R}^{-1}} W_{\mathbf{p}} + T^{\gamma}_{\zeta_1 \mathcal{I}_5 \mathbf{R}^{-1}} \partial_z W_{\mathbf{p}} \qquad (\text{using } \ \mathcal{I}_5 \mathbf{R}^{-1} = \mathbf{L} \mathcal{I}_5) \\
&= \mathcal{I}_5 T^{\gamma}_{\partial_z \zeta_1 \mathbf{R}^{-1}} W_{\mathbf{p}} + \mathcal{I}_5 T^{\gamma}_{\zeta_1 \partial_z \mathbf{R}^{-1}} W_{\mathbf{p}} + T^{\gamma}_{\zeta_1 \mathbf{L}} (\mathcal{I}_5 \partial_z W_{\mathbf{p}}) \\
&= \mathcal{I}_5 T^{\gamma}_{\partial_z \zeta_1 \mathbf{R}^{-1}} W_{\mathbf{p}} + \mathcal{I}_5 T^{\gamma}_{\zeta_1 \partial_z \mathbf{R}^{-1}} W_{\mathbf{p}} + T^{\gamma}_{\zeta_1 \mathbf{L}} (R_{-1} W + T^{\gamma}_{\chi_{\mathbf{p}}} \tilde{F}) \\
&\quad - T^{\gamma}_{\zeta_1 \mathbf{L}} T^{\gamma}_{\tau \mathbf{A}_1 + i\eta \mathbf{A}_2} W_{\mathbf{p}} - T^{\gamma}_{\zeta_1 \mathbf{L}} T^{\gamma}_{\mathbf{C}} W_{\mathbf{p}} - T^{\gamma}_{\zeta_1 \mathbf{L}} T^{\gamma}_{r'} W.
\end{aligned}$$

By symbolic calculus, note $\zeta_2 \zeta_1 = \zeta_1$,

$$\begin{aligned}
& T^{\gamma}_{\zeta_1 \mathbf{L}} T^{\gamma}_{\tau \mathbf{A}_1 + i\eta \mathbf{A}_2} W_{\mathbf{p}} \\
&= T^{\gamma}_{\zeta_1 \mathbf{L}(\tau \mathbf{A}_1 + i\eta \mathbf{A}_2)} W_{\mathbf{p}} + R_{-1} W + T^{\gamma}_{-i[\partial_\delta(\zeta_1 \mathbf{L}) \partial_x (\tau \mathbf{A}_1 + i\eta \mathbf{A}_2) + \partial_\eta (\zeta_1 \mathbf{L}) \partial_y (\tau \mathbf{A}_1 + i\eta \mathbf{A}_2)]} W_{\mathbf{p}} \\
&= T^{\gamma}_{\zeta_2 \mathbf{A}^d (\zeta_1 \mathbf{R}^{-1})} W_{\mathbf{p}} + R_{-1} W + T^{\gamma}_{-i[\partial_\delta(\zeta_1 \mathbf{L}) \partial_x (\tau \mathbf{A}_1 + i\eta \mathbf{A}_2) + \partial_\eta (\zeta_1 \mathbf{L}) \partial_y (\tau \mathbf{A}_1 + i\eta \mathbf{A}_2)]} W_{\mathbf{p}} \\
&= T^{\gamma}_{\zeta_2 \mathbf{A}^d} V - T^{\gamma}_{-i[\partial_\xi \mathbf{A}^d \partial_x (\zeta_1 \mathbf{R}^{-1}) + \partial_\eta \mathbf{A}^d \partial_y (\zeta_1 \mathbf{R}^{-1})]} W_{\mathbf{p}} + R_{-1} W \\
&\quad + T^{\gamma}_{-i[\partial_\delta(\zeta_1 \mathbf{L}) \partial_x (\tau \mathbf{A}_1 + i\eta \mathbf{A}_2) + \partial_\eta (\zeta_1 \mathbf{L}) \partial_y (\tau \mathbf{A}_1 + i\eta \mathbf{A}_2)]} W_{\mathbf{p}} \\
&= T^{\gamma}_{\zeta_2 \mathbf{A}^d} V + T^{\gamma}_{s_1'} W_{\mathbf{p}} + T^{\gamma}_{s_2'} W_{\mathbf{p}} + R_{-1} W.
\end{aligned}$$

