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Abstract

In this paper, we construct various special solutions on a convergent-divergent surface for the steady compressible complete Euler system and established the stability of the purely subsonic flows. For a given pressure \( p_0 \) prescribed at the “entry” of the surface, as the pressure \( p_1 \) at the “exit” decreases, the flow patterns on the surface change continuously as those happen in a de Laval nozzle: there appear subsonic flow, subsonic–sonic flow, transonic flow and transonic shocks. This work may help us understand subsonic flows in de Laval nozzles. Our results indicate that, to determine a subsonic flow in a two-dimensional de Laval nozzle, if the Bernoulli constant is uniform in the flow field, then this constant should not be prescribed if the pressure, density at the entry and the pressure at the exit of the nozzle are given; if the Bernoulli constant and both the pressures at the entrance and the exit are given, then the average of the density at the entrance is totally determined.

Key words: Euler system, surfaces, special solutions, nozzles, subsonic flows, transonic flows, transonic shocks
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1 Introduction

1. In recent years, there has been an increasing interest in studying various systems of conservation laws originating from mathematical physics on Riemannian manifold. Apart from the purely theoretical curiosity on what is the...
effect of the metric of a Riemannian manifold to the solvability and solutions, the motivation may be twofold. First, some physical problems can be naturally formulated on nontrivial Riemannian manifold. For example, the study of shock wave solutions in general relativity [11], and the problem of supersonic flow past infinite cones of arbitrary cross section and arbitrary angle of attack in aerodynamics [12]. Second, some physical problems can be well “approximated” by fixing them up into certain Riemannian manifolds, and the latter may be more easier to carry out a rigorous analysis. This paper is devoted to demonstrate this in some detail. We begin with a physical problem.

2. It is well known that one of the challenging problems in aerodynamics is to rigorously analyze the flow fields in a de Laval nozzle for given both the pressures at the entrance and the exit. A de Laval nozzle is a duct consisting of a convergent part and a divergent part, while the narrowest part of it is called as the *throat*. De Laval nozzles have many important applications in building wind tunnels, propulsion of jets and rocket engines etc. to transport fluid flows and control their motions [30]. Numerous physical experiments and numerical simulations have shown that for steady flows, for a given pressure $p_0$ at the entrance of the nozzle, the following steady flow patterns may occur (see [7,22,25,30]):

1. *(Subsonic flow)* If the pressure $p_1$ at the exit of the nozzle lies slightly below $p_0$, there will be a tender wind in the nozzle which flows from the entrance to the exit, and the flow field is always subsonic;

2. *(Subsonic–sonic flow and supersonic bubbles)* However, there is a $p_e < p_0$ such that if $p_1 = p_e$, the flow may accelerate to sonic at the throat, and then decelerate to subsonic. Observing shows that there may also appear local supersonic regions near the throat if $p_1$ is close to $p_e$, as that happens for higher subsonic flow past a wing [13,14,21];

3. *(Supersonic flow)* There is a $p_t < p_e$, such that the flow always accelerates in the nozzle, which is sonic at the throat and becomes supersonic after passing it;

4. *(Transonic shock)* For $p_1 \in (p_j, p_e)$ with a $p_j > p_t$, the flow is subsonic in convergent part, sonic at the throat, and becomes supersonic after passing it, and then a normal shock front appears and the flow becomes subsonic behind it, with the pressure increases then to $p_1$ at the exit. As $p_1$ decreases to $p_j$, the position of the shock front moves to the exit;

5. *(Various jets)* If $p_1 < p_j$, various very complex jet flows may occur inside and outside of the nozzle. For simplicity, we sometimes call these possible flows in a de Laval nozzle as “de Laval flows”.

For simplicity, we sometimes call these possible flows in a de Laval nozzle as “de Laval flows”.
3. So we see that, a rigorous analysis of the flow field in a de Laval nozzle by using full compressible steady Euler system is a formidable task up to now. The difficulties one may encounter include:

(1) The Euler system for subsonic–transonic steady flows is of quasi-linear elliptic–hyperbolic composite–mixed type;

(2) To determine the transonic shock is a nonlinear free boundary problem;

(3) The nozzle walls are characteristic boundaries by posing the natural inviscid and impenetrability conditions on them;

(4) The admissible boundary conditions on the entrance and exit of the nozzle are not clear.

