Stability of Transonic Jet with Strong Shock in Two-Dimensional Steady Compressible Euler Flows

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Abstract

For steady supersonic flow past a solid convex corner surrounded by quiescent gas, if the pressure of the upcoming supersonic flow is lower than the pressure of the quiescent gas, there may appear a strong shock to increase the pressure and then a transonic characteristic discontinuity to separate the supersonic flow behind the shock-front from the still gas. In this paper we prove global existence, uniqueness, and stability of such flow patterns under suitable conditions on the upstream supersonic flow and the pressure of the surrounding quiescent gas, for the two-dimensional steady complete compressible Euler system. Mathematically, a global weak solution to a characteristic free boundary problem of hyperbolic conservation laws is constructed and shown to be unique and stable under the framework of front tracking method. The main part of the proof is to reformulate the problem in the Lagrangian coordinates, then solve several typical Riemann problems and obtain estimates for wave interactions/reflections/refractions, and define a Glimm functional using suitable weights, which guarantees that the solution can be constructed by standard front tracking method. The uniqueness and stability is proved by showing that the Lyapunov functional introduced by Bressan-Liu-Yang is non-increasing.

Keywords: compressible Euler equations, steady, transonic, characteristic discontinuity, jet, shock, free boundary, front tracking, reflection, Glimm functional.

Mathematics Subject Classification 2010: 35L50, 35L65, 35R35, 76N10.

Contents

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1 Introduction

As illustrated in Figure 1, for a steady supersonic flow passing a solid convex corner surrounded by static gas, if the pressure of the quiescent gas is larger than the pressure of the upcoming supersonic flow, there may appear a shock to increase the pressure of the supersonic flow to that of the quiescent gas, and a characteristic discontinuity (also called contact discontinuity in gas dynamics, which is a combination of vortex sheet and/or entropy wave) to separate the supersonic flow behind of the shock-front from the static gas. In this paper, we are going to prove the existence and uniqueness, as well as stability of such a flow pattern using the two-dimensional steady complete compressible Euler system, under reasonable assumptions on the upstream supersonic flow and the lower downstream quiescent gas.

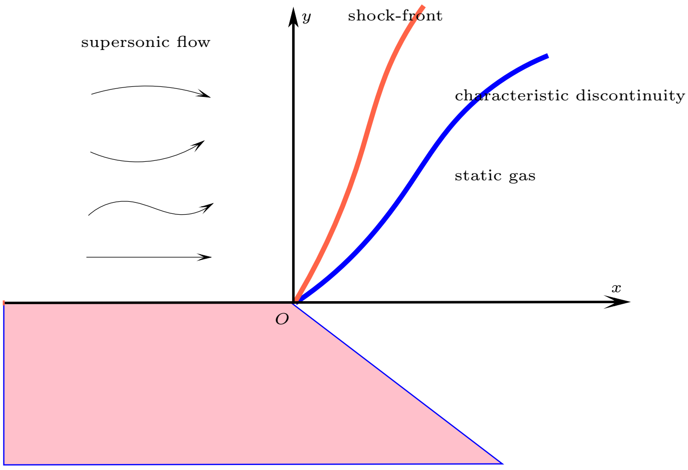

Figure 1: A characteristic discontinuity (blue line) and a shock (red line) emerged from the solid corner. The characteristic discontinuity separates quiescent gas below from supersonic flow behind the shock-front.

The Euler system that governing two-dimensional steady compressible flows consists of the following conservation laws of mass, momentum and energy:

$$\begin{cases} \partial_x(\rho u) + \partial_y(\rho v) = 0, \\ \partial_x(\rho u^2 + p) + \partial_y(\rho uv) = 0, \\ \partial_x(\rho uv) + \partial_y(\rho v^2 + p) = 0, \\ \partial_x(\rho uE + pu) + \partial_y(\rho vE + pv) = 0, \end{cases} \tag{1.1}$$

where $E = \frac{1}{2}(u^2+v^2)+e$ is the (total) energy density. The unknowns ρ, p, e and (u, v) represent the density, pressure, internal energy, and velocity of the fluid, respectively. Specifically, for a polytropic gas, the constitutive relation is $p = \kappa_0 \rho^\gamma \exp(\frac{s}{c_v})$ and $e = \frac{1}{\gamma-1}\frac{p}{\rho}$. Here s is the entropy, κ_0 and c_v are positive constants, and $\gamma > 1$ is the adiabatic exponent. The sonic speed c is determined by $c = \sqrt{\frac{\partial p}{\partial \rho}} = \sqrt{\frac{\gamma p}{\rho}}$. It is well-known that for supersonic flow $u > c$, the system (1.1) is hyperbolic in the positive x-direction.

Since the quiescent gas below the characteristic discontinuity should not be affected by the supersonic flow above it (that is, it will always be static as suggested by physics), we only need to determine the characteristic discontinuity $\mathcal{C}$, which is a free boundary, and the supersonic flow above it. Suppose that $\mathcal{C}$ is given by an equation $y = g(x), x \geq 0$ with $g(0) = 0$. By the Rankine-Hugoniot conditions across a characteristic discontinuity (see equation (3) in [5, p.3]), we have the following boundary conditions on $\mathcal{C}$:

$$p = \underline{p}_b, \qquad g'(x) = \frac{v}{u}(x, g(x)). \tag{1.2}$$

Here $\underline{p}_b$ is a constant, which is the pressure of the quiescent gas. Also, suppose the supersonic flow on $\mathcal{I} = \{(x, y) \in \mathbb{R}^2 : x = 0, y > 0\}$ is given:

$$U(0, y) = U_0(y), \quad U = (u, v, p, \rho), \quad \text{and} \quad u_0(y) > c_0(y). \tag{1.3}$$

Then we have an initial-free boundary problem of (1.1) in the planar domain bounded by $\mathcal{C}$ and $\mathcal{I}$. We call this as problem (E) below. We note that the second condition in (1.2) implies that $\mathcal{C}$ is a characteristic curve for the Euler system.

A weak entropy solution of problem (E) could be defined in the same way as in the Definition 1 of [5, p.4].

A special (weak) solution to the problem (E) could be constructed by using the well-known p-w shock polar (see [10, p. 325, p.347] or Figure 2), with $w = v/u$:

$$w = \pm \frac{\frac{p}{p_0} - 1}{\gamma M_0^2 - (\frac{p}{p_0} - 1)} \sqrt{\frac{\frac{2\gamma}{\gamma+1}(M_0^2 - 1) - (\frac{p}{p_0} - 1)}{\frac{p}{p_0} + \frac{\gamma-1}{\gamma+1}}}. \tag{1.4}$$

This curve represents all possible states (p, w) behind a shock, for giving uniform supersonic flow $U_0 = (u = u_0, v = 0, p = p_0, \rho = \rho_0)$ ahead of the shock, and $M_0 = \frac{u_0}{c_0}$ is the Mach number. Once p, w are determined, all u, v, p, ρ and the slope of the shock-front could be uniquely solved from the Rankine-Hugoniot conditions.

We could see from Figure 2 that there is an interval (p_0, p_1) so that if $\underline{p}_b$ (the pressure of the static gas) lies in (p_0, p_1), then there will be a unique $\underline{w}_b > 0$ so that $(\underline{w}_b, \underline{p}_b)$ lies on the shock polar, and it determines a supersonic shock. Let the state behind the shock be denoted by $\underline{U}_b$, and write U_0 as $\underline{U}_a$, we get a special piecewise constant solution $\underline{U} = (\underline{U}_a, \underline{U}_b)$ to problem (E), which is separated by a straight shock-front (the slope $\underline{s} = \underline{s}_b$ is already determined), and the characteristic discontinuity is given by the equation $y = \frac{\underline{v}_b}{\underline{u}_b} x$. We call such a solution as a *background solution* in the sequel.

The main result of this paper is the following theorem.

Theorem 1.1. *For a given constant supersonic data $\underline{U}_a = (\underline{u}_a, 0, \underline{p}_a, \underline{\rho}_a)$, there is a number $p_* \in (\underline{p}_a, p_1)$ determined by $\underline{U}_a$. For any background solution $\underline{U} = (\underline{U}_a, \underline{U}_b)$ so that $\underline{p}_b \in (\underline{p}_a, p_*)$, there exist constants ε_0 and C so that if the initial data U_0 in problem (E) satisfies*

$$\|U_0 - \underline{U}_a\|_{\mathrm{BV}([0,\infty))} \le \varepsilon \le \varepsilon_0, \tag{1.5}$$

then problem (E) has a unique weak entropy solution (U, g) constructed by the front tracking method, and

i) $g(x)$ is a Lipschitz function for $x \ge 0$ with $g(0) = 0$, representing the characteristic discontinuity, and

$$\|g' - \underline{v}_b/\underline{u}_b\|_{L^\infty([0,\infty))} \le C\varepsilon; \tag{1.6}$$

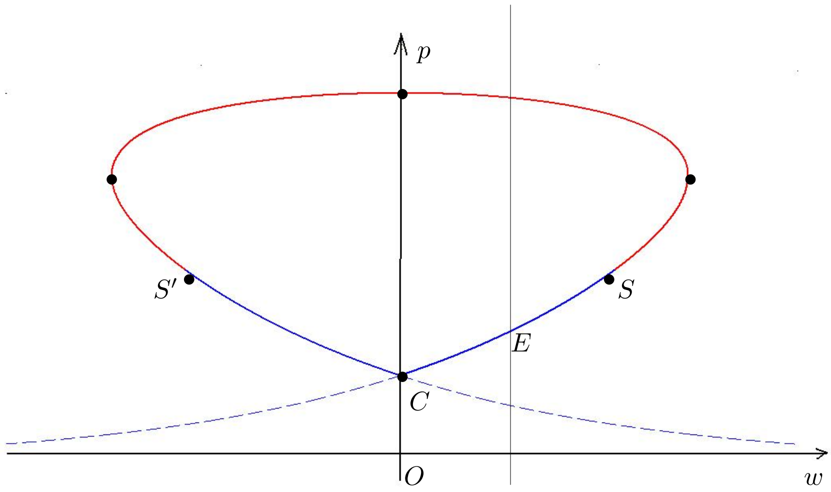

Figure 2: A p-w shock polar. $C = (0, p_0)$ represents the upcoming supersonic flow; the open arc $\widehat{CS}$ and $\widehat{CS'}$ represent all possible supersonic shocks. $E = (w, \underline{p}_b)$ represents a supersonic shock for given pressure $\underline{p}_b$ behind the shock.

ii) there exists a vector $\underline{U}_0 \in \mathbb{R}^4$ *so that* $\lim_{y\to\infty} U_0(y) = \underline{U}_0$, *and*

$$U - \underline{U}_0 \in C([0,\infty); L^1(g(x),\infty)); \tag{1.7}$$

iii) there is a Lipschitz function $y = s(x)$ *representing the shock-front, with* $s(0) = 0$, *and*

$$\|s' - \underline{s}_b\|_{L^\infty([0,\infty))} \leq C\varepsilon; \tag{1.8}$$

$$\|(U - \underline{U}_a)(x,\cdot)\|_{\mathrm{BV}((s(x),\infty))} \leq C\varepsilon, \quad \forall x > 0; \tag{1.9}$$

$$\|(U - \underline{U}_b)(x,\cdot)\|_{\mathrm{BV}(g(x),s(x))} \leq C\varepsilon, \quad \forall x > 0. \tag{1.10}$$

Furthermore, reformulating this problem in Lagrangian coordinates, then the L^1*–stability holds in the sense that*

$$\left\|V^1(\xi) - V^2(\xi)\right\|_{L^1(\mathbb{R}^+)} \leq C \left\|V^1(0) - V^2(0)\right\|_{L^1(\mathbb{R}^+)}$$

for any two solutions V^1, V^2 *with initial data* $V^1(0)$, $V^2(0)$ *respectively.*

We remark that the existence of $\underline{U}_0$ in ii) follows easily from (1.5) and a property of bounded variation (BV) functions. The number p_* could be determined by the inequality (4.26) in section 4.4. We only claim the L^1 stability in the Lagrangian coordinates, since it is not clear how to express the stability for such free boundary problems in the Eulerian coordinates.

We review some related works in the literature. In [5, 6], the authors had studied transonic characteristic discontinuity for the case that the still gas pressure $\underline{p}_b$ is very close to the pressure of the upstream supersonic flows, hence there will not appear any strong shock in the downstream supersonic flow. A complete existence and well-posedness theory has been established. A similar problem is the supersonic shock attached to the vertex of a slim wedge which is against uniform supersonic flow [8], for which only the solid wall of the wedge is perturbed and the wall is a known characteristic boundary. The authors constructed a solution by the Glimm scheme and later Chen and Li established well-posedness in [7]. Wang and Zhang also studied the problem of steady supersonic flow past a curved cone in [17]. Another well studied problem is the piston problem for unsteady Euler equations (and its generalizations), see for example, [4, 12, 18] and references therein.

We note that for the piston problem or for the problem of supersonic flow past a wedge, there will not appear a free boundary to separate supersonic flow from subsonic flow, across which the Euler system changes type. To handle the characteristic free boundary, we use the Lagrangian coordinates as in [6], which was introduced by S. Chen in the studies of transonic shocks (see [9, 15, 19]). The advantage of working in the Lagrangian coordinates is that the characteristic free boundary becomes fixed and flattened, and also the Euler system simplifies to be strictly hyperbolic. The equivalence of weak entropy solution in the Eulerian coordinates and Lagrangian coordinates is guaranteed by a theorem of Wagnar [16], which is also repeated in [6, p.1724]. Accordingly we only need to study an initial-boundary value problem in the Lagrangian coordinates. We then study several typical Riemann problems and obtain estimates on wave interactions/reflections/refractions, and finally construct Glimm functional by adapting suitable weights on wave strengths. We remark that waves might reflect from the boundary and the strong shock infinitely many times, so it is crucial that they become weaker after reflections. However, in our case the reflection coefficient K_2 off the characteristic boundary is strictly larger than 1. So what makes the flow pattern we described above to be stable is that the reflection coefficient C_{13} off the strong shock is quite small so that $|K_2C_{13}| < 1$. This is the reason why we need the parameter p_* in Theorem 1.1. Since we also need to consider the refractions of waves above the strong shock due to perturbations of the upcoming supersonic flow, the determination of various weights in the Glimm functional is more complicate.

The rest of the paper is organized as follows. In section 2 we reformulate problem (E) to be problem (L) in the Lagrangian coordinates. In section 3 we solve four typical Riemman problems, and section 4 is devoted to establishing some waves interaction estimates. In section 5 we construct a Glimm functional necessary for the front tracking method. Once a suitable Glimm functional is identified, we could construction the approximate solutions by front tracking and apply the compactness arguments to establish the convergence of approximate solutions to a global entropy solution. Thus the existence claimed in Theorem 1.1 is proved. Finally, in section 6 we prove the uniqueness and L^1 stability part of Theorem 1.1 by adapting the arguments in [2, 6] and [7].

2 Problem (L) in Lagrangian coordinates

In this section, we reformulate problem (E) in the Lagrangian coordinates (ξ, η) given by $(x, y) \mapsto (\xi, \eta) = (x, \eta(x, y))$ with

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{pmatrix} 1 & 0 \\ -\rho v & \rho u \end{pmatrix}.$$

For details, please see [6, p.1713]. This transform is Lipschitz continuous and one–to–one provided that $\rho u > 0$. The transonic characteristic discontinuity is transformed to the positive ξ-axis, and $\mathcal{I}$ becomes the positive η-axis, see Figure 3.

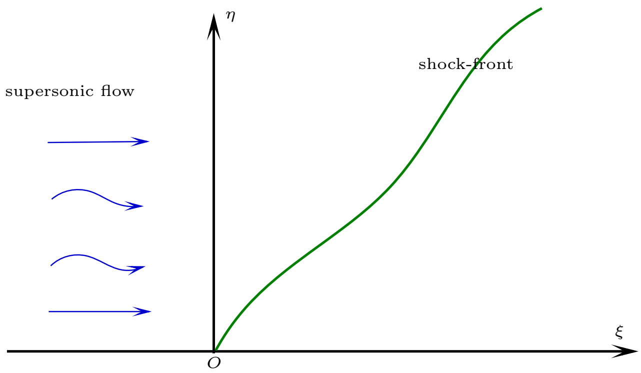

Figure 3: The problem in the Lagrangian coordinates (ξ, η). The characteristic discontinuity becomes the positive ξ-axis.

The Euler system (1.1) in the (ξ, η) coordinates may be written in divergence form as

$$\begin{cases} \partial_\xi\Big(\dfrac{1}{\rho u}\Big) - \partial_\eta\big(\dfrac{v}{u}\big) = 0, \\ \partial_\xi(u + \frac{p}{\rho u}) - \partial_\eta\big(\frac{pv}{u}\big) = 0, \\ \partial_\xi v + \partial_\eta p = 0, \end{cases} \tag{2.1}$$

or, as a symmetric system for $U = (u, v, p)^\top$,

$$A\partial_\xi U + B\partial_\eta U = 0, \tag{2.2}$$

with

$$A = \begin{pmatrix} u & 0 & \frac{1}{\rho} \\ 0 & u & 0 \\ \frac{1}{\rho} & 0 & \frac{u}{\rho^2 c^2} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & -v \\ 0 & 0 & u \\ -v & u & 0 \end{pmatrix}, \tag{2.3}$$

and the conservation of energy becomes $\partial_\xi(\frac{u^2+v^2}{2}+\frac{c^2}{\gamma-1})=0$; that is,

$$\frac{u^2+v^2}{2}+\frac{c^2}{\gamma-1}=b(\eta). \tag{2.4}$$

This is called the Bernoulli law. We note that it holds even across shock-fronts by Rankine-Hugoniot conditions. Hence we observed that $b(\eta)$ is a given function determined by the initial data. In the following we focus on system (2.1) with ρ determined by $U=(u,v,p)^\top$ through (2.4).

For later usage, we repeat here the eigenvalues λ of (2.2):

$$\begin{aligned}
\lambda_1 &= \frac{\rho c^2 u}{u^2-c^2}\Big(\frac{v}{u}-\sqrt{M^2-1}\Big)<0, && (2.5)\\
\lambda_2 &= 0, && (2.6)\\
\lambda_3 &= \frac{\rho c^2 u}{u^2-c^2}\Big(\frac{v}{u}+\sqrt{M^2-1}\Big)>0, && (2.7)
\end{aligned}$$

where $M=\frac{\sqrt{u^2+v^2}}{c}$ is the Mach number of the flow. Then, for $u>c$, system (2.2) is strictly hyperbolic. The associated right-eigenvectors are

$$\begin{aligned}
r_1 &= \kappa_1(\frac{\lambda_1}{\rho}+v,-u,-\lambda_1 u)^\top, && (2.8)\\
r_2 &= (u,v,0)^\top, && (2.9)\\
r_3 &= \kappa_3(\frac{\lambda_3}{\rho}+v,-u,-\lambda_3 u)^\top, && (2.10)
\end{aligned}$$

where $\kappa_j<0$ can be chosen so that $r_j\cdot\nabla\lambda_j\equiv 1$, since the j-th characteristic fields ($j=1,3$) are genuinely nonlinear. Note that the second characteristic field is always linearly degenerate: $r_2\cdot\nabla\lambda_2=0$ and ξ-axis is a characteristic curve.