Here we have set

$$\begin{aligned}
s_1' &= -i[\partial_\delta(\zeta_1) \mathbf{L} \partial_x (\tau \mathbf{A}_1 + i\eta \mathbf{A}_2) + \partial_\eta (\zeta_1) \mathbf{L} \partial_y (\tau \mathbf{A}_1 + i\eta \mathbf{A}_2)] \\
&\quad + i[\partial_\xi \mathbf{A}^d \partial_x (\zeta_1) \mathbf{R}^{-1} + \partial_\eta \mathbf{A}^d \partial_y (\zeta_1) \mathbf{R}^{-1}], \\
s_2' &= -i\zeta_1 [\partial_\delta(\mathbf{L}) \partial_x (\tau \mathbf{A}_1 + i\eta \mathbf{A}_2) + \partial_\eta (\mathbf{L}) \partial_y (\tau \mathbf{A}_1 + i\eta \mathbf{A}_2)] \\
&\quad + i\zeta_1 [\partial_\xi \mathbf{A}^d \partial_x (\mathbf{R}^{-1}) + \partial_\eta \mathbf{A}^d \partial_y (\mathbf{R}^{-1})] := \zeta_1 S
\end{aligned}$$

belonging to Γ^0_1. So s_1' vanishes on support of ζ_1 and out side $\mathcal{V}_{\mathbf{p}}$. We also have

$$T^{\gamma}_{s_2'} W_{\mathbf{p}} = T^{\gamma}_{\zeta_2 S \mathbf{R} (\zeta_1 \mathbf{R}^{-1})} W_{\mathbf{p}} = T^{\gamma}_{\zeta_2 S \mathbf{R}} V + R_{-1} W, \qquad \zeta_2 S \mathbf{R} \in \Gamma^0_1.$$

Similarly, we get

$$\begin{aligned} T^\gamma_{\zeta_1\mathbf{L}}T^\gamma_{\mathbf{C}}W_{\mathbf{p}} &= T^\gamma_{(\zeta_2\mathbf{LCR})(\zeta_1\mathbf{R}^{-1})}W_{\mathbf{p}} + R_{-1}W \\ &= T^\gamma_{\zeta_2\mathbf{LCR}}V + R_{-1}W, \qquad \zeta_2\mathbf{LCR}\in\Gamma^0_1, \\ T^\gamma_{\zeta_1\partial_z\mathbf{R}^{-1}}W_{\mathbf{p}} &= T^\gamma_{\zeta_2\mathbf{R}\partial_z\mathbf{R}}V + R_{-1}W, \quad \zeta_2\mathbf{R}\partial_z\mathbf{R}\in\Gamma^0_1, \end{aligned}$$

and

$$T^\gamma_{\zeta_1\mathbf{L}}T^\gamma_{r'}W = T^\gamma_{\zeta_1\mathbf{L}r'}W + R_{-1}W.$$

So in all, we find that V solves

$$\mathcal{I}_5\partial_z V + T^\gamma_{\zeta_2\mathbf{A}^d}V + T^\gamma_{\zeta_2\mathbf{E}}V + T^\gamma_{\tilde r}W + R_{-1}W = T^\gamma_{\zeta_1\mathbf{L}}T^\gamma_{\chi_{\mathbf{p}}}\tilde F. \tag{4.73}$$

Here $\zeta_2\mathbf{E},\ \tilde r\in\Gamma^0_1$, and $\tilde r$ also vanishes in a neighborhood of $\Upsilon_{\mathbf{p}}$ and outside $\mathcal{V}_{\mathbf{p}}$; R_{-1} is an operator of order -1.

Finally, we write down the explicit form of (4.73). There are two "differential" equations

$$\partial_z V_5 + T^\gamma_{\zeta_2 e}V_5 + T^\gamma_{\alpha^0}V = \mathcal{F}_5, \tag{4.74}$$

$$\partial_z V_4 + T^\gamma_{\zeta_2 e'}V_4 + T^\gamma_{\zeta_2\mu\tau}V_2 + T^\gamma_{i\zeta_2\mu\eta}V_3 + T^\gamma_{\alpha^0}V = \mathcal{F}_4, \tag{4.75}$$

and three "algebraic" equations

$$T^\gamma_{\zeta_2\omega}V_1 + T^\gamma_{\alpha^0}V = \mathcal{F}_1, \tag{4.76}$$

$$T^\gamma_{\zeta_2\omega}V_2 + T^\gamma_{-\zeta_2\alpha_1}V_5 + T^\gamma_{\alpha^0}V = \mathcal{F}_2, \tag{4.77}$$

$$T^\gamma_{\zeta_2\omega}V_3 + T^\gamma_{\zeta_2\beta_1}V_5 + T^\gamma_{\alpha^0}V = \mathcal{F}_3, \tag{4.78}$$

with the lower order terms being coupled. Here for shortness, we set

$$\mathcal{F} = (\mathcal{F}_1,\cdots,\mathcal{F}_5)^T = -T^\gamma_{\tilde r}W + R_{-1}W + T^\gamma_{\zeta_1\mathbf{L}}T^\gamma_{\chi_{\mathbf{p}}}\tilde F, \tag{4.79}$$

and α^m is a generic symbol of class Γ^m_1 for an integer m.