4. Thus, aiming to fully understand the de Laval flows, a reasonable program is to construct appropriate approximate models to attack these difficulties separately.

For transonic shocks, some important works have been carried out by many authors. Based on a special transonic shock solution (“the first class” as called in [28]), Canić, Keyfitz, Lieberman [3] initiated the study of stability of transonic shocks by using the transonic small disturbance equation, then Chen, Feldman (see [4,5] etc.) and Xin, Yin [27] carried out their research on the stability of transonic shocks in ducts for full potential equations, then Chen [6] firstly studied this problem in a straight duct for Euler system, and Yuan [29] established the instability of this first class of transonic shock for Euler flows. Thus it is shown that the first class of transonic shock is not physical. Then based on [7], in [28] Yuan established the second class of transonic shocks in divergent nozzles and in [20] Liu and Yuan showed that they are stable. This result may partly explain the transonic shock phenomena in de Laval nozzles.

Various innovative ideas and methods are developed in these works, and many interesting phenomena are observed, such as a new type of nonlocal elliptic problem [20].

The study of transonic shocks shows that various special solutions play a very significant role in understanding flows in de Laval nozzles. That is, the general philosophy of utilizing special solutions as building blocks to construct more general solutions, such as using various self-similar solutions of Riemann problem to study general Cauchy problems of hyperbolic systems of conservation laws [2], also works here.

5. There has already lots of important works on de Laval flows based on the well known quasi-one-dimensional model [22,25] reduced by supposing the nozzle is slowly–varying and the state of the flow is independent of the direc-
tions perpendicular to the axis of the nozzle. Although this model is somewhat rough at first glance, it works surprisingly well [25]. The reason might be that, as showed in [7,28], in certain circumstances, the special solutions obtained by quasi-one-dimensional model are exact solutions of the Euler system! In a series of paper [9,10,17,18], Liu and Glimm etc. have already studied this quasi-one-dimensional model by applying theory of hyperbolic balance laws. See also [15]. Their results also conform well to the experiments, thus demonstrate validity of quasi-one-dimensional model and the importance of certain special solutions of Euler equations in studying nozzle flows.

In the elegant monographs [13,14] by Kuz’mín, by assuming the existence of certain exact solutions, transonic flows are also studied by using von Kármán equation and Chaplygin equation. These monographs also provide many numerical results. Morawetz contributes greatly to the study of transonic flows in many ways, see, for example, [21], where she also presents many outstanding open problems and important methods on transonic flows, some of which, such as compensated compactness, do not depend on special solutions.

Purely subsonic plane flows have been studied for a long time by using full potential equation. See [1,14,26] and references therein. Recently Liu [19] studied instability of subsonic flow in a straight duct. However, it seems that no works have been done in analyzing subsonic de Laval flows by using Euler system before.

6. Now, a natural question is, can we construct special solutions which can describe various flow patterns in a de Laval nozzle for stationary two-dimensional compressible complete Euler system?

This might be very difficult in physical spaces and no any result in physical space is known, at least to the author’s knowledge. However, on a convergent-divergent surface (i.e., a two-dimensional Riemannian manifold), the answer is positive and the construction of special solutions is indeed not so hard.

We note that in [23], L. M. Sibner and R. J. Sibner considered isentropic irrotational flows on a two-dimensional axially symmetric torus, and obtained special solutions to the full potential equation, which exhibit similar phenomena as that in a de Laval nozzle. However, they considered only compact manifold without boundary; the admissible boundary conditions and the stability of these solutions are not studied.

Motivated by the discovery of Sibners, we consider in this paper compressible Euler flows on a convergent-divergent surface (for example, the wall of a three-dimensional de Laval nozzle, see also (9) for such a surface). We then established special solutions as in [23] (see Lemma 2) and especially studied stability of the uniform subsonic flows (see Theorem 5, 6 and 7) based on methods developed in [29] and [19]. Further studies on other flow patterns,
such as transonic flows, will be reported in forthcoming papers.

7. The study of compressible Euler flows on a surface may help us to understand the flows in a two-dimensional de Laval nozzle in many ways.