The fact that $\kappa_j<0$ is a consequence of the following lemma.

Lemma 2.1. *If λ is the first or third eigenvalue of* (2.2) *(i.e. $|\lambda A-B|=0$) , and $r=\kappa(\frac{\lambda}{\rho}+v,-u,-\lambda u)^\top$ is the corresponding eigenvector, where $r\cdot\nabla\lambda\equiv 1$. Then*

$$\frac{2}{u}\left[\left(1-\frac{u^2}{c^2}\right)\frac{\lambda}{\rho u}+w\right]=(1+\gamma)\frac{u^2}{c^2}\frac{\lambda}{\rho u}\frac{\lambda^2}{\rho c^2}\cdot\kappa. \tag{2.11}$$

Proof. The factor κ is determined by $r\cdot\nabla\lambda\equiv 1$, so we need to calculate $\nabla\lambda=(\frac{\partial\lambda}{\partial u},\frac{\partial\lambda}{\partial v},\frac{\partial\lambda}{\partial p})^\top$. Since λ_i ($i=1,3$) is the root of

$$(1-\frac{u^2}{c^2})(\frac{\lambda}{\rho u})^2+2\frac{\lambda}{\rho u}w+w^2+1=0, \tag{2.12}$$

we could differentiate it to obtain

$$
\begin{aligned}
(a_1,a_2,a_3)^\top &\triangleq 2\left[(1-\frac{u^2}{c^2})\frac{\lambda}{\rho u}+w\right]\frac{1}{\rho u}\nabla\lambda \\
&= -2\left[(1-\frac{u^2}{c^2})\frac{\lambda}{\rho u}+w\right]\lambda\nabla\left(\frac{1}{\rho u}\right)-2\left(\frac{\lambda}{\rho u}+w\right)\nabla w-(\frac{\lambda}{\rho u})^2\nabla\left(1-\frac{u^2}{c^2}\right).
\end{aligned}
$$

We use $(a_1,a_2,a_3)^\top$ instead of $\nabla\lambda$ to take the scalar product with $r=\kappa\left(\frac{\lambda}{\rho}+v,-u,-\lambda u\right)^\top$, and it follows that

$$
\kappa u\left[\left(\frac{\lambda}{\rho u}+w\right)a_1-a_2-\lambda a_3\right]=2\left[\left(1-\frac{u^2}{c^2}\right)\frac{\lambda}{\rho u}+w\right]\frac{1}{\rho u}.
$$

Observing that

$$
\begin{aligned}
&w\frac{\partial(\frac{1}{\rho u})}{\partial u}-\frac{\partial(\frac{1}{\rho u})}{\partial v}=-\frac{w}{\rho u^2},\quad w\frac{\partial w}{\partial v}+\frac{\partial w}{\partial u}=0,\\
&w\frac{\partial\left(1-\frac{u^2}{c^2}\right)}{\partial u}-\frac{\partial\left(1-\frac{u^2}{c^2}\right)}{\partial v}=-\frac{2v}{c^2},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\frac{2\left[\left(1-\frac{u^2}{c^2}\right)\frac{\lambda}{\rho u}+w\right]\frac{1}{\rho u}}{\kappa u}\\
=\ &2\lambda\left[(1-\frac{u^2}{c^2})\frac{\lambda}{\rho u}+w\right]\left[-\frac{\lambda}{\rho u}\partial_u(\frac{1}{\rho u})+\frac{w}{\rho u^2}+\lambda\partial_p(\frac{1}{\rho u})\right]\\
&-2\left[\left(1-\frac{u^2}{c^2}\right)\frac{\lambda}{\rho u}+w\right]\frac{1}{u}\left(\frac{\lambda}{\rho u}+w\right)\frac{\lambda}{\rho u}+(\frac{\lambda}{\rho u})^2\left[-\frac{\lambda}{\rho u}\partial_1\left(1-\frac{u^2}{c^2}\right)+\frac{2v}{c^2}\right]\\
=\ &-2\left[\left(1-\frac{u^2}{c^2}\right)\frac{\lambda}{\rho u}+w\right]\frac{\lambda}{\gamma p}\frac{\lambda}{\rho u}+(\frac{\lambda}{\rho u})^2\left[\left(\frac{\lambda}{\rho u}+w\right)\frac{2u}{c^2}+\frac{u^2}{c^2}\frac{(\gamma-1)\lambda}{\gamma p}\right].
\end{aligned}
$$

Multiplying both sides by $\frac{\rho u}{\lambda^2}$ yields

$$
\frac{2}{\kappa}\left[\left(1-\frac{u^2}{c^2}\right)\frac{\lambda}{\rho u}+w\right]\frac{1}{\lambda^2 u}=(1+\gamma)\frac{u^2}{c^2}\frac{\lambda}{\rho u}\frac{1}{\gamma p}.
$$

Therefore we proved (2.11). □

Suppose the equation of the shock-front is $\eta=s(\xi)$. The Rankine-Hugoniot conditions of the system (2.1) read (recall we set $w=v/u$, and as usual, $[\cdot]$ here represents jump of a quantity across the shock-front):

$$s'\left[\frac{1}{\rho u}\right]+[w]=0, \tag{2.13}$$

$$s'\left[u+\frac{p}{\rho u}\right]+[pw]=0, \tag{2.14}$$

$$s'[v]-[p]=0. \tag{2.15}$$

We now formulate problem (E) as the following problem (L), which is an initial-boundary value problem for equations (2.1):

$$\begin{cases} (2.1) & \text{in } \ \xi>0,\eta>0,\\ U(0,\eta)=U_0(\eta) & \text{on } \ \xi=0,\eta>0,\\ p=\underline{p}_b & \text{on } \ \xi>0,\eta=0. \end{cases} \tag{2.16}$$

The weak entropy solutions of problem (2.16) can be defined in the standard way via the integration by parts. The equivalence of weak solutions of problem (E) and problem (L) is given by Theorem 2 in [16]. We remark that the initial data $U_0(\xi)$ here could totally be solved from the initial data $U_0(y)$ in problem (E), see the proof of Lemma 6.1 in [6, p.1725]. If (1.5) holds, then it follows that

$$\|U_0-\underline{U}_a\|_{\mathrm{BV}((0,\infty))}\le C_0\varepsilon\le C_0\varepsilon_0, \tag{2.17}$$

with a positive constant C_0 depending only on $\underline{U}_a$. Also, once U is solved from problem (L), we can recover the characteristic discontinuity in Eulerian coordinates by

$$g(x)=\int_0^x \frac{v}{u}(\xi,0)\,\mathrm{d}\xi. \tag{2.18}$$

Hence, to prove Theorem 1.1, we only need to prove the following theorem.

Theorem 2.2. *Under the assumptions of Theorem 1.1, there exist constants ε_0 and C so that if the initial data U_0 in problem (L) satisfies*

$$\|U_0-\underline{U}_a\|_{\mathrm{BV}([0,\infty))}\le \varepsilon\le\varepsilon_0, \tag{2.19}$$

then problem (L) has a unique weak entropy solution U constructed by the front tracking method, and

i) there exists a vector $\underline{U}_0\in\mathbb{R}^4$ so that $\lim_{y\to\infty}U_0(y)=\underline{U}_0$, and

$$U-\underline{U}_0\in C([0,\infty);L^1(0,\infty)); \tag{2.20}$$

ii) there is a Lipschitz function $\eta = s(\xi)$ representing the shock-front, with $s(0) = 0$, and

$$\|s' - \underline{s}\|_{L^\infty([0,\infty))} \le C\varepsilon; \tag{2.21}$$
$$\|(U - \underline{U}_a)(\xi, \cdot)\|_{\mathrm{BV}((s(\xi),\infty))} \le C\varepsilon, \quad \forall \xi > 0; \tag{2.22}$$
$$\|(U - \underline{U}_b)(\xi, \cdot)\|_{\mathrm{BV}((0,s(\xi)))} \le C\varepsilon, \quad \forall \xi > 0. \tag{2.23}$$

Here $\underline{s}$ is the slope of the shock-front of the background solution $\underline{U}$ in its Lagrangian coordinates;

iii) for any two initial data U_0^1 and U_0^2 satisfy (2.19) *and assumptions in Theorem 1.1, it holds that*

$$\left\|U^1(\xi) - U^2(\xi)\right\|_{L^1(\mathbb{R}^+)} \le C \left\|U_0^1 - U_0^2\right\|_{L^1(\mathbb{R}^+)}$$

for any $\xi \ge 0$. Here U^1 and U^2 are the solutions of problem (L) corresponding to U_0^1 and U_0^2, respectively, constructed by front tracking method.

The rest of this paper is devoted to proving this theorem.

3 Solvability of Riemann problems

In this section we study solvability of several typical Riemann problems that appear in the process of front tracking. In the sequel, we will call the shock like that appearing in the background solution as the strong shock or the main shock, and all other waves/discontinuities that comes from the upstream supersonic flows or wave interactions as weak waves. Because the magnitude of the strong shock is much larger than the strengths of the other weak waves, they should be treated differently.

3.1 Standard Riemann problem generating only weak waves

We now consider the standard Riemann problem, that is, system (2.1) with the piecewise constant (supersonic) initial data

$$U|_{\xi=\tau} = \begin{cases} U_+ & \eta > \eta_0, \\ U_- & \eta < \eta_0, \end{cases} \tag{3.1}$$

where U_+ and U_- are the constant states which are regarded as the *above/right* state and *below/left* state with respect to the line $\eta = \eta_0$, respectively (cf. Figure 4).

The following solvability result is the well-known Lax's theorem (see, for example, Theorem 5.17 in [13, p.196] or Theorem 9.4.1 in [11, p.279]).

Lemma 3.1. *For given constant supersonic data U_0, there exists $\epsilon > 0$ such that for any states U_-, U_+ lie in the ball $O_\epsilon(U_0) \subset \mathbb{R}^4$ with radius ϵ and center U_0, the above Riemann*

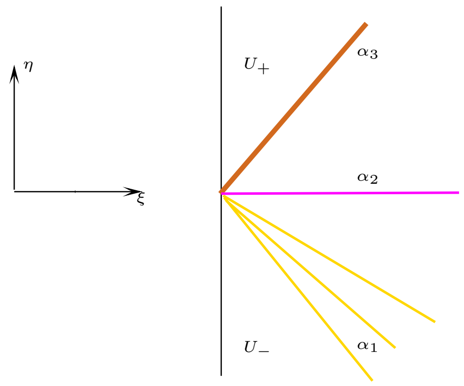

Figure 4: A solution to the standard Riemann problem.

problem admits a unique admissible solution consisting of three elementary waves. In addition, the state U_+ can be represented by

$$U_+ = \Phi(\alpha_3, \alpha_2, \alpha_1; U_-). \tag{3.2}$$

Here Φ is the wave curves determined by the system (2.1) *near U_0.*

It is well-known that one can use the parameters α_j to bound $|U_+ - U_-|$: there is a positive constant B depending continuously on U_0 and ϵ so that for $U_\pm$ connected by (3.2), there holds

$$\frac{1}{B}\sum_{j=1}^{3}|\alpha_j| \le |U_+ - U_-| \le B\sum_{j=1}^{3}|\alpha_j|.$$

For later applications, it is also important to express the Riemann solver from right (upper) state U_+ to left (lower) state U_- rather than the usual way given above. For $U_+ = \Phi_j(\alpha_j; U_-)$, we may have a C^2 map $U_- = \Psi_j(\alpha_j; U_+)$ with $\Psi_j(0; U) = U$ and $\partial_{\alpha_j}\Psi_j(0; U) = -r_j(U)$. So for $U_+ = \Phi(\alpha_3, \alpha_2, \alpha_1; U_-)$, we may express U_- in terms of U_+ by

$$U_- = \Psi(\alpha_1, \alpha_2, \alpha_3; U_+) = \Psi_1(\alpha_1; \Psi_2(\alpha_2; \Psi_3(\alpha_3; U_+))),$$

and of course there holds $\Psi(0,0,0;U) = U$, as well as $\partial_{\alpha_j}\Psi(0,0,0;U) = -r_j(U)$.

3.2 Boundary Riemann problems that generating only weak waves

We show the following boundary Riemann problem with the boundary data $p = \underline{p}_b$ on the characteristic boundary $\{\eta = 0\}$ is uniquely solvable (cf. Figure 5).

Lemma 3.2. *Consider the following boundary Riemann problem:*

$$\begin{cases} (2.1) & \textit{in}\ \ \xi > 0, \eta > 0, \\ U = U_+ & \textit{on}\ \ \xi = 0, \eta > 0, \\ p = \underline{p}_b & \textit{on}\ \ \xi > 0, \eta = 0. \end{cases} \tag{3.3}$$

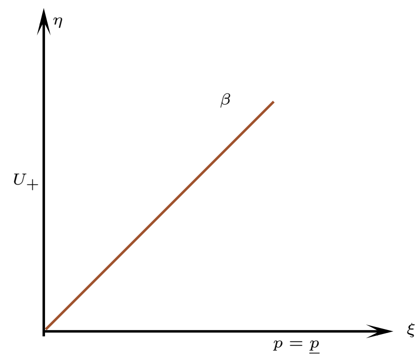

Figure 5: A 3-shock in a solution of the boundary Riemann problem.

There exists $\varepsilon > 0$ so that, if U_+ lies in the ball $O_\varepsilon(\underline{U}_\mathrm{b})$ with center $\underline{U}_\mathrm{b}$ and radius ε, then there is a unique admissible solution that contains only a 3-wave.

Proof. 1. Note that system (2.1) is strictly hyperbolic for $u > c$. For each point U with $u > c$, in its small neighborhood, we have the C^2-wave curves $\Phi_j(\alpha; U), j = 1, 2, 3$, so that $\Phi_j(0; U) = U$ and $\frac{\mathrm{d}\Phi_j}{\mathrm{d}\alpha}|_{\alpha=0} = r_j(U)$. $\Phi_j(\alpha; U)$ is connected to U from the upper side by a simple wave of j-family with strength $|\alpha|$: for $\alpha > 0$ and $j = 1, 3$, this wave is a rarefaction wave; while for $\alpha < 0$ and $j = 1, 3$, this wave is a shock. For $j = 2$, the wave is always a characteristic discontinuity.

For our purpose, we note that there is also a C^2-curve $\Psi_3(\beta; U)$ which consists of those states that can be connected to U from below by a 3-wave of strength β. We have $\Psi_3(0; U) = U$ and $\frac{\mathrm{d}\Phi_j}{\mathrm{d}\beta}|_{\beta=0} = -r_3(U)$.

2. We set $U = (u, v, p)^\top$ in Lagrangian coordinates and use $U_{(3)}$ to represent the third argument of the vector U (i.e. p). Then, to solve the boundary Riemann problem, it suffices to show that there exists a unique β so that $(\Psi_3(\beta; U_+))_{(3)} = \underline{p}_b$. Therefore, we consider the following function:

$$L(\beta; U_+) = \Psi_3(\beta; U_+))_{(3)} - \Psi_3(0; \underline{U}_\mathrm{b}))_{(3)}.$$

It is clear that $L(0; \underline{U}_\mathrm{b}) = 0$, and

$$\frac{\partial L(0; \underline{U}_\mathrm{b})}{\partial \beta} = -r_3(\underline{U}_\mathrm{b})_{(3)} = K_0 \triangleq \Big(\kappa_3 \lambda_3 u\Big)\Big|_{\underline{U}_\mathrm{b}} < 0.$$

By the implicit function theorem, there exists $\epsilon > 0$ such that, for $U_+ \in O_\epsilon(\underline{U}_\mathrm{b})$, there is a function $\beta = \beta(U_+)$ so that $L(\beta(U_+); U_+) = 0$. Then using the Taylor expansion up to second order (recall that Ψ_j is C^2 except at $\beta = 0$), we obtain the following estimate:

$$\beta = K(p_+ - \underline{p}_b) + O(1)|U_+ - \underline{U}_\mathrm{b}|^2, \tag{3.4}$$

with a constant $K = -1/K_0 > 0$ depending only on $\underline{U}_\mathrm{b}$. □

3.3 Boundary Riemann problem generating a strong shock

We have shown the existence of background solutions. Now we consider the boundary Riemann problem when U_+ is a small perturbations of the background state $\underline{U}_a$. A strong shock should resolve this problem.

Lemma 3.3. *For the boundary Riemann problem* (3.3)*, there exist constants C and $\varepsilon > 0$ so that, if U_+ lies in the ball $O_\varepsilon(\underline{U}_{\mathrm{a}})$, then there is a unique admissible solution that contains only a strong* 3*-shock with speed s'. The state of the flow behind the shock-front lies in $O_{C\varepsilon}(\underline{U}_{\mathrm{b}})$, and $|s' - \underline{s}| \le C\varepsilon$.*

Proof. 1. Let s' be a parameter, we may write the Rankine-Hugoniot conditions (2.13) as $U = \Upsilon(s'; U_+)$, with U being the state of the gas behind the shock-front. Set

$$L(s', U_+) = \Upsilon(s'; U_+)_{(3)} - \underline{p} = \big(\Upsilon(s'; U_+) - \Upsilon(\underline{s}; \underline{U}_{\mathrm{a}})\big)_{(3)}.$$

Recall that $\underline{U}_{\mathrm{b}} = \Upsilon(\underline{s}; \underline{U}_{\mathrm{a}})$, we have $L(\underline{s}, \underline{U}_{\mathrm{a}}) = 0$.