4.8.2. Energy estimates. We now derive energy estimates of solutions to the equations (4.74)–(4.78).

Estimate on V_5. Let us start from V_5. Using (4.74), it holds

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}z}(\Lambda^{1,\gamma}V_5,\Lambda^{1,\gamma}V_5) = 2\mathrm{Re}\,(\Lambda^{1,\gamma}\partial_z V_5,\Lambda^{1,\gamma}V_5) \\ &= I + J := \Big[2\mathrm{Re}\,(T^\gamma_{-\zeta_2 e\lambda^{2,\gamma}}V_5, V_5)\Big] \\ &\quad + \Big[(R_2V_5,V_5) - 2\mathrm{Re}\,(\Lambda^{1,\gamma}T^\gamma_{\alpha^0}V,\Lambda^{1,\gamma}V_5) + 2\mathrm{Re}\,(\Lambda^{1,\gamma}\mathcal{F}_5,\Lambda^{1,\gamma}V_5)\Big]. \end{aligned} \tag{4.80}$$

Just using the Cauchy–Schwartz inequality, it is easy to obtain that

$$|J| \le C(\|V_5\|^2_{1,\gamma} + \varepsilon\,\|V_5\|^2_{3/2,\gamma} + C_\varepsilon\,\|V\|^2_{1/2,\gamma} + C_\varepsilon\,\|\mathcal{F}\|^2_{1/2,\gamma} + \varepsilon\,\|V_5\|^2_{3/2,\gamma}). \tag{4.81}$$

To handle I, we still need the localized Gårding's inequality ([11, Theorem B.8]). Recall we have shown that there is a positive constant c so that $-(\mathrm{Re}\,e)/\lambda^{1,\gamma} > c$ in the support of ζ_4. Then we have

$$-\zeta_3^2\zeta_2\lambda^{2,\gamma}\mathrm{Re}\,e \ge c\zeta_3^2\zeta_2\lambda^{3.\gamma}.$$

So there is a constant $C > 0$ and $\gamma_0 \geq 1$ such that for all $\gamma \geq \gamma_0$ and smooth V_5, it holds

$$\mathrm{Re}\,(T^\gamma_{-e\zeta_2\lambda^2,\gamma} T^\gamma_{\zeta_3} V_5, T^\gamma_{\zeta_3} V_5) \geq \frac{c}{2} \left\| T^\gamma_{\zeta_3} V_5 \right\|^2_{\frac{3}{2},\gamma} - C \left\| V_5 \right\|^2_{\frac{1}{2},\gamma}. \tag{4.82}$$

Note that $T^\gamma_{\zeta_3} V_5 = V_5 + R_{-2} W_{\mathbf{p}}$, we see

$$\begin{aligned} &\frac{c}{2} \left\| T^\gamma_{\zeta_3} V_5 \right\|^2_{\frac{3}{2},\gamma} - C \left\| V_5 \right\|^2_{\frac{1}{2},\gamma} \geq C_1 \left\| V_5 \right\|^2_{3/2,\gamma} - C_2 \left\| W_{\mathbf{p}} \right\|^2_{-1/2,\gamma} - C \left\| V_5 \right\|^2_{\frac{1}{2},\gamma} \\ \geq &\frac{C_1}{2} \left\| V_5 \right\|^2_{3/2,\gamma} + \frac{C_1}{2} \gamma \left\| V_5 \right\|^2_{1,\gamma} - C_2 \frac{1}{\gamma} \left\| W_{\mathbf{p}} \right\|^2_{L^2} - C \frac{1}{\gamma} \left\| V_5 \right\|^2_{1,\gamma}, \end{aligned} \tag{4.83}$$

and

$$\begin{aligned} &\mathrm{Re}\,(T^\gamma_{-e\zeta_2\lambda^2,\gamma} T^\gamma_{\zeta_3} V_5, T^\gamma_{\zeta_3} V_5) \\ \leq &\mathrm{Re}\,(T^\gamma_{-e\zeta_2\lambda^2,\gamma} V_5, V_5) + C \left\| W_{\mathbf{p}} \right\|_{L^2} \left\| V_5 \right\|_{1,\gamma} + \left\| W_{\mathbf{p}} \right\|_{L^2} \left\| W_{\mathbf{p}} \right\|_{-1,\gamma} \\ \leq &\mathrm{Re}\,(T^\gamma_{-e\zeta_2\lambda^2,\gamma} V_5, V_5) + C \frac{1}{\gamma} \left\| W_{\mathbf{p}} \right\|^2_{L^2} + \frac{C_1 \gamma}{2} \left\| V_5 \right\|^2_{1,\gamma}. \end{aligned} \tag{4.84}$$