(1) For a nozzle, it is clear that the boundary conditions on the walls should be the slip condition: the velocity of the flow is perpendicular to the normal of the wall. (This implies that the wall is a characteristic boundary.) However, it is not clear that, what should be the physically appropriate admissible boundary conditions on the entrance and the exit of the nozzle even for potential flow equation. (One often gives potential there [14,26], but since potential is not physical, such conditions are not quite well in understanding the physical phenomena.) By considering the steady subsonic Euler system, which is a hyperbolic-elliptic composite system, the situation may be more perplex, since there is still no general theory on boundary value problems of such systems.

However, since the wall is characteristic, that is, it is not independent of the equations, so roughly speaking it might not influence the boundary conditions on the entrance and the exit. Then we may cancel the walls by considering flows on a surface, while the convergent–divergent effect of the walls is replaced by the non-flat metric of the surface.

Our analysis shows that, for subsonic steady Euler flows, for given the pressure, density at the entrance, and the pressure at the exit of a de Laval nozzle, if we assume that the Bernoulli constant is the same in the whole flow field, then this constant should not be prescribed (see Theorem 5). Otherwise, there may be no any solution. Similarly, if we are given Bernoulli constant and both the pressures at the entrance and the exit, then the density at the entrance should not be totally given: it should contain an unknown constant to be solved, that is, the average of the density at the entrance has already been determined by the other given datum.

Such admissible boundary conditions at the entrance of the nozzle might be instructive for later works on studying other de Laval flows (transonic flows etc.) since all such flows are subsonic near the entrance.

(2) It is noticed for a long time by studying isentropic irrotational flows that, for given both the pressures at the entrance and the exit of a nozzle, the boundary value problem of full potential equation is not well-posed [13,14,26]. As showed in [19] and in this paper, this is also true for Euler System. One needs an integral condition to solve a first order elliptic system. It turns out that, this is closely connected to the total mass flux in the nozzle. For given pressure and density at the entrance, if the pressure at the exit is in certain range, one may adjust the mass flux of the flow (or the Bernoulli constant) to satisfy this integral condition to realize a subsonic flow. Since the flow may be “choked” and there is a maximum mass flux, such an integral condition is
naturally related to the bifurcation phenomena for de Laval flows.

8. Now there are several comments on the method we used to prove the stability of subsonic flows in this paper. It grew from the study of transonic shocks [20,29] and then used by Liu to study subsonic flow in a slowly-varying two-dimensional duct. The key point is that by introducing the Lagrangian transformation and characteristic decomposition one may reduce the Euler system to a $2 \times 2$ system coupled with two algebraic equations (i.e., the Bernoulli’s law and invariance of entropy along streamlines for $C^1$ flows). Since the $2 \times 2$ system is uniformly elliptic for subsonic flows and we are dealing with a small perturbation problem, we can solve the nonlinear problem by using solely Banach fixed point theorem. Compare to the study of the transonic shocks in [20,29], a difference one should pay special attention to is that, since the mass flux $\eta_0$ is unknown, then in Lagrangian coordinate, we encounter a “semi-fixed” boundary value problem as called in [20]. That is, we consider a boundary value problem on the rectangular $N_L = [0,1] \times [0, \eta_0]$ with $\eta_0$ unknown.

9. This paper is organized as follows. In section 2 we first introduce steady compressible Euler system on a Riemannian manifold and then write it explicitly on a given convergent–divergent surface. By using conservation of mass we can employ Lagrangian coordinates and characteristic decomposition to reduce the Euler equations to a $2 \times 2$ system and the Bernoulli’s law and invariance of entropy for $C^1$ flows on streamlines. In section 3 we construct special solutions which behave like those flows in de Laval nozzles by solving several algebraic equations. Section 4 to section 6 are devoted to study the stability of the constructed subsonic flows. In section 4 we state the main theorems and the nonlinear boundary value problems to be solved. In section 6 we solve the linearized problems. Finally, in section 6, by using Banach fixed point theorem to an appropriately constructed nonlinear mapping, we then prove the three theorems stated in section 4.