2. Next we compute $\partial_{s'} L(\underline{s}, \underline{U}_{\mathrm{a}}) = \partial_{s'} \Upsilon(\underline{s}, \underline{U}_{\mathrm{a}})_{(3)}$. Consider U as a function of s' for fixed U_+ in (2.13), we have

$$\begin{aligned}
\left[\frac{1}{\rho u}\right] - \frac{s'}{\rho u}\left(\frac{1}{u}\frac{\partial u}{\partial s'} + \frac{1}{\rho}\frac{\partial \rho}{\partial s'}\right) + \frac{\partial w}{\partial s'} &= 0,\\
\left[u + \frac{p}{\rho u}\right] + s'\left(\frac{\partial u}{\partial s'} + \frac{1}{\rho u}\frac{\partial p}{\partial s'} - \frac{p}{\rho u}\left(\frac{1}{u}\frac{\partial u}{\partial s'} + \frac{1}{\rho}\frac{\partial \rho}{\partial s'}\right)\right) + p\frac{\partial w}{\partial s'} + w\frac{\partial p}{\partial s'} &= 0,\\
[v] + s'\frac{\partial v}{\partial s'} - \frac{\partial p}{\partial s'} &= 0.
\end{aligned}$$

Note that

$$\frac{\partial w}{\partial s'} = \frac{1}{u}\frac{\partial v}{\partial s'} - \frac{v}{u^2}\frac{\partial u}{\partial s'}$$

and from the Bernoulli law:

$$u\frac{\partial u}{\partial s'} + v\frac{\partial v}{\partial s'} + \frac{\gamma}{\gamma - 1}\left(\frac{1}{\rho}\frac{\partial p}{\partial s'} - \frac{p}{\rho^2}\frac{\partial \rho}{\partial s'}\right) = 0,$$

that is,

$$\frac{1}{\rho}\frac{\partial \rho}{\partial s'} = \frac{\gamma - 1}{c^2}\left(u\frac{\partial u}{\partial s'} + v\frac{\partial v}{\partial s'} + \frac{\gamma}{\gamma - 1}\frac{1}{\rho}\frac{\partial p}{\partial s'}\right).$$

Then we get that

$$\begin{aligned}
\left(\frac{s'}{\rho u}\left(1 + (\gamma - 1)\frac{u^2}{c^2}\right) + \frac{v}{u}\right)\frac{1}{u}\frac{\partial u}{\partial s'} - \left(1 - \frac{s'}{\rho}\frac{\gamma - 1}{c^2}v\right)\frac{1}{u}\frac{\partial v}{\partial s'} + \frac{s'}{\rho u}\frac{1}{p}\frac{\partial p}{\partial s'} &= \left[\frac{1}{\rho u}\right],\\
\left(\frac{s'u}{\gamma}\left(1 - \frac{c^2}{u^2}\right) - \frac{pv}{u}\right)\frac{1}{u}\frac{\partial u}{\partial s'} + \left(p - s'\frac{\gamma - 1}{\gamma}v\right)\frac{1}{u}\frac{\partial v}{\partial s'} + pw\frac{1}{p}\frac{\partial p}{\partial s'} &= -\left[u + \frac{p}{\rho u}\right],\\
s'u\frac{1}{u}\frac{\partial v}{\partial s'} - p\frac{1}{p}\frac{\partial p}{\partial s'} &= -[v].
\end{aligned}$$

We consider this as a linear algebraic system for the unknowns $\frac{1}{u}\frac{\partial u}{\partial s'}$, $\frac{1}{u}\frac{\partial v}{\partial s'}$ and $\frac{1}{p}\frac{\partial p}{\partial s'}$. The determinant of the coefficient matrix is

$$\Delta \triangleq -s'pu\left(1+w^2+2w\left(\frac{s'}{\rho u}\right)+(1-\frac{u^2}{c^2})\left(\frac{s'}{\rho u}\right)^2\right).$$

We claim that $\Delta|_{U=\underline{U}_{\mathrm{b}},s'=\underline{s}}\neq 0$. Suppose this is not true, then set $t=s'/(\rho u)$ and there must hold $(1-\frac{u^2}{c^2})t^2+2wt+w^2+1=0$, or, recall $t>0$ for the background solution, that $t=\left.\frac{w+\sqrt{\frac{u^2+v^2}{c^2}-1}}{\frac{u^2}{c^2}-1}\right|_{\underline{U}_{\mathrm{b}}}$. In other words, it holds that

$$\underline{s}=\left.\frac{\rho uc^2}{u^2-c^2}(w+\sqrt{M^2-1})\right|_{\underline{U}_{\mathrm{b}}}=\lambda_3(\underline{U}_{\mathrm{b}}).$$

This means that the background shock-front is a characteristic curve, while we know that this is impossible since such shocks satisfy the Lax entropy condition (see [11, p.242] or [13, p.189]), which requires that $\lambda_3(\underline{U}_{\mathrm{b}})>\underline{s}$. From this we also see that $\Delta<0$ at the background state.

We also obtain that (at the background state $U=\underline{U}_{\mathrm{b}}$, $s'=\underline{s}$)

$$\Delta_3\triangleq\det\begin{pmatrix}\left(\frac{\underline{s}}{\rho u}\left(1+(\gamma-1)\frac{u^2}{c^2}\right)+\frac{v}{u}\right) & -\left(1-\frac{\underline{s}}{\rho}\frac{\gamma-1}{c^2}v\right) & -\frac{w}{\underline{s}}\\ \left(\frac{\underline{s}u}{\gamma}\left(1-\frac{c^2}{u^2}\right)-\frac{pv}{u}\right) & p-\frac{\gamma-1}{\gamma}\underline{s}v & \frac{pw}{\underline{s}}\\ 0 & \underline{s}u & -v\end{pmatrix}=\underline{s}uv\left(\frac{\underline{s}v}{p}\frac{\gamma-1}{\gamma}-2\right).$$

It is nonzero. Otherwise we should have $[p]=\underline{s}v=\frac{2\gamma p}{\gamma-1}$, that is, $p>p-\underline{p}_{\mathrm{a}}=\frac{2\gamma p}{\gamma-1}>p$, a contradiction! Actually we see that $\Delta_3<0$.

Therefore by Cramer's rule, we have $\partial_{s'}L(\underline{s};\underline{U}_{\mathrm{a}})=\frac{\partial p}{\partial s'}=\left(p\frac{\Delta_3}{\Delta}\right)>0$ at the background state.

3. Hence by implicit function theorem, we could find a C^1 function $s'=\mathscr{S}(U_+)$ so that $\underline{s}=\mathscr{S}(\underline{U}_{\mathrm{a}})$ and $L(\mathscr{S}(U_+);U_+)=0$ in a neighborhood $O_\varepsilon(\underline{U}_{\mathrm{a}})$. Then by continuity of Υ on s' and U_+, we find a constant $C>0$ so that $\Upsilon(\mathscr{S}(U_+);U_+)\in O_{C\varepsilon}(\underline{U}_{\mathrm{b}})$. Since $\lambda_3(\Upsilon(\underline{s};\underline{U}_{\mathrm{a}}))>\underline{s}$, by continuity we also have $\lambda_3(\Upsilon(\mathscr{S}(U_+);U_+))>\mathscr{S}(U_+)$ for $U_+\in O_{C\varepsilon}(\underline{U}_{\mathrm{b}})$. Thus the discontinuity we constructed is a 3-shock. $\square$

3.4 Riemann problem involving a strong shock

Lemma 3.4. *There exist positive constants C,ε so that for $U_+\in O_\varepsilon(\underline{U}_{\mathrm{a}})$ and $U_-\in O_{C\varepsilon}(\underline{U}_{\mathrm{b}})$, there is uniquely one admissible solution to the Riemann problem with initial data* (3.1), *which contains a strong 3-shock and two weak waves of the first and second characteristic family.*

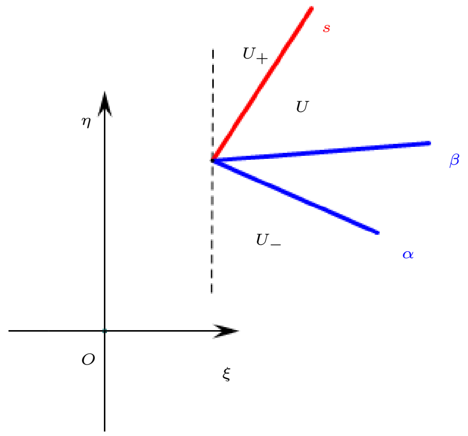

Figure 6: The solution of Riemann problem (3.1) consists of a strong shock (red line) and two weak waves (blue line).

Proof. 1. As illustrated in Figure 6, if such a solution does exist, it must satisfy

$$U_+ = \Upsilon(s, \Phi_2(\beta, \Phi_1(\alpha, U_-))),$$

where $U_+ = \Upsilon(s, U)$ is the Rankine-Hugoniot conditions, and Φ_i represents the wave curve of the i-th family in a neighborhood of $\underline{U}_{\mathrm{b}}$. In the following, we will employ the implicit function theorem to complete the proof.

2. Set

$$L(s, \beta, \alpha; U_-, U_+) = \Upsilon(s, \Phi_2(\beta, \Phi_1(\alpha, U_-))) - U_+.$$

We already know that $L(\underline{s}, 0, 0; \underline{U}_{\mathrm{b}}, \underline{U}_{\mathrm{a}}) = 0$, hence there are constants C and ε such that for $U_+ \in O_\varepsilon(\underline{U}_{\mathrm{a}})$ and $U_- \in O_{C\varepsilon}(\underline{U}_{\mathrm{b}})$, we can solve uniquely one triple (s, β, α) depending continuously on (U_+, U_-), if the Jacobian $\frac{\partial L}{\partial(s,\beta,\alpha)}$ is nonsingular at $(s, \beta, \alpha; U_-, U_+) = (\underline{s}, 0, 0; \underline{U}_{\mathrm{b}}, \underline{U}_{\mathrm{a}})$. We see that

$$\frac{\partial L}{\partial(s, \beta, \alpha)} = \left(\frac{\partial \Upsilon(s, U)}{\partial s}, \frac{\partial \Upsilon(s, U)}{\partial U} \cdot \frac{\partial U}{\partial \beta}, \frac{\partial \Upsilon(s, U)}{\partial U} \cdot \frac{\partial U}{\partial \alpha} \right), \tag{3.5}$$

and

$$\left. \frac{\partial U}{\partial \beta} \right|_{(\underline{s},0,0,\underline{U}_{\mathrm{b}},\underline{U}_{\mathrm{a}})} = r_2(\underline{U}_{\mathrm{b}}), \tag{3.6}$$

$$\left. \frac{\partial U}{\partial \alpha} \right|_{(\underline{s},0,0,\underline{U}_{\mathrm{b}},\underline{U}_{\mathrm{a}})} = r_1(\underline{U}_{\mathrm{b}}). \tag{3.7}$$

Note that at the background state $(\underline{s}, 0, 0, \underline{U}_{\mathrm{b}}, \underline{U}_{\mathrm{a}})$, the state U appeared above should be $\underline{U}_{\mathrm{b}}$.

3. In order to evaluate $\frac{\partial \Upsilon(s,U)}{\partial s}$ and $\frac{\partial \Upsilon(s,U)}{\partial U}$ at $(\underline{s}, 0, 0, \underline{U}_{\mathrm{b}}, \underline{U}_{\mathrm{a}})$, we differentiate the Rankine–

Hugoniot conditions $U_+ = \Upsilon(s, U)$ with respect to s and U respectively.

The Rankine–Hugoniot conditions can be written in a symmetric form as follow:

$$sF(U_+) + G(U_+) = sF(U) + G(U), \tag{3.8}$$

where

$$F(U) = \begin{pmatrix} \frac{1}{\rho u} \\ u + \frac{p}{\rho u} \\ v \end{pmatrix}, \quad \text{and} \quad G(U) = \begin{pmatrix} w \\ pw \\ -p \end{pmatrix}. \tag{3.9}$$

Differentiation of (3.8) leads to

$$Q(U_+)\frac{\partial U_+}{\partial U} = Q(U), \tag{3.10}$$

$$Q(U_+)\frac{\partial U_+}{\partial s} = F(U) - F(U_+), \tag{3.11}$$

where

$$Q(U) = s\frac{\partial F(U)}{\partial U} + \frac{\partial G(U)}{\partial U}. \tag{3.12}$$

If $Q(U_+)$ (with $U_+ = \underline{U}_{\mathrm{a}}$) is nonsingular, we could multiply it on the left-hand side of (3.5) to simplify some computations later. Fortunately, $Q(U_+)$ is indeed nonsingular unless s equals either λ_1, λ_2 or λ_3, which is impossible due to the Lax entropy conditions of shocks.

4. We now verify the claim. A straightforward and direct calculation yields

$$\frac{\partial F(U)}{\partial U} = \begin{pmatrix} \frac{\partial}{\partial u}(\frac{1}{\rho u}) & \frac{\partial}{\partial v}(\frac{1}{\rho u}) & \frac{\partial}{\partial p}(\frac{1}{\rho u}) \\ p\frac{\partial}{\partial u}(\frac{1}{\rho u}) + 1 & p\frac{\partial}{\partial v}(\frac{1}{\rho u}) & p\frac{\partial}{\partial p}(\frac{1}{\rho u}) + \frac{1}{\rho u} \\ 0 & 1 & 0 \end{pmatrix}, \tag{3.13}$$

$$\frac{\partial G(U)}{\partial U} = \begin{pmatrix} \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} & 0 \\ p\frac{\partial w}{\partial u} & p\frac{\partial w}{\partial v} & w \\ 0 & 0 & -1 \end{pmatrix}, \tag{3.14}$$

hence

$$Q(U) = \begin{pmatrix} s\frac{\partial}{\partial u}(\frac{1}{\rho u}) + \frac{\partial w}{\partial u} & s\frac{\partial}{\partial v}(\frac{1}{\rho u}) + \frac{\partial w}{\partial v} & s\frac{\partial}{\partial p}(\frac{1}{\rho u}) \\ ps\frac{\partial}{\partial u}(\frac{1}{\rho u}) + p\frac{\partial w}{\partial u} + s & sp\frac{\partial}{\partial v}(\frac{1}{\rho u}) + p\frac{\partial w}{\partial v} & ps\frac{\partial}{\partial p}(\frac{1}{\rho u}) + \frac{s}{\rho u} + w \\ 0 & s & -1 \end{pmatrix}. \tag{3.15}$$

We also introduce a matrix $M_{21}(-p) = \begin{pmatrix} 1 & 0 & 0 \\ -p & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Set $Q'(U) = M_{21}(-p)Q(U)$. In the following we actually multiply $Q'(U)$ on the left-hand side of (3.5) instead of $Q(U)$. We calculate that

$$
\begin{aligned}
Q'(U) &= \begin{pmatrix} s\frac{\partial}{\partial u}(\frac{1}{\rho u}) + \frac{\partial w}{\partial u} & s\frac{\partial}{\partial v}(\frac{1}{\rho u}) + \frac{\partial w}{\partial v} & s\frac{\partial}{\partial p}(\frac{1}{\rho u}) \\ s & 0 & \frac{s}{\rho u} + w \\ 0 & s & -1 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\gamma-1}{\gamma}\cdot\frac{s}{p} - \frac{1}{u}(\frac{s}{\rho u} + w) & -\frac{\gamma-1}{\gamma}\cdot\frac{s}{p}w + \frac{1}{u} & -\frac{1}{p}\cdot\frac{s}{\rho u} \\ s & 0 & \frac{s}{\rho u} + w \\ 0 & s & -1 \end{pmatrix}, \qquad (3.16)
\end{aligned}
$$

and

$$\det Q(U) = \det Q'(U) = \frac{s}{u}(1 - \frac{u^2}{c^2})(\frac{s}{\rho u} - \frac{\lambda_1}{\rho u})(\frac{s}{\rho u} - \frac{\lambda_3}{\rho u}). \qquad (3.17)$$

Hence $\det Q(U_+) \neq 0$.

5. Now our task is to verify that $Q(U_+)\frac{\partial L}{\partial(s,\beta,\alpha)}$ is nonsingular if $(s, \beta, \alpha; U_-, U_+) = (\underline{s}, 0, 0; \underline{U}_{\mathrm{b}}, \underline{U}_{\mathrm{a}})$. Noticing (3.10) and (3.11), we have

$$Q(U_+)\frac{\partial L}{\partial(s,\beta,\alpha)} = \left(F(U) - F(U_+), Q(U)\frac{\partial U}{\partial \beta}, Q(U)\frac{\partial U}{\partial \alpha}\right). \qquad (3.18)$$

In the following, we will write $(\underline{s}, \underline{U}_{\mathrm{b}}, \underline{U}_{\mathrm{a}})$ as (s, U, U_0) for simplicity. So the value of (3.18) at $(\underline{s}, 0, 0; \underline{U}_{\mathrm{b}}, \underline{U}_{\mathrm{a}})$ is simply $(F(U) - F(U_0), Q(U)r_2(U), Q(U)r_1(U))$. We find that the employment of $M_{21}(-p)$ will bring us a great deal of convenience. So we instead consider the matrix

$$(M_{21}(-p)[F(U) - F(U_0)], Q'(U)r_2(U), Q'(U)r_1(U)). \qquad (3.19)$$

We start to calculate $M_{21}(-p)[F(U) - F(U_0)]$. According to (3.9) and Rankine-Hugoniot conditions, we have

$$F(U)_{(1)} - F(U_0)_{(1)} = -\frac{1}{s}(G(U)_{(1)} - G(U_0)_{(1)}) = -\frac{w}{s}, \qquad (3.20)$$

$$F(U)_{(2)} - F(U_0)_{(2)} = -\frac{1}{s}(G(U)_{(2)} - G(U_0)_{(2)}) = -\frac{pw}{s}, \qquad (3.21)$$

$$F(U)_{(3)} - F(U_0)_{(3)} = v - v_0 = v. \qquad (3.22)$$

Thus, $F(U) - F(U_0) = (-\frac{w}{s}, -\frac{pw}{s}, v)^\top$ and

$$M_{21}(-p)[F(U) - F(U_0)] = (-\frac{w}{s}, 0, v)^\top. \qquad (3.23)$$

The second column $Q'(U)r_2(U)$ as well can be easily obtained:

$$Q'(U)r_2(U) = \left(-\frac{s}{\rho u}[1 + M^2(\gamma - 1)], su, sv\right)^\top. \qquad (3.24)$$

As for $Q'(U)r_1(U)$, recall by setting $r_1'(U) = (v, -u, 0)^\top$ and $r_1''(U) = (\frac{\lambda_1}{\rho}, 0, -\lambda_1)^\top$, then $r_1(U) = \kappa_1(U)(r_1'(U) + r_1''(U))$; and

$$Q'(U)r_1'(U) = \left(-(\frac{s}{\rho u} + w^2 + 1), sv, -su\right)^\top, \tag{3.25}$$

$$Q'(U)r_1''(U) = \left(-[(1 - \frac{u^2}{c^2})\frac{s}{\rho u} + w]\frac{\lambda_1}{\rho u}, -\lambda_1 v, \lambda_1 u\right)^\top. \tag{3.26}$$

All together we get that

$$\begin{aligned} Q'(U)r_1(U) &= \kappa_1(U)\begin{pmatrix} -(1-\frac{u^2}{c^2})\frac{s}{\rho u}\frac{\lambda_1}{\rho u} - \frac{\lambda_1}{\rho u}w - \frac{s}{\rho u}w - w^2 - 1 \\ (s-\lambda_1)v \\ (s-\lambda_1)(-u) \end{pmatrix} \\ &= \kappa_1(U)\begin{pmatrix} [(1-\frac{u^2}{c^2})\frac{\lambda_3}{\rho u} + w](\frac{s}{\rho u} - \frac{\lambda_1}{\rho u}) \\ (s-\lambda_1)v \\ (s-\lambda_1)(-u) \end{pmatrix}. \end{aligned} \tag{3.27}$$

Therefore we find

$$\begin{aligned} &(M_{21}(-p)[F(U) - F(U_0)], Q'(U)r_2(U), Q'(U)r_1(U)) \\ &= \begin{pmatrix} -\frac{w}{s} & -\frac{s}{\rho u}[1 + M^2(\gamma-1)] & \kappa_1(U)[(1-\frac{u^2}{c^2})\frac{\lambda_3}{\rho u} + w](\frac{s}{\rho u} - \frac{\lambda_1}{\rho u}) \\ 0 & su & \kappa_1(U)(s-\lambda_1)v \\ v & sv & \kappa_1(U)(s-\lambda_1)(-u) \end{pmatrix}. \end{aligned} \tag{3.28}$$

This matrix can be factored as the product $M_{11}(\frac{1}{\rho u})AM_{22}(s)M_{33}(\kappa_1(U)(s-\lambda_1))$, with the 3×3 matrix $M_{ii}(x)$ obtained by replacing the (i,i) entry of I_3 by x, and

$$A = \begin{pmatrix} -\frac{\rho v}{s} & -[1 + M^2(\gamma-1)] & (1-\frac{u^2}{c^2})\frac{\lambda_3}{\rho u} + w \\ 0 & u & v \\ v & v & -u \end{pmatrix}. \tag{3.29}$$

Clearly, none of the matrices $M_{ii}(i = 1, 2, 3)$ are singular. So we complete the proof if we show that $\det A \neq 0$. This is true since

$$\begin{aligned} \frac{s}{\rho u}\frac{\det A}{uv} &= 1 + w^2 - w[2 + M^2(\gamma-1)]\frac{s}{\rho u} - (1-\frac{u^2}{c^2})\frac{\lambda_1}{\rho u} \\ &= [(1 - \frac{\gamma-1}{\gamma}\cdot\frac{sv}{p})\frac{\lambda_3}{\rho u} + \frac{s}{\rho u}](1-\frac{u^2}{c^2})\frac{\lambda_1}{\rho u} \neq 0. \end{aligned} \tag{3.30}$$

. $\square$

4 Estimates on interactions of waves

In this section we consider four cases of wave interactions appearing in the front tracking algorithm which lead to changes of flow field, namely: i) collisions of two weak waves; ii) reflection of a weak wave off the boundary; iii) reflection of a weak wave from the strong shock; iv) refraction of a weak wave by the strong shock from above.