Therefore, combining (4.82), (4.83) with (4.84), and choosing γ_0 further large, we find

$$I \geq C_1 (\gamma \left\| V_5 \right\|^2_{1,\gamma} + \left\| V_5 \right\|^2_{3/2,\gamma}) - C_2 \frac{1}{\gamma} \left\| W_{\mathbf{p}} \right\|^2_{L^2}. \tag{4.85}$$

Now by plugging (4.81) and (4.85) into (4.80), and integrating over $z \in (0, \infty)$, and remember by our assumption that V should vanish for $z \to \infty$, we get

$$\begin{aligned} &\left\| V_5(0) \right\|^2_{1,\gamma} + C_1 \gamma \left\| V_5 \right\|^2_{L^2(H_{1,\gamma})} + C_1 \left\| V_5 \right\|^2_{L^2(H_{3/2,\gamma})} \\ \leq &\frac{C}{\gamma} \left\| W_{\mathbf{p}} \right\|^2_{L^2(L^2)} + C (\left\| V \right\|^2_{L^2(H_{1/2,\gamma})} + \left\| \mathcal{F} \right\|^2_{L^2(H_{1/2,\gamma})}). \end{aligned} \tag{4.86}$$

Estimate on V_1. Next we consider estimate of V_1. Acting $\Lambda^{1/2,\gamma}$ to equation (4.76) and then taking $L^2(\mathbb{R}^2)$ inner product with $\Lambda^{1/2,\gamma} V_1$, we get, for the real part, that

$$\begin{aligned} &\mathrm{Re}\,(\Lambda^{1/2,\gamma} T^\gamma_{\zeta_2\omega} V_1, \Lambda^{1/2,\gamma} V_1) \\ = &-\mathrm{Re}\,(\Lambda^{1/2,\gamma} T^\gamma_{\alpha^0} V, \Lambda^{1/2,\gamma} V_1) + \mathrm{Re}\,(\Lambda^{1/2,\gamma} \mathcal{F}_1, \Lambda^{1/2,\gamma} V_1). \end{aligned} \tag{4.87}$$

For the right-hand side, it is easy to be bounded by

$$C' (\left\| V \right\|^2_{1/2,\gamma} + \varepsilon \gamma \left\| V_1 \right\|^2_{1/2,\gamma} + \frac{C_\varepsilon}{\gamma} \left\| \mathcal{F} \right\|^2_{1/2,\gamma}). \tag{4.88}$$

For the left-hand side of (4.87), we have

$$\begin{aligned}
&\mathrm{Re}\,(\Lambda^{1/2,\gamma}T^{\gamma}_{\zeta_2\omega}V_1,\Lambda^{1/2,\gamma}V_1)\\
=&\,\mathrm{Re}\,(T^{\gamma}_{\zeta_2\omega}\Lambda^{1/2,\gamma}V_1,\Lambda^{1/2,\gamma}V_1)+\mathrm{Re}\,(R_{1/2}V_1,\Lambda^{1/2,\gamma}V_1)\\
=&\,\frac{1}{2}((T^{\gamma}_{\zeta_2\omega}+(T^{\gamma}_{\zeta_2\omega})^*)\Lambda^{1/2,\gamma}V_1,\Lambda^{1/2,\gamma}V_1)+\mathrm{Re}\,(R_{1/2}V_1,\Lambda^{1/2,\gamma}V_1)\\
=&\,\frac{1}{2}(T^{\gamma}_{\zeta_2(\omega+\bar\omega)}\Lambda^{1/2,\gamma}V_1,\Lambda^{1/2,\gamma}V_1)+\mathrm{Re}\,(R_{1/2}V_1,\Lambda^{1/2,\gamma}V_1)\\
&+(R_0\Lambda^{1/2,\gamma}V_1,\Lambda^{1/2,\gamma}V_1)\\
=&\,\gamma(T^{\gamma}_{\zeta_2}\Lambda^{1/2,\gamma}V_1,\Lambda^{1/2,\gamma}V_1)+\mathrm{Re}\,(R'_{1/2}V_1,\Lambda^{1/2,\gamma}V_1)\\
=&\,\gamma(\Lambda^{1/2,\gamma}(V_1+R_{-2}W_{\mathbf{p}}),\Lambda^{1/2,\gamma}V_1)+\mathrm{Re}\,(R''_{1/2}V_1,\Lambda^{1/2,\gamma}V_1)\\
\geq&\,(\gamma-C)\,\|V_1\|^2_{1/2,\gamma}-\gamma|(R_{-3/2}W_{\mathbf{p}},\Lambda^{1/2,\gamma}V_1)|\\
\geq&\,(\gamma-C)\,\|V_1\|^2_{1/2,\gamma}-\gamma\,\|W_{\mathbf{p}}\|_{-3/2,\gamma}\,\|V_1\|_{1/2,\gamma}\\
\geq&\,(\gamma-C)\,\|V_1\|^2_{1/2,\gamma}-\frac{1}{\sqrt{\gamma}}\,\|W_{\mathbf{p}}\|_{L^2}\,\|V_1\|_{1/2,\gamma}\\
\geq&\,(\gamma-C')\,\|V_1\|^2_{1/2,\gamma}-\frac{C}{\gamma}\,\|W_{\mathbf{p}}\|^2_{L^2}\,.
\end{aligned}\tag{4.89}$$

