Transonic Flows in Nozzles

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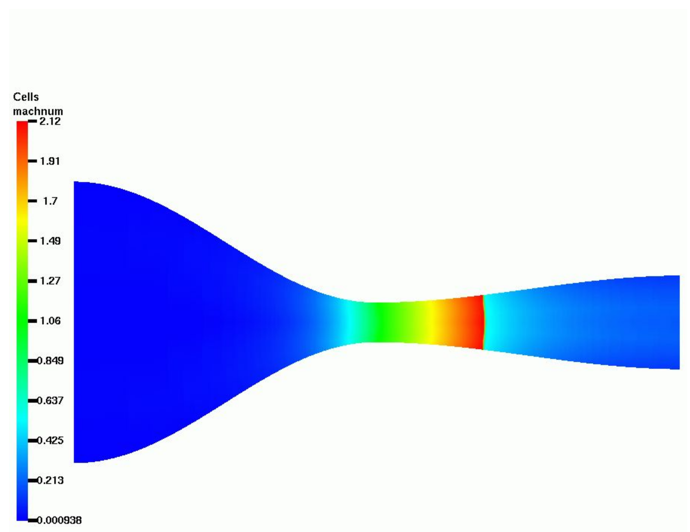
Introduction

♠ We consider steady compressible polytropic gas flows in a convergent–divergent nozzle. It is well–known from experiments and numerical simulations in gas dynamics that, for given appropriately total pressure (pressure at the entry) and back pressure (pressure at the exit), subsonic–supersonic continuous flow may appear near the throat of the nozzle, while a supersonic–subsonic transonic shock may appear in the divergent part of the nozzle. (See the following picture, which belongs to P. Solín et.al. [8].)



♠ This poster is to introduce many results obtained by the author and his collaborators — Gui-Qiang Chen (Northwestern University, USA), Shuxing Chen (Fudan University, China), Yue He (Nanjing Normal University, China), Li Liu (Shanghai Jiaotong University, China), Pan Liu (East China Normal University, China) — on the existence, stability and uniqueness of these transonic flows by using various different models. We are sorry that many achievements in this direction by other people might not be specifically indicated here.

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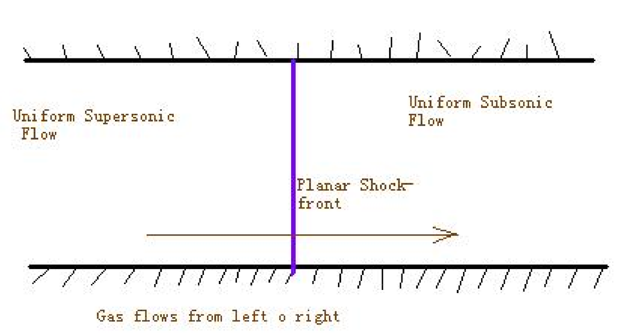
Transonic Shocks in Straight Duct

♠ Existence In straight duct with constant cross–section (see the picture below), one can easily construct a class of uniform transonic shocks with planar shock–fronts by using the Rankine–Hugoniot conditions of either full Euler system or potential flow equation (cf. [10]).



♠ Stability: Two–Dimensional Full Euler Flow For finitely long duct with given back pressure, [9] shows that these special transonic shocks are not stable with respect to perturbations of the supersonic incoming flow, the walls of the duct, and the back pressure. G.-Q. Chen, J. Chen and M. Feldman studied the case of infinitely long duct.

♠ Stability: Three–Dimensional Full Euler Flow For finitely long duct with quadratic cross–section and given back pressure, [2] shows that these special transonic shocks are not stable with respect to perturbations of supersonic incoming flow (which requires to have certain symmetric properties) and the back pressure. The boundary value problem is well–posed only if the back pressure is given with a freedom (containing a constant to be solved with the solution simultaneously).

The instability is closely connected to boundary value problems like the Neumann problem of Poisson equation.

♠ For the stability studied via potential flow equation, see papers of G.-Q. Chen, M. Feldman, Z. Xin, H. Yin etc.

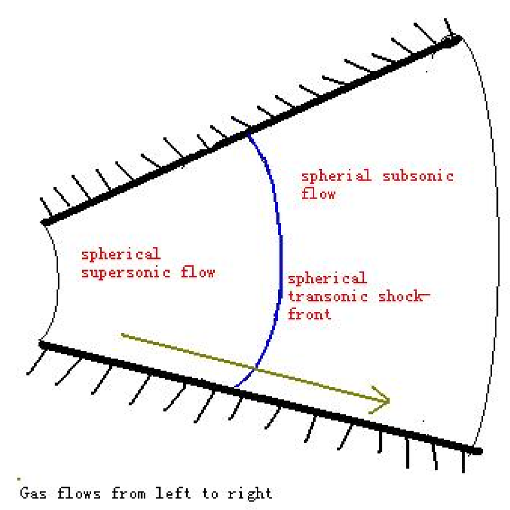
♠ Uniqueness: Three–Dimensional Potential Flow In [1] we proved that, for given uniform supersonic flow at the entry of a finitely long straight duct with arbitrary C^3 cross–section, and a uniform back pressure at the exit of the duct, either there is no any transonic shock in the duct, or there exist a family of transonic shocks in the duct which are translations of the special uniform transonic shocks we constructed before.