2 Reduction of Euler System

1. Let $\mathcal{M}$ be a Riemannian manifold with a metric $g$. The equations governing the motions of compressible steady perfect fluids on $\mathcal{M}$ are the following conservation laws (Euler system)[24]:

$$\nabla \cdot (\rho u) = 0, \quad \text{(conservation of mass)} \quad (1)$$
$$\nabla \cdot (\rho u \otimes u) + \nabla p = 0, \quad \text{(conservation of momentum)} \quad (2)$$
$$\nabla \cdot (\rho \mathcal{E} u) = 0. \quad \text{(conservation of energy)} \quad (3)$$

Here $u$ is a vector field on $\mathcal{M}$ represents the velocity of the fluid flows; $p, \rho, \mathcal{E}$
are the scalar pressure, density of mass, and density of energy respectively. $\nabla$- and $\nabla$ are the divergence and gradient operator on $(\mathcal{M}, g)$. Equation (2) may be simplified as

$$\rho \nabla u \cdot u + \nabla p = 0, \quad (4)$$

with $\nabla_X V$ the covariant derivative on $(\mathcal{M}, g)$.

We consider in this paper polytropic gas; that is, the equation of state is

$$p = A(S) \rho^\gamma, \quad (5)$$

where $\gamma > 1$ is the adiabatic exponent, $S$ is the entropy, and $A(S) = \exp(S/C_v)$ for some constant $C_v$. Then the speed of sound is

$$a = \sqrt{\gamma p / \rho}. \quad (6)$$

For $C^1$ flows, the equation (3) may be replaced by

$$\nabla u S = 0; \quad (7)$$

or by the following Bernoulli’s law

$$\frac{1}{2} |u|^2 + \frac{a^2}{\gamma - 1} = c_1, \quad (8)$$

which also holds along streamlines, i.e., the integral curves of the vector field $u$. Here $c_1$ is a number which may be different on different streamline, but remains the same on one streamline even across a shock front.

We say a flow is subsonic (supersonic, sonic) at a point $p \in \mathcal{M}$ if $|u(p)| < a(p)$ ($|u(p)| > a(p)$, $|u(p)| = a(p)$). Flows that are everywhere subsonic (supersonic) on $\mathcal{M}$ are called subsonic (supersonic) flows. If the flow is sonic somewhere and subsonic (supersonic) at the rest points, then the flow is called subsonic-sonic (supersonic-sonic). Finally, if both the sets of points where the flow is subsonic, sonic, supersonic are non-empty, then the flow is called transonic.

2. Now we introduce a convergent–divergent surface as follows. Let $n(r)$ be a positive smooth function on $[0, 1]$. We take $\mathcal{M}$ to be the surface
with the induced metric of $\mathbb{R}^3$. Then $(r, \theta) \in N := [0, 1] \times [0, 2\pi)$ is a coordinate system on $\mathcal{M}$. Direct calculation shows that the metric is

$$ds^2 = (1 + n'(r)^2)dr^2 + n(r)^2d\theta^2;$$

that is,

$$g = \begin{pmatrix}
1 + n'(r)^2 & 0 \\
0 & n(r)^2
\end{pmatrix}. \quad (11)$$

Then by setting $u = (u, v)$ with $u, v$ the velocity component in $r-$ and $\theta-$ direction, the Euler system may be written explicitly as

$$\begin{align*}
\partial_r \left( n(r) \sqrt{1 + n'(r)^2} \rho u \right) + \partial_\theta \left( n(r) \sqrt{1 + n'(r)^2} \rho v \right) &= 0, \\
ud_r u + v \partial_\theta u + \frac{1}{1 + n'(r)^2} \cdot \frac{1}{\rho} \partial_r \rho u &= \frac{n'(r)}{1 + n'(r)^2} \left( n(r) v^2 - n''(r) u^2 \right), \\
u \partial_\theta v + v \partial_\theta v + \frac{1}{n(r)^2} \cdot \frac{1}{\rho} \partial_\theta \rho v &= -\frac{2n'(r)}{n(r)^2} uv, \\
\frac{1}{2} \left( (1 + n'(r)^2) u^2 + n(r)^2 v^2 \right) + \frac{\rho^2}{\rho(r)} &= c_1. \quad (12)
\end{align*}$$

3. Next we write the above Euler system in Lagrangian coordinates $(\xi, \eta)$ by using conservation of mass as follows.