From (4.87), (4.88) and (4.89), by taking ε small and $\gamma_0\geq 1$ rather large, we obtain for $\gamma\geq\gamma_0$ that

$$\gamma\,\|V_1\|^2_{1/2,\gamma}\leq C(\|V\|^2_{1//2,\gamma}+\frac{1}{\gamma}\,\|\mathcal{F}\|^2_{1/2,\gamma}+\frac{1}{\gamma}\,\|W_{\mathbf{p}}\|^2_{L^2})$$

for a constant C independent of γ and V. Integrating this for $z\in(0,\infty)$, we have

$$\gamma\,\|V_1\|^2_{L^2(H_{1/2,\gamma})}\leq C(\|V\|^2_{L^2(H_{1/2,\gamma})}+\frac{1}{\gamma}\,\|\mathcal{F}\|^2_{L^2(H_{1/2,\gamma})}+\frac{1}{\gamma}\,\|W_{\mathbf{p}}\|^2_{L^2(L^2)}).\tag{4.90}$$

Estimates on V_2 and V_3. Comparing to the estimate of V_1, we need to consider further the term $\mathrm{Re}\,(\Lambda^{1/2,\gamma}T^{\gamma}_{-\zeta_2\alpha_1}V_5,\Lambda^{1/2,\gamma}V_2)$ (or $\mathrm{Re}\,(\Lambda^{1/2,\gamma}T^{\gamma}_{\zeta_2\beta_1}V_5,\Lambda^{1/2,\gamma}V_3)$). We take the first as an example. It is easy to see that we have the estimate

$$|\mathrm{Re}\,(\Lambda^{1/2,\gamma}T^{\gamma}_{-\zeta_2\alpha_1}V_5,\Lambda^{1/2,\gamma}V_2)|\leq C\,\|V_2\|_{1/2,\gamma}\,\|V_5\|_{3/2,\gamma}\leq C\varepsilon\gamma\,\|V_2\|^2_{1/2,\gamma}+\frac{C_\varepsilon}{\gamma}\,\|V_5\|^2_{3/2,\gamma}\,.$$

Then totally similar as for V_1, from (4.77) we get

$$\gamma\,\|V_2\|^2_{1/2,\gamma}\leq C\varepsilon\gamma\,\|V_2\|^2_{1/2,\gamma}+\frac{C_\varepsilon}{\gamma}\,\|V_5\|^2_{3/2,\gamma}+C(\|V\|^2_{1/2,\gamma}+\frac{1}{\gamma}\,\|\mathcal{F}\|^2_{1,\gamma}+\frac{1}{\gamma}\,\|W_{\mathbf{p}}\|^2_{L^2}).$$

Choosing ε small and note $\gamma\geq 1$, integrating for $z\in(0,\infty)$, and using (4.86) to bound the term $\|V_5\|^2_{3/2,\gamma}$, one gets

$$\gamma\,\|V_2\|^2_{L^2(H_{1/2,\gamma})}\leq C(\|V\|^2_{L^2(H_{1/2,\gamma})}+\frac{1}{\gamma}\,\|\mathcal{F}\|^2_{L^2(H_{1/2,\gamma})}+\frac{1}{\gamma}\,\|W_{\mathbf{p}}\|^2_{L^2(L^2)}).\tag{4.91}$$

The same inequality holds if V_2 is replaced by V_3 on the left-hand side.

Estimate on V_4. By equation (4.75), we have

$$\frac{\mathrm{d}}{\mathrm{d}z}(V_4, V_4) = 2\mathrm{Re}\,(\partial_z V_4, V_4) := \sum_{j=1}^{5} I_j = -2\mathrm{Re}\,(T^{\gamma}_{\zeta_2 e'} V_4, V_4) - 2\mathrm{Re}\,(T^{\gamma}_{\zeta_2 \mu\tau} V_2, V_4)$$
$$-2\mathrm{Re}\,(T^{\gamma}_{\zeta_2 i\mu\eta} V_3, V_4) - 2\mathrm{Re}\,(T^{\gamma}_{\alpha^0} V, V_4) + 2\mathrm{Re}\,(\mathcal{F}_4, V_4). \tag{4.92}$$