4.1 Collision of weak waves

The interaction of two waves can be resolved by solutions of the corresponding Riemann problems (cf. Figure 7). More importantly, we have the following well-known interaction estimates of weak waves (see Theorem 9.9.1 in [11, p.312] or (6.11) in [13, p.212]):

Lemma 4.1. *Suppose that U_+, U_m and U_- are three states in a small neighborhood of U_0 with $U_+ = \Phi(\alpha_3, \alpha_2, \alpha_1; U_m)$, $U_m = \Phi(\beta_3, \beta_2, \beta_1; U_-)$, and $U_+ = \Phi(\gamma_3, \gamma_2, \gamma_1; U_-)$. Then*

$$\gamma_j = \alpha_j + \beta_j + O(1)\triangle(\alpha, \beta), \tag{4.1}$$

where $\triangle(\alpha,\beta) = |\beta_3|(|\alpha_1| + |\alpha_2|) + |\beta_2||\alpha_1| + \sum_{j=1,3} \triangle_j(\alpha, \beta)$, with

$$\triangle_j(\alpha,\beta) = \begin{cases} 0, & \alpha_j \geq 0,\ \beta_j \geq 0, \\ |\alpha_j||\beta_j|, & \text{otherwise.} \end{cases}$$

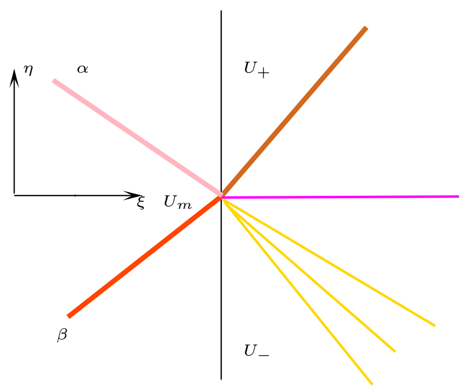

Figure 7: Interaction of two waves α, β.

4.2 Reflection of weak waves off boundary

Next we consider the reflection of a weak 1-wave off the boundary $\{\eta = 0\}$, that could only produce a weak 3-wave (cf. Figure 8).

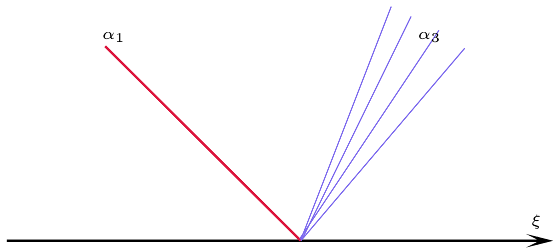

Figure 8: A 1-wave α_1 meets the boundary $\eta = 0$ and reflects to produce a 3-wave α_3.

Lemma 4.2. *Suppose that U^l, U^m, and U^r are three states in $O_\epsilon(\underline{U}_{\rm b})$ for sufficiently small ϵ, with $U^m = \Phi_1(\alpha_1; U^l) = \Phi_3(\alpha_3; U^r)$. Then*

$$\alpha_3 = -K_2\alpha_1 + M_2|\alpha_1|^2, \tag{4.2}$$

with the constant $K_2 > 1$ and the quantity M_2 bounded in $O_\epsilon(\underline{U}_{\rm b})$.

Proof. We have $U_r = \Psi_3(\alpha_3, \Phi_1(\alpha_1, U_l))$. By boundary conditions, it holds $(U_r)_{(3)} = (U_l)_{(3)}$. So we construct a function

$$L(\alpha_3, \alpha_1) = (\Psi_3(\alpha_3, \Phi_1(\alpha_1, U_l)) - U_l)_{(3)}.$$

Obviously $L(0,0) = 0$, and

$$\frac{\partial L}{\partial \alpha_3}(0,0) = -(r_3(U_l))_{(3)}, \quad \frac{\partial L}{\partial \alpha_1}(0,0) = (r_1(U_l))_{(3)}.$$

So expanding the function $\alpha_3 = \alpha_3(\alpha_1)$ at $\alpha_1 = 0$ leads to (4.2), with

$$K_2 = \frac{\partial \alpha_3}{\partial \alpha_1}(0) = -\frac{(r_1(U_l))_{(3)}}{(r_3(U_l))_{(3)}}.$$

We note that

$$\begin{aligned} -\frac{(r_1(\underline{U}_b))_{(3)}}{(r_3(\underline{U}_b))_{(3)}} &= -\frac{\kappa_1\lambda_1}{\kappa_3\lambda_3}(\underline{U}_b) = \left(\frac{\lambda_3}{\lambda_1}(\underline{U}_b)\right)^2 \\ &= \left(\frac{\sqrt{M^2-1}+w}{\sqrt{M^2-1}-w}\bigg|_{\underline{U}_b}\right)^2 > 1. \end{aligned}$$

Here we used the identity $\kappa_3/\kappa_1 = -\lambda_1^3/\lambda_3^3$ proved in (2.11), and the fact that $w > 0$ for the background state $\underline{U}_b$. Therefore, $K_2 > 1$ in a neighborhood of $\underline{U}_b$ as claimed. $\square$

4.3 Reflection of weak waves off strong shock

Now we consider the case that a weak 3-wave meets the strong shock and then reflects to produce two weak waves and a deflected strong shock. See Figure 9.

Lemma 4.3. *Suppose that* $U_+ \in O_\varepsilon(\underline{U}_{\rm a})$ *and* $U_-, U_-^l, U_-^r \in O_\varepsilon(\underline{U}_{\rm b})$ *for sufficiently small* ε, *with* $U_- = \Psi_3(\alpha_3, U_-^l) = \Psi_1(\beta_1, \Psi_2(\beta_2, U_-^r))$, *and* $U_+ = \Upsilon(s_k, U_-^l) = \Upsilon(s_{k+1}, U_-^r)$. *Then*

$$s_{k+1} - s_k = C_{33}\alpha_3 + C_3'(|\alpha_3(s_k - \underline{s})| + |\alpha_3|^2), \tag{4.3}$$

$$\beta_2 = C_{23}\alpha_3 + C_2'(|\alpha_3(s_k - \underline{s})| + |\alpha_3|^2), \tag{4.4}$$

$$\beta_1 = C_{13}\alpha_3 + C_1'(|\alpha_3(s_k - \underline{s})| + |\alpha_3|^2), \tag{4.5}$$

where C_{13} *and* C_{23}, C_{33} *are constants depending only on the background solution* $\underline{U}$, *and particularly,*

$$C_{13} = -\frac{(s-\lambda_3)}{(s-\lambda_1)} \cdot \frac{\left[\left(1 - \frac{\gamma-1}{\gamma}\frac{sv}{p}\right)\lambda_1 + s\right]}{\left[\left(1 - \frac{\gamma-1}{\gamma}\frac{sv}{p}\right)\lambda_3 + s\right]} \cdot \frac{\lambda_1^2}{\lambda_3^2}\Bigg|_{U=\underline{U}_b, s=\underline{s}}. \tag{4.6}$$

Also, C_i' $(i = 1, 2, 3)$ *are bounded quantities with bounds determined by the background solution.*

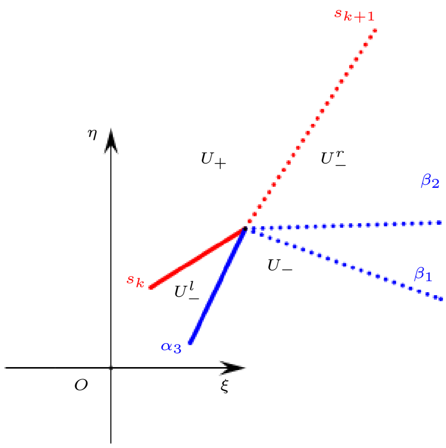

Figure 9: A weak 3–wave (blue solid line) meets the strong shock s_k (red solid line) from below and reflects to produce a 1–wave β_1 and a 2–wave β_2 (blue dot line), while the strong shock is deflected (red dot line) to s_{k+1}.

Proof. 1. According to Lemma 3.4, there exists a vector-valued C^2 function f, such that

$(s_{k+1}-\underline{s},\beta_2,\beta_1)=f(s_k-\underline{s},\alpha_i)$. Since

$$f(x,y)=-f(0,0)+f(x,0)+f(0,y)+xy\int_0^1\int_0^1\frac{\partial^2 f}{\partial x\partial y}(tx,sy)\,\mathrm{d}t\,\mathrm{d}s,$$

and note that $f(0,0)=\mathbf{0}$, while D^2f is bounded, we have

$$\begin{aligned}f(s_k-\underline{s},\alpha_i)&=-f(0,0)+f(s_k-\underline{s},0)+f(0,\alpha_i)+O(\alpha_i(s_k-\underline{s}))\\&=f(s_k-\underline{s},0)+\frac{\partial f}{\partial\alpha_i}\alpha_i+O(\alpha_i^2+\alpha_i(s_k-\underline{s}))\\&=(s_k-\underline{s},0)+C\alpha_i+C'(\alpha_i^2+\alpha_i(s_k-\underline{s})).\end{aligned}\tag{4.7}$$

So in the following, we need to calculate

$$C=\frac{\partial(s_{k+1}-\underline{s},\beta_2,\beta_1)}{\partial(\alpha_3,\alpha_2,\alpha_1)}\Big|_{(\underline{s},0,0;\underline{U}_{\mathrm{b}},\underline{U}_{\mathrm{a}})}=\begin{pmatrix}C_{33}&C_{32}&C_{31}\\C_{23}&C_{22}&C_{21}\\C_{13}&C_{12}&C_{11}\end{pmatrix}.\tag{4.8}$$

Of course we will see that only the first column $(C_{33},C_{23},C_{13})^\top$ is necessary, because the other two columns will turn out to be $(0,1,0)^T$ and $(0,0,1)^T$, which matches the fact that α_1 and α_2 could never meet the strong shock from below.

2. By chain rule we have

$$\frac{\partial U_-}{\partial\alpha_i}=\frac{\partial U_-}{\partial(s_{k+1},\beta_2,\beta_1)}\cdot\frac{\partial(s_{k+1}-\underline{s},\beta_2,\beta_1)}{\partial\alpha_i}=(\frac{\partial U_-}{\partial U_-^r}\frac{\partial U_-^r}{\partial s_{k+1}},\frac{\partial U_-}{\partial\beta_2},\frac{\partial U_-}{\partial\beta_1})\cdot\frac{\partial(s_{k+1}-\underline{s},\beta_2,\beta_1)}{\partial\alpha_i}.\tag{4.9}$$

Evaluating this equality at the state $\alpha_i=0$, $\beta_1=\beta_2=0$ and $s_k=s_{k+1}=\underline{s}$ (so $\frac{\partial U_-}{\partial U_-^r}=I_3$, and $U_-^r=U_-$), because of (3.10), we get

$$Q(U_-)\frac{\partial U_-}{\partial\alpha_i}=(Q(U_-^r)\frac{\partial U_-^r}{\partial s_{k+1}},Q(U_-)\frac{\partial U_-}{\partial\beta_2},Q(U_-)\frac{\partial U_-}{\partial\beta_1})\cdot\frac{\partial(s_{k+1}-\underline{s},\beta_2,\beta_1)}{\partial\alpha_i}.\tag{4.10}$$

If we plug the background state $U_-=\underline{U}_{\mathrm{b}}$ into (4.10), it follows that

$$-Q(\underline{U}_{\mathrm{b}})r_i(\underline{U}_{\mathrm{b}})=(F(\underline{U}_{\mathrm{a}}-F(\underline{U}_{\mathrm{b}}),-Q(\underline{U}_{\mathrm{b}})r_2(\underline{U}_{\mathrm{b}}),-Q(\underline{U}_{\mathrm{b}})r_1(\underline{U}_{\mathrm{b}}))\cdot C_i.\tag{4.11}$$

Here we have used the fact that $U_-^r=\Upsilon(s_{k+1},U_+)$ which follows from $U_+=\Upsilon(s_{k+1},U_-^r)$.

For convenience, we write (4.11) all together as

$$-Q(U)(r_3(U),r_2(U),r_1(U))=(F(U_0)-F(U),-Q(U)r_2(U),-Q(U)r_1(U))\cdot C,\tag{4.12}$$

which is equivalent to

$$(F(U)-F(U_0), Q(U)r_2(U), Q(U)r_1(U))\cdot C = Q(U)(r_3(U), r_2(U), r_1(U)). \tag{4.13}$$

3. Now we find that the effect of C is just to convert the first column $F(U)-F(U_0)$ to $Q(U)r_3(U)$, while the second and the third column of C should be $(0,1,0)^\top$ and $(0,0,1)^\top$ respectively. So the first column of C, that is, $C_3 = (C_{33}, C_{23}, C_{13})^\top$, is our only concern.

To solve C, we resort again to $M_{21}(-p)$ and multiply it to both sides of (4.13) from the left. This leads to

$$(M_{21}(-p)(F(U)-F(U_0)), Q'(U)r_2(U), Q'(U)r_1(U))\cdot C = Q'(U)(r_3(U), r_2(U), r_1(U)), \tag{4.14}$$

which is equivalent to

$$\begin{aligned} & A\cdot M_{22}(s)\cdot M_{33}(\kappa_1(U)(s-\lambda_1))\cdot C \\ = \ & B\cdot M_{11}(\kappa_3(s-\lambda_3))M_{22}(s)\cdot M_{33}(\kappa_1(U)(s-\lambda_1)). \end{aligned} \tag{4.15}$$

Here, A is the same as in (3.29), and

$$B = \begin{pmatrix} (1-\frac{u^2}{c^2})\frac{\lambda_1}{\rho u}+w & -[1+M^2(\gamma-1)] & (1-\frac{u^2}{c^2})\frac{\lambda_3}{\rho u}+w \\ v & u & v \\ -u & v & -u \end{pmatrix}. \tag{4.16}$$

Because only $(A^{-1}B)_{(1)}$ (which is the first column of $A^{-1}B$) is necessary, we will not bother with the rest of columns. In accordance with C, we write A as (A_3, A_2, A_1) and B as (B_3, B_2, B_1), while $A_i = (a_{3i}, a_{2i}, a_{1i})^\top, B_i = (b_{3i}, b_{2i}, b_{1i})^\top$ are column vectors. Notice that $A_2 = B_2$ and $A_1 = B_1$. Then

$$\begin{aligned} (A^{-1}B)_{(1)} &= A^{-1}B_3 = A^{-1}A_1 + A^{-1}(B_3 - A_1) = A^{-1}A_1 + A^{-1}(B_3 - B_1) \\ &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (1-\frac{u^2}{c^2})(\frac{\lambda_1}{\rho u} - \frac{\lambda_3}{\rho u})\cdot\frac{1}{|A|}\cdot\begin{pmatrix} -u^2-v^2 \\ v^2 \\ -uv \end{pmatrix} \\ &= \begin{pmatrix} -(1-\frac{u^2}{c^2})(\frac{\lambda_1}{\rho u}-\frac{\lambda_3}{\rho u})\frac{u^2+v^2}{|A|} \\ (1-\frac{u^2}{c^2})(\frac{\lambda_1}{\rho u}-\frac{\lambda_3}{\rho u})\frac{v^2}{|A|} \\ 1-(1-\frac{u^2}{c^2})(\frac{\lambda_1}{\rho u}-\frac{\lambda_3}{\rho u})\frac{uv}{|A|} \end{pmatrix}. \end{aligned} \tag{4.17}$$

According to (4.16), we can solve C_3 out as follows:

$$
\begin{aligned}
&M_{33}\left(\frac{1}{\kappa_1(U)(s-\lambda_1)}\right)M_{22}\left(\frac{1}{s}\right)(A^{-1}B)_{(1)}(\kappa_3(U)(s-\lambda_3))\\
&=\begin{pmatrix}
-\frac{1}{|A|}\left(1-\frac{u^2}{c^2}\right)\left(\frac{\lambda_1}{\rho u}-\frac{\lambda_3}{\rho u}\right)(u^2+v^2)\kappa_3(U)(s-\lambda_3)\\
\\
\frac{1}{s}\frac{1}{|A|}\left(1-\frac{u^2}{c^2}\right)\left(\frac{\lambda_1}{\rho u}-\frac{\lambda_3}{\rho u}\right)v^2\kappa_3(U)(s-\lambda_3)\\
\\
\frac{\kappa_3(U)(s-\lambda_3)}{k_1(U)(s-\lambda_1)}\left[1-\frac{1}{|A|}\left(1-\frac{u^2}{c^2}\right)\left(\frac{\lambda_1}{\rho u}-\frac{\lambda_3}{\rho u}\right)uv\right]
\end{pmatrix}.
\end{aligned}
\tag{4.18}
$$