Let

$$\begin{align*}
\frac{d\theta(r,h)}{dr} &= \frac{\xi}{\nu}(r, \hat{\theta}(r, h)), \\
\hat{\theta}(0, h) &= h. \quad (13)
\end{align*}$$

Then $\theta = \hat{\theta}(r, h)$ is a streamline. Set

$$\eta = \eta(r, h) = \int_{\theta(r,0)}^{\hat{\theta}(r,h)} n(r) \sqrt{1 + n'(r)^2} \rho(r, \theta) u(r, \theta) \, d\theta. \quad (14)$$

By the first equation in (12), one gets $\partial \eta(r,h) / \partial r \equiv 0$. Thus $\eta = \eta(h) := \eta(0, h)$ and we see $\eta(0) = 0$. Direct computation also shows that
\[
\frac{\partial \eta}{\partial h} = n(0)\sqrt{1 + n'(0)^2} \rho u(0, h).
\]  

(15)

So if \(\rho u(0, h) \neq 0\) for all \(h \in [0, 2\pi]\), we get the inverse function \(h = h(\eta)\). Set \(\theta(r, \eta) := \tilde{\theta}(r, h(\eta))\). Then (14) is the following identity:

\[
\eta = \int_{\theta(r, 0)}^{\theta(r, \eta)} n(r)\sqrt{1 + n'(r)^2} \rho u(r, s) \, ds.
\]  

(16)

Differentiate this with respect to \(\eta\), one gets \(\partial \theta / \partial \eta = 1/(n(r)\sqrt{1 + n'(r)^2} \rho u)\), while (13) indicates \(\partial \theta / \partial r = v/u\).

Now we introduce the Lagrangian coordinates \((\xi, \eta)\) by

\[
\begin{aligned}
\begin{cases}
  r = \xi, \\
  \theta = \tilde{\theta}(\xi, \eta).
\end{cases}
\end{aligned}
\]  

(17)

Then the above definitions and computations show that

\[
\frac{\partial (r, \theta)}{\partial (\xi, \eta)} = \begin{pmatrix}
1 & 0 \\
\frac{v}{u} & \frac{1}{n(r)\sqrt{1 + n'(r)^2} \rho u}
\end{pmatrix},
\]  

(18)

so

\[
\frac{\partial (\xi, \eta)}{\partial (r, \theta)} = \begin{pmatrix}
1 & 0 \\
-n(r)\sqrt{1 + n'(r)^2} \rho v & n(r)\sqrt{1 + n'(r)^2} \rho u
\end{pmatrix},
\]  

(19)

thus

\[
\begin{aligned}
\begin{cases}
\partial_r = \partial_\xi - n(\xi)\sqrt{1 + n'(\xi)^2} \rho v \partial_\eta, \\
\partial_\theta = n(\xi)\sqrt{1 + n'(\xi)^2} \rho u \partial_\eta.
\end{cases}
\end{aligned}
\]  

(20)

By a lengthy computation, (12) may be written in Lagrangian coordinates as a symmetric system:

\[
A \partial_\xi U + B \partial_\eta U + D = 0,
\]  

(21)

where
\[
A = \begin{pmatrix}
    u(1 + n'^2) & 0 & \frac{1}{\rho} \\
    0 & un^2 & 0 \\
    \frac{1}{\rho} & 0 & \frac{u}{(\rho a)^2}
\end{pmatrix}, \quad B = n\sqrt{1 + n'^2} \begin{pmatrix}
    0 & 0 & -v \\
    0 & 0 & u \\
    -v & u & 0
\end{pmatrix},
\]
\[
D = \begin{pmatrix}
    n'(u^2n'' - v^2n) \\
    2unu' \\
    \frac{(n\sqrt{1 + n'^2})'}{n\sqrt{1 + n'^2}} \frac{u}{\rho}
\end{pmatrix}, \quad U = \begin{pmatrix}
    u \\
    v \\
    \rho
\end{pmatrix}.
\]

By (14), the domain \( N \) in Lagrangian coordinates is \( N_L := [0, 1] \times [0, \eta_0) \) with

\[
\eta_0 = \int_0^{2\pi} n_0 \sqrt{1 + n'(0)^2} \rho(0, \theta)u(0, \theta) \, d\theta.
\]

Note that \( \eta = 0 \) and \( \eta = \eta_0 \) are the same streamline passing the point \((0, 0) \in N\), so the periodical boundary conditions should be posed on them. That is, \( U \) should be periodic with respect to \( \eta \) with period \( \eta_0 \).