Using the localized Gårding's inequality (recall $\mathrm{Re}\, e' > c\lambda^{1,\gamma}$ by our construction), we have

$$-I_1 \geq C_1 \|V_4\|^2_{1/2,\gamma} - C_2(\|W_{\mathbf{p}}\|^2_{-3/2,\gamma} + \|V_4\|^2_{-1/2,\gamma} + \|V_4\|_{1/2,\gamma} \|W_{\mathbf{p}}\|_{-3/2,\gamma})$$
$$\geq (\frac{C_1}{2} - \frac{1}{\gamma^2}) \|V_4\|^2_{1/2,\gamma} - C \|W_{\mathbf{p}}\|^2_{-3/2,\gamma}.$$

We also see that

$$|I_2|+|I_3| \leq C \|V_4\|_{1/2,\gamma} (\|V_2\|_{1/2,\gamma}+\|V_3\|_{1/2,\gamma}) \leq C\varepsilon \|V_4\|^2_{1/2,\gamma}+C_\varepsilon(\|V_2\|^2_{1/2,\gamma}+\|V_3\|^2_{1/2,\gamma}),$$

and

$$|I_4| \leq \|V_4\|_{1/2,\gamma} \|V\|_{-1/2,\gamma} \leq \frac{C}{\gamma} \|V\|^2_{1/2,\gamma},$$
$$|I_5| \leq \|V_4\|_{1/2,\gamma} \|\mathcal{F}\|_{-1/2,\gamma} \leq C\varepsilon \|V_4\|^2_{1/2,\gamma} + \frac{C_\varepsilon}{\gamma^2} \|\mathcal{F}\|^2_{1/2,\gamma}.$$

So by integrating (4.92) with respect to z over $(0,\infty)$, and for γ large, ε small, we get

$$\|V_4\|^2_{L^2(H_{1/2,\gamma})} \leq C \left(\|V_4(0)\|^2_{L^2} + \|W_{\mathbf{p}}\|^2_{L^2(H_{-3/2,\gamma})} + \frac{1}{\gamma^2} \|\mathcal{F}\|^2_{L^2(H_{1/2,\gamma})} \right)$$
$$+C \left(\|V_2\|^2_{L^2(H_{1/2,\gamma})} + \|V_3\|^2_{L^2(H_{1/2,\gamma})} + \frac{1}{\gamma} \|V\|^2_{L^2(H_{1/2,\gamma})} \right).$$

Multiplying this by γ and using (4.91), we find

$$\gamma \|V_4\|^2_{L^2(H_{1/2,\gamma})} \leq C\gamma \|V_4(0)\|^2_{L^2}$$
$$+C \left(\|V\|^2_{L^2(H_{1/2,\gamma})} + \frac{1}{\gamma} \|\mathcal{F}\|^2_{L^2(H_{1/2,\gamma})} + \frac{1}{\gamma} \|W_{\mathbf{p}}\|^2_{L^2(L^2)} \right). \tag{4.93}$$

Estimate of $V_4(0)$. Finally, we use boundary conditions to estimate $\gamma \|V_4(0)\|^2_{L^2}$. We write $W_{\mathbf{p}} = (w_1, w_2, w_3, w_4, w_5)^T$ and $V = (V_1, \cdots, V_5)^T$. Then the boundary condition in (4.33) for $W_{\mathbf{p}}$ reads

$$w_5 - w_4 = T^{\gamma}_{\chi_{\mathbf{p}}} \tilde{G}.$$

We could easily solve that

$$\mathbf{R}^{-1} = \begin{pmatrix} 1 & & \\ & I_2 & -R_1 L_2 \\ & & L_2 \end{pmatrix},$$

hence

$$V_1 = w_1,\ V_2 = T^{\gamma}_{\zeta_1} w_2 - T^{\gamma}_{\zeta_1 r_1} w_5,\ V_3 = T^{\gamma}_{\zeta_1} w_3 - T^{\gamma}_{\zeta_1 r_3} w_5,$$
$$V_4 = T^{\gamma}_{\zeta_1} w_5,\ V_5 = T^{\gamma}_{\zeta_1} w_4 + T^{\gamma}_{\zeta_1 m_4} w_5. \tag{4.94}$$

It follows that

$$\begin{aligned}T^\gamma_{\zeta_2(1-s_1)}V_4+T^\gamma_{-\zeta_2}V_5&=T^\gamma_{\zeta_2(1+m_4)}T^\gamma_{\zeta_1}w_5-T^\gamma_{\zeta_2}T^\gamma_{\zeta_1}w_4-T^\gamma_{\zeta_2}T^\gamma_{\zeta_1 m_4}w_5\\&=T^\gamma_{\zeta_1}(w_5-w_4)+R_{-1}(w_4+w_5)=T^\gamma_{\zeta_1}T^\gamma_{\chi_{\mathbf{p}}}\tilde{G}+R_{-1}(w_4+w_5).\end{aligned}$$