From (3.30), we know that

$$
\begin{aligned}
&\frac{s}{\rho u}\frac{|A|}{uv}-\left(1-\frac{u^2}{c^2}\right)\left(\frac{\lambda_1}{\rho u}-\frac{\lambda_3}{\rho u}\right)\frac{s}{\rho u}\\
&=\left[\left(1-\frac{\gamma-1}{\gamma}\frac{sv}{p}\right)\frac{\lambda_3}{\rho u}+\frac{s}{\rho u}\right]\left(1-\frac{u^2}{c^2}\right)\frac{\lambda_1}{\rho u}-\left(1-\frac{u^2}{c^2}\right)\left(\frac{\lambda_1}{\rho u}-\frac{\lambda_3}{\rho u}\right)\frac{s}{\rho u}\\
&=\left[\left(1-\frac{\gamma-1}{\gamma}\frac{sv}{p}\right)\frac{\lambda_1}{\rho u}+\frac{s}{\rho u}\right]\left(1-\frac{u^2}{c^2}\right)\frac{\lambda_3}{\rho u},
\end{aligned}
\tag{4.19}
$$

hence,

$$
\begin{aligned}
1-\frac{1}{|A|}\left(1-\frac{u^2}{c^2}\right)\left(\frac{\lambda_1}{\rho u}-\frac{\lambda_3}{\rho u}\right)uv&=\frac{\frac{s}{\rho u}\frac{|A|}{uv}-\left(1-\frac{u^2}{c^2}\right)\left(\frac{\lambda_1}{\rho u}-\frac{\lambda_3}{\rho u}\right)\frac{s}{\rho u}}{\frac{s}{\rho u}\frac{|A|}{uv}}\\
&=\frac{\left[\left(1-\frac{\gamma-1}{\gamma}\frac{sv}{p}\right)\frac{\lambda_1}{\rho u}+\frac{s}{\rho u}\right]\left(1-\frac{u^2}{c^2}\right)\frac{\lambda_3}{\rho u}}{\left[\left(1-\frac{\gamma-1}{\gamma}\frac{sv}{p}\right)\frac{\lambda_3}{\rho u}+\frac{s}{\rho u}\right]\left(1-\frac{u^2}{c^2}\right)\frac{\lambda_1}{\rho u}}\\
&=\frac{\left[\left(1-\frac{\gamma-1}{\gamma}\frac{sv}{p}\right)\lambda_1+s\right]\lambda_3}{\left[\left(1-\frac{\gamma-1}{\gamma}\frac{sv}{p}\right)\lambda_3+s\right]\lambda_1}.
\end{aligned}
\tag{4.20}
$$

In Lemma 2.1 we showed that

$$
\frac{2}{u}\left[\left(1-\frac{u^2}{c^2}\right)\frac{\lambda}{\rho u}+w\right]=(1+\gamma)\frac{u^2}{c^2}\frac{\lambda}{\rho u}\frac{\lambda^2}{\rho c^2}\cdot\kappa(U),
$$

therefore

$$
\frac{\kappa_3}{\kappa_1}=\frac{\left(1-\frac{u^2}{c^2}\right)\frac{\lambda_3}{\rho u}+w}{\left(1-\frac{u^2}{c^2}\right)\frac{\lambda_1}{\rho u}+w}\cdot\frac{\lambda_1^3}{\lambda_3^3}=-\frac{\lambda_1^3}{\lambda_3^3},
\tag{4.21}
$$

and

$$C_{13} = -\frac{(s-\lambda_3)}{(s-\lambda_1)} \cdot \frac{\left[\left(1-\frac{\gamma-1}{\gamma}\frac{sv}{p}\right)\lambda_1 + s\right]}{\left[\left(1-\frac{\gamma-1}{\gamma}\frac{sv}{p}\right)\lambda_3 + s\right]} \cdot \frac{\lambda_1^2}{\lambda_3^2}, \tag{4.22}$$

$$C_{23} = \frac{1}{s}\frac{1}{|A|}\left(1-\frac{u^2}{c^2}\right)\left(\frac{\lambda_1}{\rho u}-\frac{\lambda_3}{\rho u}\right) v^2\kappa_3(U)(s-\lambda_3), \tag{4.23}$$

$$C_{33} = -\frac{1}{|A|}\left(1-\frac{u^2}{c^2}\right)\left(\frac{\lambda_1}{\rho u}-\frac{\lambda_3}{\rho u}\right)(u^2+v^2)\kappa_3(U)(s-\lambda_3). \tag{4.24}$$

□

4.4 On product of reflection coefficients K_2C_{13}

Recall the reflection coefficient K_2 appeared in Lemma 4.2, we have that

$$|C_{13}K_2| \;=\; \left|\frac{(s-\lambda_3)}{(s-\lambda_1)} \cdot \frac{\left[\left(1-\frac{\gamma-1}{\gamma}\frac{sv}{p}\right)\lambda_1 + s\right]}{\left[\left(1-\frac{\gamma-1}{\gamma}\frac{sv}{p}\right)\lambda_3 + s\right]}\right|. \tag{4.25}$$

To construct a Glimm functional, it is crucial to have $|C_{13}K_2| < 1$, which means waves are weakened after consequential reflections.

Note here that, by the Rankine-Hugoniot conditions,

$$a \triangleq 1 - \frac{\gamma-1}{\gamma}\frac{sv}{p} = \frac{1}{\gamma} + \frac{\gamma-1}{\gamma}\frac{p_0}{p} \in (\frac{1}{\gamma}, 1).$$

By Lax entropy condition, we also have $0 < s < \lambda_3$. So

$$\left|\frac{s-\lambda_3}{s-\lambda_1}\cdot\frac{a\lambda_1+s}{a\lambda_3+s}\right| < 1 \Leftrightarrow \begin{cases} (s-\lambda_3)(a\lambda_1+s) < (s-\lambda_1)(a\lambda_3+s), & ① \\ -(s-\lambda_1)(a\lambda_3+s) < (s-\lambda_3)(a\lambda_1+s), & ② \end{cases}$$

In fact, ① always holds for $s > 0$, while

$$② \quad \Leftrightarrow \quad s^2 + (a-1)\cdot\tfrac{\lambda_1+\lambda_3}{2}s - a\lambda_1\lambda_3 > 0.$$

Let

$$s' = (\frac{u^2}{c^2}-1)\frac{s}{\rho u} = (\frac{u^2}{c^2}-1)\frac{p-p_0}{\rho u^2 w}$$

and $\lambda_i' = (\frac{u^2}{c^2}-1)\frac{\lambda_i}{\rho u}$. Note that $\frac{\lambda_1'+\lambda_3'}{2} = w$, $\lambda_1'\lambda_3' = w^2+1-M^2$, so (recall $M = (u^2+v^2)/c^2 > 1$)

$$② \quad \Leftrightarrow \quad s'^2 + (a-1)ws' - a(w^2+1-M^2) > 0.$$

Therefore we see that $|C_{13}K_2| < 1$ holds if and only if $u > c$ and

$$\left(\frac{u^2}{c^2}-1\right)\left(\frac{p-p_0}{\rho u^2}\right)^2-(\gamma-1)\frac{v^2}{c^2}\left(\frac{p-p_0}{\rho u^2}\right)^2+\left(\frac{1}{\gamma}+\frac{\gamma-1}{\gamma}\frac{p_0}{p}\right)w^2(1+w^2)>0,$$

and the last inequality could further be simplified as

$$\left(\frac{p-p_0}{p}\right)^2\left(\frac{c^2}{u^2}\right)\left[1-\frac{c^2}{u^2}-(\gamma-1)w^2\right]+\gamma\left(1+(\gamma-1)\frac{p_0}{p}\right)w^2(1+w^2)>0. \tag{4.26}$$

We recall here that $p_0 = \underline{p}_a$ and $p = \underline{p}_b$, $u = \underline{u}_b$, $c = \underline{c}_b$, $w = \underline{w}_b > 0$. By Rankine-Hugoniot conditions, for given $\underline{U}_a$, we could write the left-hand side of (4.26) as a function of pressure, namely $f(p)$. Furthermore, entropy condition implies that $p > p_0$. So $f(p) > 0$ holds obviously if $p > p_0$ and $p - p_0$ is small (which implies $w > 0$ small from the p-w shock polar). Therefore by continuity of f, there is a $p_* > p_0$ so that $f(p) > 0$ for all $p \in (p_0, p_*)$. Here p_* is determined by $\underline{U}_a$. Then for the background solution $(\underline{U}_a, \underline{U}_b)$, it holds $|C_{13}K_2| < 1$. This estimate also holds for the approximate solution U^δ constructed by the front tracking method, since it should be a small perturbation from the background solution.

4.5 Refraction of weak wave by strong shock

We now consider the case that weak waves meet the strong shock from above (cf. Figure 10).

Lemma 4.4. *Suppose that $U_+, U_+^l \in O_\varepsilon(\underline{U}_\mathrm{a})$ and $U_-, U_-^r \in O_\varepsilon(\underline{U}_\mathrm{b})$, for sufficiently small ε, with*

$$U_+ = \Phi_1(\alpha_1, \Phi_2(\alpha_2, \Phi_3(\alpha_3, \Upsilon(s_k, U_-)))) = \Upsilon(s_{k+1}, \Phi_2(\beta_2, \Phi_1(\beta_1, U_-))).$$

Then (with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$)

$$s_{k+1}-s_k=\sum_{j=1}^{3}\bar{C}_{3j}\alpha_j+\bar{C}_3'(|\alpha|^2+|\alpha||s_k-\underline{s}|), \tag{4.27}$$

$$\beta_2=\sum_{j=1}^{3}\bar{C}_{2j}\alpha_j+\bar{C}_2'(|\alpha|^2+|\alpha||s_k-\underline{s}|), \tag{4.28}$$

$$\beta_1=\sum_{j=1}^{3}\bar{C}_{1j}\alpha_j+\bar{C}_1'(|\alpha|^2+|\alpha||s_k-\underline{s}|), \tag{4.29}$$

for some constants $\bar{C}_{ij}$ depending only on $\underline{U}_\mathrm{a}$ and $\underline{U}_\mathrm{b}$. Also, $\bar{C}_i'$ $(i = 1, 2, 3)$ are bounded quantities with bounds determined by the background solution.

Proof. This proof is similar to that of Lemma 4.3, but simpler since we do not need the

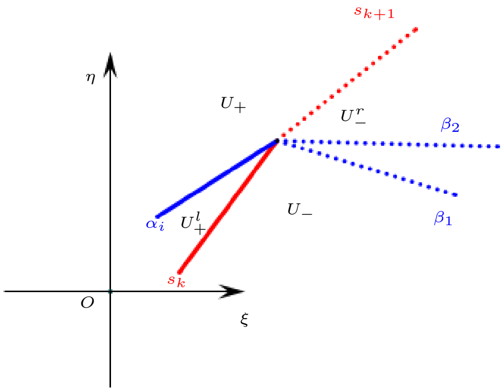

Figure 10: A weak wave α_i (blue solid line) of the i-th family meets the strong shock (red solid line) from above, and then penetrates it to produce a 1–wave β_1 and a 2–wave β_2 (blue dot line), and a deflected strong shock s_{k+1} .

specific expressions of the coefficients $\bar{C}$. By (4.7) and (4.8), the goal is to solve

$$\bar{C} = \frac{\partial(s_{k+1} - \underline{s}, \beta_2, \beta_1)}{\partial(\alpha_3, \alpha_2, \alpha_1)}\Big|_{(\underline{s},0,0;\underline{U}_{\mathrm{b}},\underline{U}_{\mathrm{a}})},$$

which is the Jacobian of $(s_{k+1} - \underline{s}, \beta_2, \beta_1) = f(s_k - \underline{s}, \alpha_3, \alpha_2, \alpha_1)$ with respect to $(\alpha_3, \alpha_2, \alpha_1)$.

According to the chain rule, we get

$$\begin{aligned}\frac{\partial U_+}{\partial(\alpha_3, \alpha_2, \alpha_1)} &= \frac{\partial U_+}{\partial(s_{k+1}, \beta_2, \beta_1)} \cdot \frac{\partial(s_{k+1} - \underline{s}, \beta_2, \beta_1)}{\partial(\alpha_3, \alpha_2, \alpha_1)} \\ &= (\frac{\partial U_+}{\partial s_{k+1}}, \frac{\partial U_+}{\partial U_-^r} \cdot \frac{\partial U_-^r}{\partial \beta_2}, \frac{\partial U_+}{\partial U_-^r} \cdot \frac{\partial U_-^r}{\partial \beta_1}) \cdot \frac{\partial(s_{k+1} - \underline{s}, \beta_2, \beta_1)}{\partial(\alpha_3, \alpha_2, \alpha_1)}. \end{aligned} \tag{4.30}$$

Multiplying both sides of (4.30) by $Q(U_+)$ from the left, then it becomes (using (3.10) and (3.11))

$$Q(U_+)\frac{\partial U_+}{\partial(\alpha_3, \alpha_2, \alpha_1)} = (F(U_-^r) - F(U_+), Q(U_-^r) \cdot \tfrac{\partial U_-^r}{\partial \beta_2}, Q(U_-^r) \cdot \tfrac{\partial U_-^r}{\partial \beta_1}) \cdot \tfrac{\partial(s_{k+1} - \underline{s}, \beta_2, \beta_1)}{\partial(\alpha_3, \alpha_2, \alpha_1)}.$$

Evaluating this equation at the background state, we get (comparing to (4.12))

$$Q(U_0)(r_3(U_0), r_2(U_0), r_1(U_0)) = (F(U) - F(U_0), Q(U)r_2(U), Q(U)r_1(U)) \cdot \bar{C}. \tag{4.31}$$

As in the derivation of (4.15), direct computation yields that

$$\begin{array}{rl} & M_{21}(p_0-p)M_{21}(p_0)M_{21}(-p_0)Q(U_0)(r_3(U_0),r_2(U_0),r_1(U_0)) \\ = & M_{11}(\frac{1}{\rho u})\cdot A\cdot M_{22}(s)\cdot M_{33}(\kappa_1(U)(s-\lambda_1))\cdot\bar{C}. \end{array} \tag{4.32}$$

Here A is given by (3.29). Since $\det A\neq 0$, we could solve uniquely the coefficients $\bar{C}$. □

5 The Glimm functional

In this section, we will construct a Glimm functional denoted by $G(\xi)$ which controls the distance from the approximate solutions to the background solution in the sense of total variation and show $G(\xi)$ is decreasing with respect to the 'time' ξ. There are mainly two classes of terms to handle: one is the total variation, and the other is the interaction potential.

5.1 Definition of Glimm functional

For the total variation, we define the following functionals without weights:

$$\begin{aligned} T_0^a(\xi) &\triangleq \sum_{\text{above}}(|\alpha_1|+|\alpha_2|+|\alpha_3|),\\ T_0^s(\xi) &\triangleq |s_k-\underline{s}|,\\ T_0^b(\xi) &\triangleq \sum_{\text{below}}(|\beta_1|+|\beta_2|+|\beta_3|). \end{aligned}$$

Here, 'above' (respectively, 'below') means the summation is over all the weak waves at the time ξ above (respectively, below) the strong shock, and α_k means the wave is of the k-th family. Then we define

$$T_0(\xi)\triangleq T_0^a(\xi)+T_0^s(\xi)+T_0^b(\xi),$$

which controls the total variation of the flow field at time ξ and the perturbation of the slope of the strong shock-front. It should be small to guarantee the solvability of Riemann problems in section 3.

To handle the wave reflections/refractions, we introduce the following weighted functionals:

$$\begin{aligned} T^a(\xi) &\triangleq \sum_{\text{above}}A(|\alpha_1|+|\alpha_2|+|\alpha_3|),\\ T^s(\xi) &\triangleq C_s|s_k-\underline{s}|,\\ T^b(\xi) &\triangleq \sum_{\text{below}}(B_1|\beta_1|+B_2|\beta_2|+B_3|\beta_3|), \end{aligned}$$

and

$$T(\xi) \triangleq T^a(\xi) + T^s(\xi) + T^b(\xi).$$

Here A, C_s, B_1, B_2, B_3 are positive constants to be chosen. They should depend only on the background solution.

To handle the wave interactions, we firstly define the sets:

$$\begin{aligned}
\mathbb{A}(\xi) &\triangleq \{(\alpha,\beta)|\alpha \text{ and } \beta \text{ are waves above the strong shock approaching to each other}\},\\
\mathbb{B}(\xi) &\triangleq \{(\alpha,\beta)|\alpha \text{ and } \beta \text{ are waves below the strong shock approaching to each other}\},\\
\mathbb{A}_s(\xi) &\triangleq \{\alpha|\alpha \text{ is a weak above the strong shock}\},\\
\mathbb{B}_s(\xi) &\triangleq \{\beta|\beta \text{ is a 3-wave lying below the strong shock}\}.
\end{aligned}$$

Here the meaning of two weak waves *approaching* is the same as that introduced by Glimm (see [11, p.311] or [13, p.215]). We also see that $\mathbb{A}_s$ (respectively, $\mathbb{B}_s$) is actually the set of weak waves that approaching the strong shock from above (respectively, below).

We then define

$$Q^a(\xi) \triangleq \sum_{(\alpha,\beta)\in\mathbb{A}} |\alpha|\cdot|\beta|, \quad Q^b(\xi) \triangleq \sum_{(\alpha,\beta)\in\mathbb{B}} |\alpha|\cdot|\beta|$$

to be the (weak) wave interaction potential above and below the strong shock respectively, and define

$$Q^{as}(\xi) \triangleq \sum_{\alpha\in\mathbb{A}_s} |\alpha|\cdot|s_k - \underline{s}|, \quad Q^{bs}(\xi) \triangleq \sum_{\beta\in\mathbb{B}_s} |\beta|\cdot|s_k - \underline{s}|$$

to be the weak wave–strong shock interaction potential from above and below, respectively. We then define the weighted total interaction potential

$$Q(\xi) \triangleq C_a Q^a(\xi) + C_b Q^b(\xi) + C_{as} Q^{as}(\xi) + C_{bs} Q^{bs}(\xi).$$

Finally, the Glimm functional is given by

$$G(\xi) \triangleq T(\xi) + Q(\xi).$$

5.2 Changes of Glimm functional

Now our aim is to find appropriate coefficients $A, B_1, B_2, B_3, C_s, C_a,\ C_b, C_{as}, C_{bs}$ that depending only on the background solution, so that by choosing $T_0(\xi)$ small (this can be done by requiring that ε_0, the perturbation of the initial data, to be small), then $G(\xi)$ is non-increasing; that is, $G(\xi_+) - G(\xi_-) \leq 0$ for any finite time ξ.

In the following, we need to analyze five different cases.