In the case of infinitely long straight duct with arbitrary C^3 cross–section, for given uniform supersonic flow at the entry, and assuming the flow behind the shock–front is always subsonic, we showed that the only transonic shock solution is the special uniform transonic shock modulo a translation.

The proof depends on maximum principles of elliptic equations, an extreme principle for solutions of elliptic equations in unbounded domain, and judicious choices of special solutions as comparison functions. This method of proving uniqueness of free boundary problems of elliptic equations may be used to other problem which has a family of special solutions with fine structure (cf. [5]).

Transonic Shocks with Spherical Symmetry

♠ Existence: Full Euler Flow The instability of uniform transonic shocks in duct shows that to understand transonic shocks in divergent nozzles, one should construct and study other special solutions—transonic shocks with spherical symmetry. This amounts to solve discontinuous solutions of several ordinary differential equations, see [10]. For potential flow, this has been done by R. Courant and K. O. Friedrichs. See also papers of E. Endres, H. K. Jenssen and M. Williams on the study of spherically symmetric viscous shocks.



♠ Stability: Two–Dimensional Euler Flow In [4], we showed the spherical symmetric transonic shocks are stable with respect to the perturbations of incoming supersonic flow at the entry, and the back pressure.

New Ingredients: (a) Nonclassical nonlocal elliptic problems arise due to the interaction of the "elliptic part" and "hyperbolic part" of the steady subsonic Euler system; (b) Determination of the position of the shock–front by using integral–like solvability conditions of boundary value problems like the Neumann problem of Poisson equation.

Z. Xin, H. Yin etc. also studied the stability of this class of spherical transonic shocks in nozzles for two–dimensional Euler flows, but requiring the open angle to be small.

♠ Uniqueness: Three–Dimensional Potential Flow In [5], upon generalizing the methods in [1], we showed that for given spherically symmetric supersonic data at the entry of a divergent finitely long straight nozzle with arbitrary smooth cross–section, and spherically symmetric back pressure, the solution must be the spherically symmetric transonic shock constructed before.

Subsonic–Supersonic Flows in Approximate Nozzles

♠ Quasi–One–Dimensional Model as Compressible Flows in Riemannian Manifold From mathematical point of view, the quasi–one–dimensional model of nozzle flows in gas dynamics may be regarded as compressible flows in Riemannian manifold like $(\mathscr{M}, G)$:

$$\mathscr{M} = \{(x, y) \in [-1, 1] \times \mathbf{S}^1\},$$
$$G = dx \otimes dx + n^2(x) dy \otimes dy,$$

with $\mathbf{S}^1$ the unit circle, $n(x)$ a smooth positive strictly convex function on $[-1, 1]$ satisfying $n'(x) < 0$ on $[-1, 0)$ and $n'(x) > 0$ on $(0, 1]$. We call such manifold as convergent–divergent approximate nozzle, since the length of cross-section is decreasing for $x \in [-1, 0]$ and then increasing for $x \in [0, 1]$.

Example: The potential flow equation on $(\mathscr{M}, G)$ is given by

$$\mathrm{div}_G\ (\rho\ \mathrm{grad}_G\ \psi) = 0,$$
$$\frac{\gamma - 1}{2}(\mathrm{grad}_G\ \psi, \mathrm{grad}_G\ \psi)_G + \rho^{\gamma-1} = 1,$$

with ψ the velocity potential function, ρ the density, $\gamma > 1$, and div_G, grad_G, $(\cdot, \cdot)_G$ respectively the divergence operator, gradient operator and inner product with respect to the metric G.

♠ Existence: Subsonic–Supersonic Euler Flows in Convergent–Divergent Approximate Nozzles This is done by constructing symmetric special solutions (i.e., those solutions depend only on x). See [11]. See also L. M. Sibner and R. J. Sibner [7] for earlier works on potential flows in torus.

♠ Stability: Subsonic–Supersonic Potential Flows in Convergent–Divergent Approximate Nozzle In [12], by utilizing methods of A. Kuz'min [3], we proved that the symmetric subsonic–supersonic potential flow is stable with respect to the perturbation of potential function ψ at the entry $\{x = -1\}$.

♠ Uniqueness: Subsonic–Sonic Potential Flows in Convergent Approximate Nozzle The left part of $\mathscr{M}$ (those with $-1 \le x \le 0$) is called convergent approximate nozzle. In [6] we showed that, for given appropriate total pressure such that a symmetric subsonic–sonic flow exists in the nozzle, and the flow at the exit $\{x = 0\}$ is sonic, then if the flow is accelerating at the exit (i.e., $\partial_{xx}\psi > 0$), then any solution of this boundary value problem must be the symmetric subsonic–sonic flow.

New Ingredient: A generalized Hopf boundary point lemma for degenerate elliptic operators, which is applicable to characteristic degenerate boundary point.

Two Unsolved Big Problems

♠ Stability and uniqueness of spherically symmetric transonic shocks in three–dimensional divergent nozzles for the full Euler system with given supersonic upstream flow and back pressure.
♠ Existence and uniqueness of subsonic–supersonic potential flows in two–dimensional physical convergent–divergent nozzles.

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