Recall $1-s_1=-\frac{l_4}{i\eta\mu}\omega$, by acting the first-order operator $T^\gamma_{-\frac{i\eta\mu}{l_4}\zeta_3}$, we find

$$T^\gamma_{\zeta_2\omega}V_4+T^\gamma_{\zeta_2\frac{i\mu\eta}{l_4}}V_5=\mathcal{G}:=T^\gamma_{-\zeta_3\frac{i\mu\eta}{l_4}}T^\gamma_{\zeta_1}T^\gamma_{\chi_{\mathbf{p}}}\tilde{G}+R_0(w_4+w_5),\ \text{ on }\{z=0\}.\tag{4.95}$$

Taking $L^2(\mathbb{R}^2)$ inner product with V_4, then the real part satisfies

$$\begin{aligned}&\operatorname{Re}(T^\gamma_{\zeta_2\omega}V_4(0),V_4(0))\\&=-\operatorname{Re}(T^\gamma_{\zeta_2\frac{i\mu\eta}{l_4}}V_5(0),V_4(0))+\operatorname{Re}(\mathcal{G},V_4(0))\le C(\|V_5(0)\|_{1,\gamma}+\|\mathcal{G}\|_{L^2})\,\|V_4(0)\|_{L^2}\\&\le\varepsilon\gamma\,\|V_4(0)\|^2_{L^2}+\frac{C_\varepsilon}{\gamma}(\|V_5(0)\|^2_{1,\gamma}+\|\mathcal{G}\|^2_{L^2}),\end{aligned}\tag{4.96}$$

while

$$\begin{aligned}\operatorname{Re}(T^\gamma_{\zeta_2\omega}V_4(0),V_4(0))&=\gamma(V_4(0),V_4(0))+\gamma(R_{-2}w_5(0),V_4(0))+(R_0V_4(0),V_4(0))\\&\ge\gamma\,\|V_4(0)\|^2_{L^2}-C\gamma\,\|R_{-2}w_5(0)\|_{L^2}\,\|V_4(0)\|_{L^2}-C\,\|V_4(0)\|^2_{L^2}\\&\ge(\gamma-C)\,\|V_4(0)\|^2_{L^2}-\frac{C}{\gamma^2}\,\|w_5(0)\|^2_{L^2}.\end{aligned}\tag{4.97}$$

So from (4.96) and (4.97), for $\gamma_0\ge 1$ large and $\gamma\ge\gamma_0$, we have

$$\gamma\,\|V_4(0)\|^2_{L^2}\le\frac{C}{\gamma}(\|V_5(0)\|^2_{1,\gamma}+\left\|\tilde{G}\right\|^2_{1,\gamma}+\|w_4(0)\|^2_{L^2}+\|w_5(0)\|^2_{L^2}).\tag{4.98}$$

Conclusion: estimate near poles. From the estimates (4.86), (4.90), (4.91), (4.93) and (4.98), we have

$$\begin{aligned}&\gamma\sum_{j=1}^{5}\|V_j\|^2_{L^2(H_{1/2,\gamma})}+\gamma\,\|V_5(0)\|^2_{L^2}+\gamma\,\|V_4(0)\|^2_{L^2}\\&\le C\Big(\|V\|^2_{L^2(H_{1/2,\gamma})}+\frac{1}{\gamma}(\|\mathcal{F}\|^2_{L^2(H_{1/2,\gamma})}+\|W_{\mathbf{p}}\|^2_{L^2(L^2)}+\left\|\tilde{G}\right\|^2_{1,\gamma}\\&\qquad+\underbrace{\|w_4(0)\|^2_{L^2}+\|w_5(0)\|^2_{L^2}}_{\|W^{\mathrm{nc}}_{\mathbf{p}}(0)\|^2_{L^2}})\Big).\end{aligned}$$

By choosing γ large, it follows

$$\gamma\,\|V\|^2_{L^2(H_{1/2,\gamma})}+\gamma\,\|V_5(0)\|^2_{L^2}+\gamma\,\|V_4(0)\|^2_{L^2}\le\mathcal{H}\tag{4.99}$$

with

$$\mathcal{H}:=\frac{C}{\gamma}\Big(\|\mathcal{F}\|^2_{L^2(H_{1/2,\gamma})}+\|W\|^2_{L^2(L^2)}+\left\|\tilde{G}\right\|^2_{1,\gamma}+\|W^{\mathrm{nc}}(0)\|^2_{L^2}\Big).$$