5.2.1 Case 1: A weak wave α_1 hits the boundary.

We assume that at time ξ, a 1-wave α_1 meets the boundary $\{\eta = 0\}$ and reflects to a 3-wave β_3, see Figure 11. For this case, $T^a(\xi), T^s(\xi), Q^a(\xi), Q^{as}(\xi)$ experience no change. We then check one by one the rest of the three terms as follows, where have we used the corresponding wave interaction estimates established in the previous section.

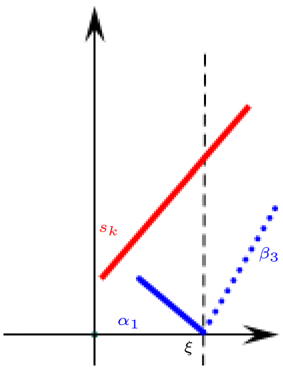

Figure 11: Case 1.

$$
\begin{aligned}
T^b(\xi_+) - T^b(\xi_-) &= B_3|\beta_3| - B_1|\alpha_1| \le B_3(|K_2|\alpha_1 + O(1)|\alpha_1|^2) - B_1|\alpha_1| \\
&= (B_3K_2 - B_1)|\alpha_1| + B_3O(1)|\alpha_1|^2,
\end{aligned}
$$

$$
\begin{aligned}
Q^{bs}(\xi_+) - Q^{bs}(\xi_-) &= |\beta_3| \cdot |s_k - \underline{s}| \\
&\le |K_2||\alpha_1| \cdot |s_k - \underline{s}| + O(1)|\alpha_1|^2| \cdot |s_k - \underline{s}|,
\end{aligned}
$$

$$
\begin{aligned}
Q^b(\xi_+) - Q^b(\xi_-) &= \sum_{(\alpha,\beta_3)\in\mathbb{B}} |\alpha| \cdot |\beta_3| - \sum_{(\alpha,\alpha_1)\in\mathbb{B}} |\alpha| \cdot |\alpha_1| \\
&\le \sum_{(\alpha,\beta_3)\in\mathbb{B}} |\alpha| \cdot |\beta_3| \le T_0^b(\xi) \cdot |\beta_3| \\
&= T_0^b(\xi)|K_2||\alpha_1| + T_0^b(\xi)O(1)|\alpha_1|^2.
\end{aligned}
$$

Here and below we use $O(1)$ to represent a bounded quantity with the bounds depending only on the background solution. Combine the above estimates, we have

$$\begin{aligned}
G(\xi_+) - G(\xi_-) \leq\ & (B_3K_2 - B_1 + C_bT_0^b(\xi_-)K_2)|\alpha_1| + (B_3O(1) + C_bT_0^b(\xi_-)O(1))|\alpha_1|^2 \\
& + C_{bs}K_2|\alpha_1|\cdot|s_b - \underline{s}| + C_{bs}O(1)|\alpha_1|^2\cdot|s_b - \underline{s}| \\
\leq\ & [B_3K_2 + C_bT_0^b(\xi_-)K_2 - B_1 + T_0^b(\xi_-)B_3O(1) \\
& + C_bT_0^b(\xi_-)O(1) + C_{bs}K_2T_0^s(\xi_-) + C_{bs}O(1)T_0^b(\xi_-)T_0^s(\xi_-)]|\alpha_1| \\
=\ & \Big\{[B_3K_2 - B_1] + T_0(\xi_-)\Big[B_3O(1) + C_bK_2 \\
& + C_bT_0^b(\xi_-)O(1) + C_{bs}K_2 + C_{bs}O(1)T_0(\xi_-)\Big]\Big\}|\alpha_1|.
\end{aligned} \tag{5.1}$$

5.2.2 Case 2: A weak wave α_3 hits strong shock from below

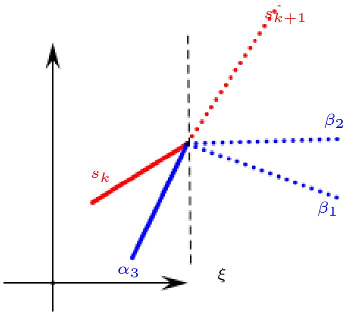

Figure 12: Case 2.

See figure 12, suppose that a weak wave a_3 hits the strong shock s_k and produces β_1, β_2, and s_{k+1}. In this case, T^a and Q^a have no any change, while

$$\begin{aligned}
T^b(\xi_+) - T^b(\xi_-) =\ & B_1|\beta_1| + B_2|\beta_2| - B_3|\alpha_3| \\
\leq\ & B_1\Big(|C_{13}||\alpha_3| + C_1'(|\alpha_3|^2 + |\alpha_3||s_k - \underline{s}|\Big) \\
& + B_2\Big(|C_{23}||\alpha_3| + C_2'(|\alpha_3|^2 + |\alpha_3||s_k - \underline{s}|\Big) - B_3|\alpha_3| \\
\leq\ & \Big(B_1|C_{13}| + B_2|C_{23}| - B_3\Big)|\alpha_3| + \Big(B_1C_1' + B_2C_2'\Big)|\alpha_3|^2 \\
& + \Big(B_1C_1' + B_2C_2'\Big)|\alpha_3|\cdot|s_k - \underline{s}|, \\
T^s(\xi_+) - T^s(\xi_-) =\ & C_s|(|s_{k+1} - \underline{s}| - |s_k - \underline{s}|) \\
\leq\ & C_s|s_{k+1} - s_k| \leq C_s\big[|C_{33}||\alpha_3| + C_3'(|\alpha_3|^2 + |\alpha_3|\cdot|s_k - \underline{s}|)\big] \\
\leq\ & C_s|C_{33}||\alpha_3| + C_sC_3'|\alpha_3|^2 + C_sC_3'|\alpha_3|\cdot|s_k - \underline{s}|
\end{aligned}$$

and

$$
\begin{aligned}
Q^b(\xi_+) - Q^b(\xi_-) = & \sum_{(\beta,\beta_1)\in\mathbb{B}} |\beta_1|\cdot|\beta| + \sum_{(\beta,\beta_2)\in\mathbb{B}} |\beta_2|\cdot|\beta| - \sum_{(\beta,\alpha_3)\in\mathbb{B}} |\alpha_3|\cdot|\beta| \\
\leq & \sum_{(\beta,\beta_1)\in\mathbb{B}} |\beta_1|\cdot|\beta| + \sum_{(\beta,\beta_2)\in\mathbb{B}} |\beta_2|\cdot|\beta| \\
\leq & T_0^b(\xi_-)\big[|C_{13}||\alpha_3| + C_1'(|\alpha_3|^2 + |\alpha_3|\cdot|s_k - \underline{s}|) \\
& + |C_{23}||\alpha_3| + C_2'(|\alpha_3|^2 + |\alpha_3|\cdot|s_k - \underline{s}|)\big] \\
= & T_0^b(\xi_-)(|C_{13}| + |C_{23}|)|\alpha_3| + T_b^0(\xi_-)(C_1' + C_2')|\alpha_3|^2 \\
& + T_0^b(\xi_-)(C_1' + C_2')|\alpha_3|\cdot|s_k - \underline{s}|, \\
Q^{bs}(\xi_+) - Q^{bs}(\xi_-) \leq & \,[\sum_{\alpha\in\mathbb{A}_s} |\alpha||s_{k+1} - \underline{s}| - \sum_{\alpha\in\mathbb{A}_s} |\alpha||s_k - \underline{s}| - |\beta_3|\cdot|s_k - \underline{s}|] \\
\leq & T_0^b(\xi_-)|s_{k+1} - s_k| - |\alpha||s_k - \underline{s}| \\
\leq & T_0^b(\xi_-)\big[|C_{33}||\alpha_3| + C_3'(|\alpha_3|^2 + |\alpha_3||s_k - \underline{s}|)\big] - |\alpha_3||s_k - \underline{s}| \\
= & T_0^b(\xi_-)|C_{33}||\alpha_3| + T_0^b(\xi_-)C_3'|\alpha_3|^2 + \big(T_0^b(\xi_-)C_3' - 1\big)|\alpha_3||s_k - \underline{s}| \\
\leq & T_0^b(\xi_-)|C_{33}||\alpha_3| + T_0^b(\xi_-)C_3'|\alpha_3|^2 + (T_0(\xi_-)C_3' - 1)|\alpha_3||s_k - \underline{s}|, \\
Q^{as}(\xi_+) - Q^{as}(\xi_-) = & \sum_{\alpha\in\mathbb{A}_s} |\alpha|(|s_{k+1} - \underline{s}| - |s_k - \underline{s}|) \\
\leq & \sum_{\alpha\in\mathbb{A}_s} |\alpha|(|s_{k+1} - s_k|) \\
\leq & T_0(\xi_-)\big(|C_{33}||\alpha_3| + C_3'(|\alpha_3|^2 + |\alpha_3|\cdot|s_k - \underline{s}|\big) \\
= & T_0(\xi_-)|C_{33}||\alpha_3| + T_0(\xi_-)C_3'|\alpha_3|^2 + T_0(\xi_-)C_3'|\alpha_3|\cdot|s_k - \underline{s}|.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
G(\xi_+) - G(\xi_-) = \; & \Big(B_1|C_{13}| + B_2|C_{23}| - B_3 + C_s|C_{33}|\Big)|\alpha_3| \\
& + \Big(B_1C_1' + B_2C_2' + C_sC_3'\Big)|\alpha_3|^2 \\
& + \Big(B_1C_1' + B_2C_2' + C_sC_3'\Big)|\alpha_3| \cdot |s_k - \underline{s}| \\
& + C_bT_0^b(\xi_-)\big(|C_{13}| + |C_{23}|\big)|\alpha_3| + T_0(\xi_-)O(1)C_{as}|\alpha_3| \\
& + C_bT_0(\xi_-)O(1)|\alpha_3|^2 + C_bT_0(\xi_-)O(1)|\alpha_3||s_k - \underline{s}| \\
& + C_{bs}T_0(\xi_-)O(1)|\alpha_3| + C_{bs}T_0(\xi_-)O(1)|\alpha_3|^2 + C_{as}T_0(\xi_-)O(1)|\alpha_3|^2 \\
& + C_{bs}\big(T_0(\xi_-)O(1) - 1\big)|\alpha_3||s_k - \underline{s}| + C_{as}T_0(\xi_-)O(1)|\alpha_3||s_k - \underline{s}| \\
= \; & \Big\{\Big[B_1|C_{13}| + B_2|C_{23}| - B_3 + C_s|C_{33}|\Big] + \big(B_1C_1' + B_2C_2' + C_sC_3'\big)T_0(\xi_-) \\
& + C_bT_0(\xi_-)O(1) + C_{as}T_0(\xi_-)O(1) + C_bT_0(\xi_-)^2O(1) + C_{bs}T_0(\xi_-)O(1) \\
& + C_{bs}T_0(\xi_-)^2O(1) + C_{as}T_0(\xi_-)^2O(1)\Big\}|\alpha_3| \\
& + \Big[B_1C_1' + B_2C_2' + C_sC_3' + C_bT_0(\xi_-)O(1) \\
& + C_{as}T_0(\xi_-)O(1) + C_{bs}(T_0(\xi_-)O(1) - 1)\Big]|\alpha_3||s_k - \underline{s}| \\
= \; & \Big\{\Big[B_1C_{13} + B_2C_{23} - B_3 + C_sC_{33}\Big] \\
& + T_0(\xi_-)\big[B_1C_1' + B_2C_2' + C_sC_3' + C_bO(1) \\
& + C_{as}O(1) + C_bT_0(\xi_-)O(1) + C_{bs}O(1) \\
& + C_{bs}T_0(\xi_-)O(1) + C_{as}T_0(\xi_-)O(1)\big]\Big\}|\alpha_3| \\
& + \Big\{B_1C_1' + B_2C_2' + C_sC_3' + C_bT_0(\xi_-)O(1) \\
& + C_{as}T_0(\xi_-)O(1) + C_{bs}\big[T_0(\xi_-)O(1) - 1\big]\Big\}|\alpha_3||s_k - \underline{s}|.
\end{aligned}
\tag{5.2}
$$

5.2.3 Case 3: Two weak waves collide below the strong shock

We assume that two waves α_i, α_j collide below the strong shock and produces $\beta_1, \beta_2, \beta_3$. So for this case, T^a, Q^a, Q^{as} and T^s have no change, but

$$
T^b(\xi_+) - T^b(\xi_-) \le O(1)(B_1 + B_2 + B_3)|\alpha_i\alpha_j|. \tag{5.3}
$$

Actually there are five cases for (i, j) in (α_i, α_j), namely $(1, 1), (2, 1), (3, 1), (3, 2), (3, 3)$. We take $(i, j) = (1, 3)$ as an example:

$$
\begin{aligned}
T^b(\xi_+) - T^b(\xi_-) = \; & B_3|\beta_3| + B_2|\beta_2| + B_1|\beta_1| - B_1|\alpha_1| - B_3|\alpha_3| \\
= \; & B_3[O(1)|\alpha_1\alpha_3|] + B_1[O(1)|\alpha_1\alpha_3|] + B_2[O(1)|\alpha_1\alpha_3|] \\
= \; & O(1)(B_1 + B_2 + B_3)|\alpha_1\alpha_3|.
\end{aligned}
$$

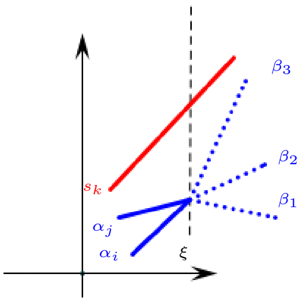

Figure 13: Case 3.

For the rest of the two terms, there hold that

$$\begin{aligned} Q^b(\xi_+) - Q^b(\xi_-) \leq & \ |\alpha_i| \cdot |\alpha_j|[O(1)T_0^b(\xi_-) - 1], \\ Q^{bs}(\xi_+) - (\xi_-) \leq & \ O(1)|\alpha_i| \cdot |\alpha_j||s_k - \underline{s}|. \end{aligned}$$

To prove this we also need to consider five cases as above. We take the case $(3,3)$ (namely, two 3-waves (α_3, γ_3) meet and one of which is a shock) as an example:

$$\begin{aligned} Q^{bs}(\xi_+) - Q^{bs}(\xi_-) = & \ [|\beta_3||s_k - \underline{s}| - |\alpha_3||s_k - \underline{s}| - |\gamma_3||s_k - \underline{s}|] \\ = & \ O(1)|\alpha_3||\gamma_3||s_k - \underline{s}|. \end{aligned}$$

Hence in this case we proved that

$$G(\xi_+) - G(\xi_-) = \Big[(B_1 + B_2 + B_3)O(1) + C_b(T_0(\xi_-)O(1) - 1) + O(1)C_{bs}T_0(\xi_-)\Big]|\alpha_i\alpha_j|. \tag{5.4}$$

5.2.4 Case 4: A weak wave hits strong shock from above

We assume that at ξ, a weak wave α_i hits the strong shock from above and produces β_1, β_2 and s_{k+1}, see Figure 14. In this case, all the seven terms in the Glimm functional experience changes. We have

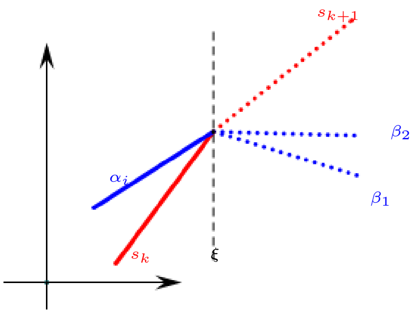

Figure 14: Case 4.

$$
\begin{aligned}
T^a(\xi_+) - T^a(\xi_-) =& \ -A|\alpha_i|, \\
T^b(\xi_+) - T^b(\xi_-) =& \ B_1|\beta_1| + B_2|\beta_2| \\
\leq& \ B_1\Big[|\bar{C}_{1i}||\alpha_i| + C_1'(|\alpha_i|^2 + |\alpha_i||s_k - \underline{s}|)\Big] \\
& +B_2\Big[|\bar{C}_{2i}||\alpha_i| + C_2'(|\alpha_i|^2 + |\alpha_i||s_k - \underline{s}|)\Big] \\
\leq& \ \Big(B_1|\bar{C}_{1i}| + B_2|\bar{C}_{2i}|\Big)|\alpha_i| + \Big(B_1C_1' + B_2C_2'\Big)|\alpha_i|^2 \\
& +(B_1C_1' + B_2C_2')|\alpha_i||s_k - \underline{s}|, \\
T^s(\xi_+) - T^s(\xi_-) =& \ C_s|(|s_{k+1} - \underline{s}| - |s_k - \underline{s}|) \leq C_s|s_{k+1} - s_k| \\
=& \ C_s|\bar{C}_{3i}||\alpha_i| + C_sC_3'|\alpha_i|^2 + C_sC_3'|\alpha_i| \cdot |s_k - \underline{s}|.
\end{aligned}
$$

Here, coefficients like $\bar{C}_{3i}$ come from Lemma 4.4. It is also easy to see that

$$
\begin{aligned}
Q^a(\xi_+) - Q^a(\xi_-) &= -\sum_{(\alpha,\alpha_i)\in\mathbb{A}} |\alpha| \cdot |\alpha_i| \leq 0, \\
Q^{as}(\xi_+) - Q^{as}(\xi_-) &= -|s_k - \underline{s}| \cdot |\alpha_i|,
\end{aligned}
$$

and

$$
\begin{aligned}
Q^b(\xi_+) - Q^b(\xi_-) =& \sum_{(\beta,\beta_1)\in\mathbb{B}} |\beta| \cdot |\beta_1| + \sum_{(\beta,\beta_2)\in\mathbb{B}} |\beta| \cdot |\beta_2| \leq T_0^b(\xi_-)(|\beta_1| + |\beta_2|) \\
\leq& \ T_0^b(\xi_-)\Big[\Big(|\bar{C}_{1i}| + |\bar{C}_{2i}|\Big)|\alpha_i| + (|C_1'| + |C_2'|)(|\alpha_i|^2 + |\alpha_i||s_k - \underline{s}|)\Big] \\
\leq& \ T_0^b(\xi_-)(|\bar{C}_{1i}| + |\bar{C}_{2i}|)|\alpha_i| + T_0^b(\xi_-)(|C_1'| + |C_2'|)|\alpha_i|^2 \\
& +T_0^b(\xi_-)(|C_1'| + |C_2'|)|\alpha_i||s_k - \underline{s}|, \\
Q^{bs}(\xi_+) - Q^{bs}(\xi_-) =& \ (|s_{k+1} - \underline{s}| - |s_k - \underline{s}|) \cdot \sum_{\beta_3\in\mathbb{B}_s} |\beta_3| \leq |s_{k+1} - s_k|T_0^b(\xi_-) \\
\leq& \ T_0^b(\xi_-)\Big[|\bar{C}_{3i}||\alpha_i| + C_3'(|\alpha_i|^2 + |\alpha_i| \cdot |s_k - \underline{s}|)\Big] \\
=& \ T_0^b(\xi_-)|\bar{C}_{3i}||\alpha_i| + T_0^b(\xi_-)C_3'|\alpha_i|^2 + T_0^b(\xi_-)C_3'|\alpha_i||s_k - \underline{s}|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
G(\xi_+)-G(\xi_-) = &\Big[-A+C_s|\bar{C}_{3i}|+B_1|\bar{C}_{1i}|+B_2|\bar{C}_{2i}|+C_bT_0(\xi_-)O(1)+C_{bs}T_0(\xi_-)O(1)\Big]|\alpha_i| \\
&+\Big[C_sC_3'+B_1C_1'+B_2C_2'+C_bT_0(\xi_-)(C_1'+C_2')+C_{bs}T_0(\xi_-)O(1)\Big]|\alpha_i|^2 \\
&+\Big[C_sC_3'+B_1C_1'+B_2C_2'+C_bT_0(\xi_-)(C_1'+C_2')+C_{bs}T_0(\xi_-)O(1)\Big]|\alpha_i||s_k-\underline{s}| \\
= &\Big\{\big[-A+C_s|\bar{C}_{3i}|+B_1|\bar{C}_{1i}|+B_2|\bar{C}_{2i}|\big] \\
&+\big[C_bO(1)+C_{bs}O(1)+C_sC_3'+B_1C_1'+B_2C_2'\big]T_0(\xi_-) \\
&+T_0(\xi_-)\big[C_bO(1)+C_{bs}O(1)\big]\Big\}|\alpha_i|+\Big\{\big[C_sC_3'-C_{as}+B_1C_1'+B_2C_2'\big] \\
&+T_0(\xi_-)\big[C_bO(1)+C_{bs}O(1)\big]\Big\}\cdot|\alpha_i||s_k-\underline{s}|.
\end{aligned}
\tag{5.5}
$$

5.2.5 Case 5: Two weak waves above the strong shock collide

Assume that at time ξ, two weak waves α_i, α_j above the strong shock collide and to produce $\beta=(\beta_1,\beta_2,\beta_3)$. In this case, T^s, T^b, Q^b, Q^{bs} have no change, see Figure 15.

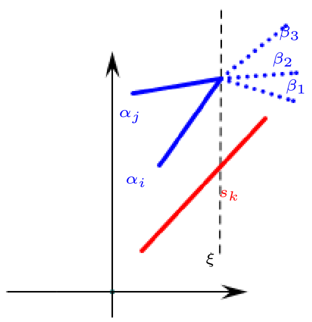

Figure 15: Case 5.

For the rest of the three terms, it is not hard to show that

$$
\begin{aligned}
T^a(\xi_+)-T^a(\xi_-) &= A(|\beta_1|+|\beta_2|+|\beta_3|-|\alpha_i|-|\alpha_j|) \\
&\leq A(|\alpha_i|+|\alpha_j|+O(1)|\alpha_i|\cdot|\alpha_j|-|\alpha_i|-|\alpha_j|) \\
&= AO(1)|\alpha_i|\cdot|\alpha_j|, \\
Q^a(\xi_+)-Q^a(\xi_-) &\leq (T^a(\xi_-))O(1)|\alpha_i|\cdot|\alpha_j|-|\alpha_i|\cdot|\alpha_j|) \\
&= (T^a(\xi_-)O(1)-1)|\alpha_i|\cdot|\alpha_j|, \\
Q^{as}(\xi_+)-Q^{as}(\xi_-) &= |s_k-\underline{s}|(\sum_{i=1}^{3}|\beta_i|-|\alpha_i|-|\alpha_j|) \\
&\leq T_0^s(\xi_-)\cdot O(1)\cdot|\alpha_i|\cdot|\alpha_j|.
\end{aligned}
$$

It follows that

$$G(\xi_+) - G(\xi_-) = \ |\alpha_i| \cdot |\alpha_j| \big\{ AO(1) + C_a[T_0(\xi_-)O(1) - 1] + C_{as}O(1)T_0(\xi_-) \big\}. \tag{5.6}$$

5.3 Determination of weights

Now we solve the coefficients $A, B_1, B_2, B_3, C_s, C_a,\ C_b, C_{as}, C_{bs}$ and determine $T_0(\xi_-)$ from (5.1), (5.2), (5.4), (5.5) and (5.6) to guarantee that $G(\xi_+) - G(\xi_-) \leq 0$.

We start from (5.2). Set $B_1 = 1$. Since $C_{13} < 1$ and $C_{13}K_2 < 1$, it holds $C_{13} < \frac{1}{K_2}$. Then we choose

$$B_3 \in (C_{13}, \frac{1}{K_2}).$$

Hence we have $B_3 > C_{13}B_1$. Therefore B_2 and C_s can be chosen small enough (depending only on B_3 and constants like C_{23}, C_{33} determined from the background solution), such that

$$B_1C_{13} + B_2C_{23} + C_sC_{33} - B_3 < -\delta,$$

where $\delta > 0$.

Up to now B_1, B_2, B_3 and C_s are chosen and henceforth fixed. Then (5.2) can be simplified as

$$\begin{aligned}
G(\xi_+) - G(\xi_-) \leq \ & \big\{ -\delta + T_0(\xi_-)[O(1) + C_bO(1) + C_{as}O(1) + C_{bs}O(1)] \\
& + T_0(\xi_-)^2[C_bO(1) + C_{as}O(1) + C_{bs}O(1)] \big\} |\alpha_3| \\
& + \big\{ O(1) + T_0(\xi_-)[C_bO(1) + C_{as}O(1)] + C_{bs}[T^0(\xi_-)O(1) - 1] \big\} |\alpha_3||s_k - \underline{s}| \\
\leq \ & \{ -\delta + T_0(\xi_-)M[1 + C_b + C_{as} + C_{bs}] + T_0(\xi_-)^2M[C_b + C_{as} + C_{bs}] \} |\alpha_3| \\
& + \{ M + T_0(\xi_-)M(C_b + C_{as}) - \tfrac{1}{2}C_{bs} \} |\alpha_3||s_k - \underline{s}| \\
\leq \ & \{ -\delta + T_0(\xi_-)M[1 + C_b + C_{as} + C_{bs}] + T_0(\xi_-)^2M[C_b + C_{as} + C_{bs}] \} |\alpha_3|,
\end{aligned} \tag{5.7}$$

provided that

$$C_{bs} > 2M + 2T_0(\xi_-)M(C_b + C_{as}). \tag{5.8}$$

Here and in the following, we have written $M = O(1)$, and assumed that

$$T_0(\xi_-)O(1) \leq \frac{1}{2}. \tag{5.9}$$

Now consider (5.1). Notice that $B_3K_2 - B_1 < 0$ because $B_1 = 1, B_3 < \frac{1}{K_2}$. Then assume that $B_3K_2 - B_1 \leq -\delta < 0$, from (5.1) we have

$$G(\xi_+) - G(\xi_-) \leq \ \Big[-\delta + T_0(\xi_-)M(1 + C_b + C_{bs}) + T_0(\xi_-)^2(C_b + C_{bs})M \Big] |\alpha_1|. \tag{5.10}$$

For (5.4), suppose that $T_0(\xi_-)$ is small enough such that (5.9) holds, and

$$C_b > 2M(1 + C_{bs}T_0(\xi_-)), \tag{5.11}$$

then (5.4) yields that

$$G(\xi_+) - G(\xi_-) \leq \left[M + MC_{bs}T_0(\xi_-) - \tfrac{1}{2}C_b\right]|\alpha_i\alpha_j| \leq 0 \tag{5.12}$$

as desired.

We then infer from (5.5) that

$$\begin{aligned} G(\xi_+) - G(\xi_-) \leq\ & [-A + M + M(C_b + C_{bs} + 1)T_0(\xi_-) + (C_b + C_{bs})MT_0(\xi_-)^2]|\alpha_i| \\ & +[M - C_{as} + T_0(\xi_-)M(C_b + C_{bs})]|\alpha_i||s_k - \underline{s}| \\ \leq\ & [-A + M + M(C_b + C_{bs} + 1)T_0(\xi_-) + (C_b + C_{bs})MT_0(\xi_-)^2]|\alpha_i| \\ \leq\ & 0, \end{aligned} \tag{5.13}$$

provided that

$$C_{as} > M + T_0(\xi_-)M(C_b + C_{bs}), \tag{5.14}$$
$$A > M + M(C_b + C_{bs} + 1)T_0(\xi_-) + (C_b + C_{bs})MT_0(\xi_-)^2. \tag{5.15}$$

Suppose $T_0(\xi_-)O(1) \leq \frac{1}{2}$, from (5.6) we have

$$G(\xi_+) - G(\xi_-) \leq \left[AM - \tfrac{1}{2}C_a + MC_{as}T_0(\xi_-)\right]|\alpha_i\alpha_j| \leq 0, \tag{5.16}$$

if

$$C_a \geq 2AM + 2MT_0(\xi_-)C_{as}. \tag{5.17}$$

We now determine C_a, C_b, C_{as}, C_{bs} and A from (5.17), (5.11), (5.14), (5.8) and (5.15). Without loss of generality, we may assume that $M > 1$ and $T_0(\xi_-)M \leq \frac{1}{12}$. Then if we choose

$$A = C_b = C_{as} = C_{bs} = 3M, \quad C_a = 7M^2,$$

all the inequalities (5.17), (5.11), (5.14), (5.8) and (5.15) are true.

Finally, for all the weights chosen above, we choose $\bar{\epsilon}_0 \in (0,1)$ small (determined only by M and δ, hence depends only on the background solution) so that $\bar{\epsilon}_0 M \leq 1/12$ and

$$\bar{\epsilon}_0 M(1 + C_b + C_{as} + C_{bs}) + \bar{\epsilon}_0^2 M(C_b + C_{as} + C_{bs}) \leq \delta.$$

Then if

$$T(\xi_-) \leq \bar{\epsilon}_0, \tag{5.18}$$

from (5.7) and (5.10), it follows that $G(\xi_+) \leq G(\xi_-)$. Thus we proved that the Glimm functional is always non-increasing.

5.4 Uniform bounds of $T_0(\xi)$

We easily see by the choice of weights that there is a constant $C > 1$ depending only on the background solution so that

$$T_0(\xi) \leq CG(\xi), \quad G(\xi) \leq C(T_0(\xi) + T_0(\xi)^2). \tag{5.19}$$

Suppose now that the initial data to problem (L) satisfies $\|U_0\|_{\mathrm{TV}((0,\infty))} \leq \varepsilon_0$, with ε_0 claimed in Theorem 2.2 to be specified below. For any positive number δ, we approximate U_0 by piecewise constant function U_0^δ with

$$\left\|U_0^\delta - U_0\right\|_{L^1((0,\infty))} \leq \delta,$$

and solve locally typical Riemann problems to obtain an approximate solution U^δ for $\xi \in (0, \xi^1)$. (At $\xi = \xi^1$, one of the five cases considered above occurs. We omit the details on splitting rarefaction waves and eliminating weak waves of higher generation order in solving Riemann problems, which does not affect the uniform estimates. See [5, 13] for details. Without loss of generality, we may also assume that at each time ξ when collision occurs, only two waves interact by changing slightly the speed of one of the wave, see [3].) By property of Riemann solutions, there holds

$$T_0(0_+) \leq C(\mathrm{TV}_{(0,\infty)} U_0^\delta) \leq C\varepsilon_0. \tag{5.20}$$

We note that it also holds from the property of Riemann problem that

$$|s_\delta'(\xi) - \underline{s}| + \mathrm{TV}_{(0,s_\delta(\xi))}(U^\delta(\xi,\cdot) - \underline{U}_b) + \mathrm{TV}_{(s_\delta(\xi),\infty)}(U^\delta(\xi,\cdot) - \underline{U}_a) \leq CT_0(\xi).$$

Here $s_\delta(\xi)$ is the equation of the strong shock in the approximate solution U^δ which can be determined step by step in the front tracking algorithm. We need to choose $\bar{\epsilon}_0$ small enough so that from this inequality each typical Riemann problem studied in section 3 could be solved.

By front tracking method, suppose then one of the five cases considered above occurs at ξ^1. By the analysis above, if $C\varepsilon_0 \leq \bar{\epsilon}_0$, then $T_0(\xi_+^1) \leq CG(\xi_+^1) \leq CG(\xi_-^1) = CG(0_+) \leq C^2(T_0(0_+) + T_0(0_+)^2) \leq 2C^3\varepsilon_0$. Hence we choose

$$\varepsilon_0 = \bar{\epsilon}_0/(4C^3),$$

and it follows that $T_0(\xi_+^1) \leq \bar{\epsilon}_0$. Therefore we can go on to solve typical Riemann problems and construct approximate solutions for $\xi > \xi^1$, until at some $\xi = \xi^2$ one of the five cases above appears. Then we have $T_0(\xi_+^2) \leq CG(\xi_+^2) \leq CG(\xi_-^2) = CG(\xi_+^1) \leq 2C^3\varepsilon_0 \leq \bar{\epsilon}_0$.

Taking such arguments for next times of collision/reflction/refraction, we infer that we can construct approximate solutions U^δ by solving typical Riemann problems and still get a uniform estimate

$$|s'_\delta(\xi) - \underline{s}| + \mathrm{TV}_{(0,s_\delta(\xi))}(U^\delta(\xi,\cdot) - \underline{U}_b) + \mathrm{TV}_{(s_\delta(\xi),\infty)}(U^\delta(\xi,\cdot) - \underline{U}_a) \le C\bar{\epsilon}_0.$$

The estimate claimed in Theorem 2.2 follows from this estimate and the finite speed of propagation for hyperbolic equations.

We also note that for the above construction to work, we need to show that for any given $T > 0$, there are only finite many times of collision/reflection/refraction happen in $(\xi, \eta) \in (0, T) \times (0, \infty)$. This is easy if we eliminate waves of high generation order as for the Cauchy problem [13, Chapter 6, p.216], and notice that a wave can only be reflected by the boundary and the strong shock for finitely many times, since the wave speed has an upper bound and the distance between the boundary and the strong shock increases (both determined by the background solution).

What is left is to show that the approximate solutions $\{U^\delta\}_{\delta>0}$ are compact in the space $C((0,\infty), L^1(0,\infty))$ (after modulo a constant state for $\eta \to +\infty$), and the limit U of a subsequence $\{U^{\delta_k}\}_k$ is actually a weak entropy solution to problem (L). This process is standard and we omit the details (cf. [5] or [8]). This completes the proof of the existence part of Theorem 2.2 (hence this part of Theorem 1.1 by Wagnar's theorem).

6 Uniqueness and stability of solutions

The main goal of this section is to establish the uniqueness of solutions to the free boundary problem (E) in the Eulerian coordinates, and the L^1 stability in the Lagrangian coordinates. Towards this aim, first we will prove the L^1-stability of solutions to the corresponding problem (L) in the Lagrangian coordinates.

6.1 The L^1–distance functional between two solutions

Given initial data U_0, let U^δ be an approximate solution constructed by a front tracking algorithm, where δ is a small parameter measuring the accuracy of the solution, which controls the following errors generated by the algorithm:

- Errors in the approximation of initial data;
- Errors in the speeds of shock, characteristic discontinuities, and rarefaction fronts;
- Errors from approximating the rarefaction waves by piecewise constant rarefaction fronts;
- Errors from removing all the fronts with generation higher than N (N is a positive integer depending on δ).

The construction of a Glimm functional as in Section 5 provides the necessary uniform estimates that guarantee the existence of a subsequence of U^δ which converges to a bounded entropy solution of problem (L) in $C([0,T]; L^1(\mathbb{R}^+))$ for any $T > 0$.

To show that the front tracking approximations, constructed for the existence analysis in Section 5, converge to a unique limit, we estimate the distance between any two δ-approximate solution U and V of problem (L). To this end, we introduce a Lyapunov functional $\Phi(U, V)$, equivalent to the L^1–distance:

$$C^{-1} \left\|U(\xi, \cdot) - V(\xi, \cdot)\right\|_{L^1} \le \varPhi(U, V) \le C \left\|U(\xi, \cdot) - V(\xi, \cdot)\right\|_{L^1},$$

and prove that the functional $\Phi(U, V)$ is *almost decreasing* along pairs of solutions:

$$\varPhi\left(U(\xi_2, \cdot), V(\xi_2, \cdot)\right) - \varPhi\left(U(\xi_1, \cdot), V(\xi_1, \cdot)\right) \le C(\xi_2 - \xi_1), \quad \text{for all } \xi_2 > \xi_1 > 0,$$

for some constant $C > 0$.

Following earlier works [2, 14], with "time" ξ fixed, at each η, we connect the state $U(\eta)$ with $V(\eta)$ in the state space by going along the Hugoniot curves S_1, C_2, and S_3. Depending on the location of the strong shock in $U(\eta)$ and $V(\eta)$, the distance between $U(\eta)$ and $V(\eta)$ is estimated along discontinuity/waves in possibly different "directions", determining the strength of the j-th Hugoniot wave $h_j(\eta)$ in the following way:

- Suppose that $U(\eta)$ and $V(\eta)$ are both in $O_\varepsilon(\underline{U}_b)$ and $O_\varepsilon(\underline{U}_a)$. Then one begins at the state $U(\eta)$ and moves along the Hugoniot curves to reach the state $V(\eta)$.
- Suppose that $U(\eta)$ is in $O_\varepsilon(\underline{U}_b)$ and $V(\eta)$ is in $O_\varepsilon(\underline{U}_a)$. Then one begins at the state $U(\eta)$ and moves along the Hugoniot curves to reach the state $V(\eta)$.
- Suppose that $V(\eta)$ is in $O_\varepsilon(\underline{U}_b)$ and $U(\eta)$ is in $O_\varepsilon(\underline{U}_a)$. Then one begins at the state $V(\eta)$ and moves along the Hugoniot curves to reach the state $U(\eta)$.

Define the L^1–weighted strengths of the waves in the solution of the Riemann problem $(U(\eta), V(\eta))$ or $(V(\eta), U(\eta))$ as follows:

$$q_j(\eta) \quad = \quad \begin{cases} w_j^b \cdot h_j(\eta) & \text{whenever } U(\eta) \text{ and } V(\eta) \text{ are both in } O_\varepsilon(\underline{U}_b), \\ w_j^m \cdot h_j(y) & \text{whenever } U(\eta) \text{ and } V(\eta) \text{ are both in different domains,} \\ w_j^a \cdot h_j(y) & \text{whenever } U(\eta) \text{ and } V(\eta) \text{ are both in } O_\varepsilon(\underline{U}_a), \end{cases} \tag{6.1}$$

with the constants w_j^b, w_j^m, and w_j^a above to be specified later on, based on the estimates of wave interactions and reflections in Section 4.