Now recall $V_4=T^\gamma_{\zeta_1}w_5=T^\gamma_{\zeta_1}T^\gamma_{\chi_{\mathbf{p}}}W_5=T^\gamma_{\chi_{\mathbf{p}}}W_5+R_{-2}W_5$, so

$$w_5=V_4+R_{-2}W_5,\quad w_4=V_5-T^\gamma_{\zeta_1 m_4}V_4+R_{-2}(W_5+W_4).$$

Using (4.99), it holds

$$\gamma \left\| w_4(0), w_5(0) \right\|_{L^2}^2 \le \gamma \left\| V_4(0) \right\|_{L^2}^2 + \gamma \left\| V_5(0) \right\|_{L^2}^2 + \frac{1}{\gamma^3} \left\| W^{\mathrm{nc}}(0) \right\|_{L^2}^2 \le 2\mathcal{H}.$$

Also, from (4.94), we could solve that

$$W_{\mathbf{p}} = R_0 V + R_{-2} W.$$

Hence

$$\gamma \left\| W_{\mathbf{p}} \right\|_{L^2(H_{1/2,\gamma})}^2 \le C(\gamma \left\| V \right\|_{L^2(H_{1/2,\gamma})}^2 + \frac{1}{\gamma^2} \left\| W \right\|_{L^2(L^2)}^2) \le 2\mathcal{H}.$$

Finally, by the definition of $\mathcal{F}$ in (4.79), we have

$$\left\| \mathcal{F} \right\|_{L^2(H_{1/2,\gamma})}^2 \le C \left(\| T_{\tilde{r}}^{\gamma} W \|_{L^2(H_{1/2,\gamma})}^2 + \frac{1}{\gamma} \left\| W \right\|_{L^2(L^2)}^2 + \frac{1}{\gamma} \left\| \tilde{F} \right\|_{L^2(H_{1,\gamma})}^2 \right).$$

Summing up, we get the desired estimate near poles:

$$\gamma \left\| T_{\chi_{\mathbf{p}}}^{\gamma} W^{\mathrm{nc}}(0) \right\|_{L^2}^2 + \gamma \left\| T_{\chi_{\mathbf{p}}}^{\gamma} W \right\|_{L^2(H_{1/2,\gamma})}^2 \le C \frac{1}{\gamma} \Big(\left\| W \right\|_{L^2(L^2)}^2 + \left\| \tilde{G} \right\|_{1,\gamma}^2 + \frac{1}{\gamma} \left\| \tilde{F} \right\|_{L^2(H_{1,\gamma})}^2 + \left\| W^{\mathrm{nc}}(0) \right\|_{L^2}^2 + \| T_{\tilde{r}}^{\gamma} W \|_{L^2(H_{1/2,\gamma})}^2 \Big). \tag{4.100}$$

4.9. Proof of Theorem 4.4. Combining estimates (4.50) and (4.100), using $W = T_{\chi_{\mathbf{u}}}^{\gamma} W + T_{\chi_{\mathbf{p}}}^{\gamma} W$, and recall $r, \tilde{r} \in \Gamma_1^0$, we have

$$\begin{aligned} \gamma \left\| W \right\|_{L^2(H_{1/2,\gamma})}^2 + \gamma \left\| W^{\mathrm{nc}}(0) \right\|_{L^2}^2 &\le \frac{C}{\gamma} \left(\frac{1}{\gamma} \left\| \tilde{F} \right\|_{L^2(H_{1,\gamma})}^2 + \left\| \tilde{G} \right\|_{1,\gamma}^2 \right) \\ &\quad + \frac{C}{\gamma} \Big(\left\| W \right\|_{L^2(L^2)}^2 + \left\| W^{\mathrm{nc}}(0) \right\|_{L^2}^2 + \| T_r^{\gamma} W \|_{L^2(H_{1/2,\gamma})}^2 + \| T_{\tilde{r}}^{\gamma} W \|_{L^2(H_{1/2,\gamma})}^2 \Big) \\ &\le \frac{C}{\gamma} \left(\frac{1}{\gamma} \left\| \tilde{F} \right\|_{L^2(H_{1,\gamma})}^2 + \left\| \tilde{G} \right\|_{1,\gamma}^2 + \frac{1}{\gamma} \left\| W \right\|_{L^2(H_{1/2,\gamma})}^2 + \left\| W^{\mathrm{nc}}(0) \right\|_{L^2}^2 + C' \left\| W \right\|_{L^2(H_{1/2,\gamma})}^2 \right). \end{aligned}$$

Therefore, by choosing $\gamma_0 \ge 1$ large and then for all $\gamma \ge \gamma_0$, we obtain

$$\gamma \left\| W \right\|_{L^2(H_{1/2,\gamma})}^2 + \gamma \left\| W^{\mathrm{nc}}(0) \right\|_{L^2}^2 \le \frac{C}{\gamma} \left(\frac{1}{\gamma} \left\| \tilde{F} \right\|_{L^2(H_{1,\gamma})}^2 + \left\| \tilde{G} \right\|_{1,\gamma}^2 \right).$$

This leads to (4.34) that was claimed in Theorem 4.4. The proof of Theorem 4.1 is also completed after employ a standard approximation argument.

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