We define the following Lyapunov functional,

$$\varPhi(U, V) = \sum_{j=1}^{3} \int_0^\infty |q_j(\eta)| W_j(\eta)\, d\eta, \tag{6.2}$$

where the weights are given by

$$W_j(\eta) = 1 + \kappa_1 A_j(\eta) + \kappa_2 \left(\mathcal{Q}(U) + \mathcal{Q}(V)\right). \tag{6.3}$$

The constants κ_1 and κ_2 are to be determined later. Here $\mathcal{Q}$ denotes the total wave interaction potential, and $A_j(\eta)$ denotes the total strength of waves in U and V, which approach the j-wave $q_j(\eta)$, defined in the following manner (for η where there is no jump in U or V):

$$A_j(\eta) = F_j(\eta)+G_j(\eta)+\begin{cases} D_j(\eta) & \text{if } j\text{-wave } q_j(\eta) \text{ is small and the } j\text{-field is genuinely nonlinear,} \\ E_j & \text{if } j = 3 \text{ and } q_j(\eta) = B \text{ is large.} \end{cases} \tag{6.4}$$

Next, we define the following global weights G_j:

$G_j(\eta) =$	U, V are both in $O_\varepsilon(\underline{U}_b)$	U, V are in distinct regions	U, V are both in $O_\varepsilon(\underline{U}_a)$
$G_1(\eta)$	B	B	B
$G_2(\eta)$	0	B	B
$G_3(\eta)$	B	0	B

Under the assumption that $\mathrm{TV}(U_0^\delta(\cdot)) + \mathrm{TV}(V_0^\delta(\cdot))$ is small enough with $U(\xi,\cdot)$, $V(\xi,\cdot)$ $\in \mathrm{BV} \cap L^1$, one concludes

$$\mathcal{M}^{-1} \left\|U(\xi,\cdot) - V(\xi,\cdot)\right\|_{L^1} \leq \sum_{j=1}^{3} \int_0^\infty |q_j(\eta)|\, d\eta \leq \mathcal{M}\|U(\xi,\cdot) - V(\xi,\cdot)\|_{L^1},$$
$$1 \leq W_j(\eta) \leq \mathcal{M}, \;\; j = 1, \ldots, 3,$$

where the constant $\mathcal{M}$ is independent of δ and "time" ξ. Here *we define the strength of any large wave of the* 3*-characteristic family to equal to some fixed number* B *(bigger than all strengths of small waves),* and the concepts "small" and "large" mean the waves that connect the states in the same or in the distinct domains $O_\varepsilon(\underline{U}_b)$ and $O_\varepsilon(\underline{U}_a)$, respectively.

The summands in (6.4) are defined as follows:

$$F_j(\eta) = \left(\sum_{\substack{\alpha\in\mathcal{J}\setminus\mathcal{S} \\ \eta_\alpha<\eta, j<k_\alpha\leq 3}} + \sum_{\substack{\alpha\in\mathcal{J}\setminus\mathcal{S} \\ \eta_\alpha>y, 1\leq k_\alpha<j}} \right)|\alpha|,$$

$$D_j(\eta) = \begin{cases} (\sum_{\alpha\in\mathcal{J}(U)\setminus\mathcal{S}(U),\eta_\alpha<\eta,k_\alpha=j} + \sum_{\alpha\in\mathcal{J}(V)\setminus\mathcal{S}(V),\eta_\alpha>\eta,k_\alpha=j})|\alpha| & \text{if } q_j(\eta) < 0, \\ (\sum_{\alpha\in\mathcal{J}(V)\setminus\mathcal{S}(V),\eta_\alpha<\eta,k_\alpha=j} + \sum_{\alpha\in\mathcal{J}(U)\setminus\mathcal{S}(U),\eta_\alpha>\eta,k_\alpha=j})|\alpha| & \text{if } q_j(\eta) > 0, \end{cases}$$

$$E_j(\eta) = \left(\sum_{\substack{\alpha\in\mathcal{J}\setminus\mathcal{S} \\ \eta_\alpha<\eta, k_\alpha=3 \\ \text{both states joined by } \alpha \text{ are located in } O_\varepsilon(\underline{U}_b)}} + \sum_{\substack{\alpha\in\mathcal{J}\setminus\mathcal{S} \\ \eta_\alpha>\eta, k_\alpha=3 \\ \text{both states joined by } \alpha \text{ are located in } O_\varepsilon(\underline{U}_a)}} \right)|\alpha|,$$

where, at each ξ, α stands for the (non-weighted) strength of the wave $\alpha \in \mathcal{J}$, located at the point η_α and belonging to the characteristic family k_α; $\mathcal{J} = \mathcal{J}(U) \cup \mathcal{J}(V)$ is the set of all waves (in U and V) and $\mathcal{S} = \mathcal{S}(U) \cup \mathcal{S}(V)$ is the set of all large (strong) shock waves (in U and V).

Consequently, there holds

$$C^{-1} \|U(\xi,\cdot) - V(\xi,\cdot)\|_{L^1} \leq \Phi(U,V) \leq C\|U(\xi,\cdot) - V(\xi,\cdot)\|_{L^1}, \tag{6.5}$$

for any $\xi \geq 0$ with the constant $C > 0$ depending only on the quantities independent of ξ: the strength of the strong shock wave and $\mathrm{TV}(U_0^\delta(\cdot)) + \mathrm{TV}(V_0^\delta(\cdot))$.

We now analyze the evolution of the Lyapunov functional Φ in the flow direction $\xi > 0$. For $j = 1,2,3$, we call $\lambda_j(\eta)$ the speed of the j-wave $q_j(\eta)$ (along the Hugoniot curve in the phase space). Then, at a "time" $\xi > 0$ which is not the interaction time of the waves either in $U(\xi) = U(\xi,\cdot)$ or $V(\xi) = V(\xi,\cdot)$, an explicit computation gives

$$\begin{aligned}
&\frac{d}{d\xi}\Phi\left(U(\xi),V(\xi)\right)\\
&= \sum_{\alpha\in\mathcal{J}}\sum_{j=1}^{3}\left(\left|q_j(\eta_\alpha^-)\right| W_j(\eta_\alpha^-) - \left|q_j(\eta_\alpha^+)\right| W_j(\eta_\alpha^+)\right)\dot{\eta}_\alpha + \sum_{j=1}^{3}\left|q_j(b)\right| W_j(b)\dot{\eta}_b\\
&= \sum_{\alpha\in\mathcal{J}}\sum_{j=1}^{3}\left(\left|q_j(\eta_\alpha^-)\right| W_j(\eta_\alpha^-)\left(\dot{\eta}_\alpha - \lambda_j(\eta_\alpha^-)\right) - \left|q_j(\eta_\alpha^+)\right| W_j(\eta_\alpha^+)\left(\dot{\eta}_\alpha - \lambda_j(\eta_\alpha^+)\right)\right)\\
&\quad + \sum_{j=1}^{3}\left|q_j(b)\right| W_j(b)\left(\dot{\eta}_b + \lambda_j(b)\right),
\end{aligned} \tag{6.6}$$

where $\dot{\eta}_\alpha$ denotes the speed of the Hugoniot wave $\alpha \in \mathcal{J}$, $b = 0^+$ stands for the points close to the characteristic boundary $\eta = 0$, and $\dot{\eta}_b$ is the slope of the boundary, which is zero.

We present the notation

$$E_{\alpha,j} = \left|q_j^+\right| W_j^+\left(\lambda_j^+ - \dot{\eta}_\alpha\right) - \left|q_j^-\right| W_j^-\left(\lambda_j^- - \dot{\eta}_\alpha\right), \tag{6.7}$$

$$E_{b,j} = \left|q_j(b)\right| W_j(b)\left(\dot{\eta}_b + \lambda_j(b)\right), \tag{6.8}$$

where $q_j^\pm = q_j(\eta_\alpha^\pm)$, $W_j^\pm = W_j(\eta_\alpha^\pm)$, and $\lambda_j^\pm = \lambda_j(\eta_\alpha^\pm)$.

Then (6.6) can be written as

$$\frac{d}{d\xi}\Phi\left(U(\xi),V(\xi)\right) = \sum_{\alpha\in\mathcal{J}}\sum_{j=1}^{3} E_{\alpha,j} + \sum_{j=1}^{3} E_{b,j}. \tag{6.9}$$

Our central aim is to prove the bounds:

$$\sum_{j=1}^{3} E_{\alpha,j} \leq \mathcal{O}(1)\delta\,|\alpha| \quad \text{ when } \alpha \text{ is a weak wave in } \mathcal{J}, \tag{6.10}$$

$$\sum_{j=1}^{3} E_{\alpha,j} \leq 0 \quad \text{ when } \alpha \text{ is a strong shock wave in } \mathcal{J}, \tag{6.11}$$

$$\sum_{j=1}^{3} E_{b,j} \leq 0 \quad \text{ near the boundary,} \tag{6.12}$$

where the quantities denoted by the Landau symbol $\mathcal{O}(1)$ are independent of the constants κ_1 and κ_2.

From (6.10)–(6.12) together with the uniform bound on the total strengths of waves, we obtain

$$\frac{d}{d\xi}\Phi\left(U(\xi), V(\xi)\right) \leq \mathcal{O}(1)\delta. \tag{6.13}$$

Integration of (6.13) over the interval $[0,\xi]$ yields

$$\Phi\left(U(\xi), V(\xi)\right) \leq \Phi\left(U(0), V(0)\right) + \mathcal{O}(1)\delta\xi, \tag{6.14}$$

which implies the uniqueness and L^1 stability as desired.

We remark that, at each interaction "time" ξ when two fronts of U or two fronts of V interact, by the Glimm interaction estimates, all the weight functions $W_j(\eta)$ decrease, if the constant κ_2 in the Lyapunov functional is taken to be sufficiently large. Furthermore, due to the self-similar property of the Riemann solutions, Φ decreases at this "time".

Next, we establish the bounds (6.10)–(6.12), particularly (6.11) and (6.12), when α is a strong shock wave in $\mathcal{J}$ and near the characteristic boundary, respectively.

6.2 L^1–stability estimates and the uniqueness theorem

By following the arguments in Bressan-Liu-Yang [2], the case that the weak wave $\alpha \in \mathcal{J} := \mathcal{J}(U) \cup \mathcal{J}(V)$, that is, when U and V are both in $O_\varepsilon(\underline{U}_b)$ or $O_\varepsilon(\underline{U}_a)$, the estimate (6.10) holds, provided that $|B/\underline{s}|$ is sufficiently small and κ_1 is sufficiently large. In what follows, we only focus on the other two cases, namely (6.11) and (6.12).

Case 1: *The first strong shock wave α in U or V is crossed.* By Lemma 4.1 and Lemma 4.3, we have the estimate:

$$h_1^+ = h_1^- + C_{13} h_3^-. \tag{6.15}$$

Moreover, the essential estimate $|C_{13}| < 1$ given in Lemma 4.3 ensures the existence of the desired weights w_1^b and w_3^b in the following way.

Lemma 6.1. *There exist w_1^b, w_4^b, and γ_b satisfying*

$$\frac{w_3^b}{w_1^b} < 1, \tag{6.16}$$

$$\frac{w_1^b}{w_3^b} C_{13} \left| \frac{\lambda_1^- - s}{\lambda_3^- - s} \right| < \gamma_b < 1. \tag{6.17}$$

With Lemma 6.1, we estimate E_j for $j = 1, 2, 3$, starting with E_1. By (6.15) and (6.17),

$$\begin{aligned} E_1 &= |q_1^-|(\lambda_1^- - \dot{\eta}_\alpha)(W_1^+ - W_1^-) + W_1^+ \left(|q_1^+|(\lambda_1^+ - \dot{\eta}_\alpha) - |q_1^-|(\lambda_1^- - \dot{\eta}_\alpha) \right) \\ &\leq \kappa_1 B \left(w_1^b |h_1^+||\lambda_1^- - \dot{\eta}_\alpha| + w_1^b C_{13} |h_3^-||\lambda_1^- - \dot{\eta}_\alpha| - w_1^m |h_1^+||(\lambda_1^+ - \dot{\eta}_\alpha)| \right) \\ &\leq \kappa_1 B \left(w_1^b |h_1^+||\lambda_1^- - \dot{\eta}_\alpha| + \gamma_b w_3^b |h_3^-||\lambda_3^- - \dot{\eta}_\alpha| - w_1^m |h_1^+||(\lambda_1^+ - \dot{\eta}_\alpha)| \right). \end{aligned}$$

For $j = 2$,

$$\begin{aligned} E_2 &= |q_2^+|(\lambda_2^+ - \dot{\eta}_\alpha)(W_2^+ - W_2^-) + W_2^- \left(|q_2^+|(\lambda_2^+ - \dot{\eta}_\alpha) - |q_2^-|(\lambda_2^- - \dot{\eta}_\alpha) \right) \\ &\leq -\kappa_1 B |q_2^+||\lambda_2^+ - \dot{\eta}_\alpha| + O(1)|q_2^-| \\ &\leq -\kappa_1 B |q_2^+||\lambda_2^+ - \dot{\eta}_\alpha| + O(1) \left(|q_2^+| + |q_3^-| \right). \end{aligned}$$

For $j = 3$,

$$\begin{aligned} E_3 &= B W_3^+ (\lambda_3^+ - \dot{\eta}_\alpha) - |q_3^-||W_3^-|(\lambda_3^- - \dot{\eta}_\alpha) \\ &\leq O(1) B |q_3^-| - \kappa_1 B |q_3^-|(\lambda_3^- - \dot{\eta}_\alpha) \\ &= O(1) B |q_3^-| - \kappa_1 B w_3^b |h_3^-|(\lambda_3^- - \dot{\eta}_\alpha). \end{aligned}$$

All together, choose w_1^m large enough relatively to w_1^b, then

$$\begin{aligned} \sum_{j=1}^{3} E_j &= -(1 - \gamma_b)\kappa_1 B w_3^b |h_3^-||\lambda_3^- - \dot{\eta}_\alpha| + O(1)|q_3^-| \\ &\quad + \kappa_1 B \left((w_1^b |h_1^+|\lambda_1^- - \dot{\eta}_\alpha| - w_1^m |h_1^+||\lambda_1^+ - \dot{\eta}_\alpha| \right) \\ &\quad - \kappa_1 B |q_2^+||\lambda_2^+ - \dot{\eta}_\alpha| + O(1)|q_2^+| \leq 0. \end{aligned}$$

Case 2: *The weak wave α between the two strong vortex sheets/entropy waves in U and V is crossed.* For $j = 1, 2$, we have

$$\begin{aligned} E_j &= |q_j^\pm|(W_j^+ - W_j^-)(\lambda_1^\pm - \dot{\eta}_\alpha) + W_j^\mp \left(|q_j^+|(\lambda_j^+ - \dot{\eta}_\alpha) - |q_j^-|(\lambda_j^- - \dot{\eta}_\alpha) \right) \\ &\leq \kappa_1 |q_j^\pm||\alpha||\lambda_j^\pm - \dot{\eta}_\alpha| + 2B\kappa_1 \left((|q_j^+| - |q_j^-|)(\lambda_j^+ - \dot{\eta}_\alpha) + |q_j^-|(\lambda_j^+ - \lambda_j^-) \right) \\ &\leq \kappa_1 |q_j^\pm||\alpha||\lambda_j^\pm - \dot{\eta}_\alpha| + B\kappa_1 \left((|q_j^+| - |q_j^-|)(\lambda_j^+ - \dot{\eta}_\alpha) + O(1)|q_\eta^-||\alpha| \right). \end{aligned}$$

For the case when $j = 3$, we have

$$\begin{aligned} E_j &= B\left(\left(W_j^+ - W_j^-\right)\left(\lambda_j^\pm - \dot\eta_\alpha\right) + W_j^\mp\left(\lambda_j^\pm - \lambda_j^\mp\right)\right) \\ &\le B\left(-\kappa_1|\alpha||\lambda_3^- - \dot\eta_\alpha| + O(1)|\alpha|\right). \end{aligned}$$

In all, we get

$$\sum_{j=1}^{3} E_j \le \kappa_1 O(1)\Big(- |\alpha| + |\alpha| \sum_{k\neq 3} |q_k^+| + |q_k^-| + \sum_{k\neq 3} |q_k^+| - |q_k^-|\Big) + O(1)|\alpha|.$$

Since $\big||q_k^+| - |q_k^-|\big| \le |q_k^+ - q_k^-| \le O(1)|\alpha|$ when $k \neq 3$, we obtain $\sum_{j=1}^{3} E_j \le 0$ if κ_1 is sufficiently large and all the weights w_j^m are small enough.

Notice that the choice of the upper or lower superscripts depends on the family number k_α.

Case 3: *The second strong shock wave α in U or V is crossed.* For the case when $j = 1, 2$, we have that

$$\begin{aligned} \sum_{j=1}^{2} E_j &= \sum_{j=1}^{2} \left(|q_j^-(W_j^+ - W_j^-)\right) + W_j^+ \left(|q_j^+|(\lambda_j^+ - \dot\eta_\alpha) - |q_j^-|(\lambda_j^- - \dot\eta_\alpha)\right) \\ &\le -\frac{2}{3}\sum_{j=1}^{2} \kappa_1 B|q_j^+||\lambda_j^+ - \dot\eta_\alpha| - \sum_{j=1}^{2} \kappa_1 B|q_j^-||\lambda_j^- - \dot\eta_\alpha|. \end{aligned}$$

For $j = 3$,

$$\begin{aligned} E_3 &= |q_3^+|W_3^+(\lambda_3^+ - \dot\eta_\alpha) - BW_3^-(\lambda_3^- - \dot y_\alpha) \\ &\le -O(1)B\sum_{j=1}^{3} |q_j^+| + (\kappa_1 B + O(1))|q_3^+||\lambda_3^+ - \dot\eta_\alpha|). \end{aligned}$$

By Lemma 4.4, we note that

$$h_i^- \approx O(1)\sum_{j=1}^{3} |p_j^+|, \qquad i = 1, 2.$$

For the weighted L^1 strength $q_i(\eta)$ in (6.1), when w_j^a $(1 \le j \le 3)$ are large enough relatively to w_j^m $(j = 1, 2)$, we can get (6.11).

Next, we note that the estimate (6.12) is done in [6]. We further note that our choice of weighted L^1 strength $w_b^1 > w_b^3$ above is consistent with the choice of weights made for 1-wave and 3-wave near the characteristic boundary $\eta = 0$ to acheive the L^1–stability estimate in [6].

Based on the estimates in Sections 3–6, for our problem (L), we can show the existence of the semigroup generated by the wave front tracking method. Actually we also obtain uniqueness of entropy solutions in a broader class, the class of viscosity solutions as defined in [1]. The only difference is that there is a strong shock in our case; nonetheless, we can still proceed with the proof as long as the convergence of the front tracking method has been done (see section 5). We refer the reader to [6] for details. This completes the proof of Theorem 2.2.

Next, we apply Wagner's theorem [16] to conclude the uniqueness of entropy solution to the free boundary problem (E) in the Eulerian coordinates. We still refer the reader to [6] for details. Thus the uniqueness claimed in Theorem 1.1 is also proved.

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