4. **Characteristic Decomposition.** Straightforward computations show that

\[
det A = \frac{un^2}{\rho^2 a^2} \left(u^2(1 + n'^2) - a^2\right).
\]

Furthermore, assuming the flow is subsonic, we may solve the generalized eigenvalues of \( B \) respect to \( A \) by

\[
det(\lambda A - B) = 0
\]

as follows:

\[
\lambda_0 = 0; \quad \lambda_\pm = \lambda_R \pm \sqrt{-\lambda_I},
\]

\[
\lambda_R = \frac{\rho u_1 \sqrt{1 + n'^2}}{a^2 - 1}, \quad \lambda_I = \frac{\sqrt{1 + n'^2} \rho u_1 \sqrt{a^2 - |u|^2}}{a(a^2 - 1)}.
\]

Here we have set

\[
u_1 = \sqrt{1 + n'^2}u, \quad v_1 = nu.
\]

The corresponding left eigenvectors \( l \) (i.e., \( \lambda A = lB \)) are
So for non-stagnation subsonic flow, the Euler system is of hyperbolic-elliptic composite type since \( \det A \neq 0 \) and there are both real and complex eigenvalues.

By multiplying the left eigenvector \( l_0 \) to (21), we have

\[
l_0 A (\partial_\xi + \lambda_0 \partial_\eta) + l_0 D = 0,
\]

or, by written explicitly,

\[
\frac{1}{2} \partial_\xi |\mathbf{u}|^2 + \frac{1}{\rho} \partial_\xi p = 0.
\]

This is exactly invariance of entropy along streamlines by using Bernoulli’s law and the equation of state (5):

\[
\partial_\xi \left( \frac{p}{\rho^\gamma} \right) = 0.
\]

Similarly, by setting

\[
w = \frac{v}{u},
\]

then multiplying \( l_\pm \) from left to (21), after a long calculations we obtain

\[
n u_1 (a^2 - u_1^2) \partial_\xi w + (1 + n^2)(a^2 - |\mathbf{u}|^2) \partial_\eta p \\
- n \sqrt{1 + n^2 u_1 v_1 \rho a^2} \partial_\eta w \\
+ \sqrt{1 + n^2} \left( n' (n'' u_1^2 - n v^2) v_1 - 2n' v (u_1^2 - a^2) - a^2 v_1 \frac{(n \sqrt{1 + n^2})'}{n \sqrt{1 + n^2}} \right) \\
= 0;
\]

\[
\left( 1 - \frac{u_1^2}{a^2} \right) \partial_\xi p - \sqrt{1 + n^2} \rho v_1 \partial_\eta p - \rho n v_1 \partial_\eta w \\
+ \rho n' (n'' u_1^2 - n v^2) - \rho u_1^2 \frac{(n \sqrt{1 + n^2})'}{n \sqrt{1 + n^2}} = 0.
\]
We remark that if the flow is supersonic and $u_1 \neq a$, then both the eigenvalues are real and the Euler system is hyperbolic, and the above characteristic decomposition method leads to the same equations (36)(38)(39). We summarizing the above results as the following lemma.

**Lemma 1** If the flow is non-sonic and $u_1 \neq a$, $\rho u \neq 0$, then the compressible steady Euler system (12) is equivalent to (36)(38)(39) and the Bernoulli’s law. Furthermore, if $|u| < a$ ($|u| > a$), then (38)(39) is elliptic (hyperbolic).

**PROOF.** One can easily check that if $|u| \neq a$, $\rho u \neq 0$ and $u_1 \neq a$ hold, then the three left eigenvectors are linearly independent. □

3 Special Solutions

In this section we construct exact special solutions of (12) which depend only on $r$. To be specific, we assume the following hold for the profile of the surface.

\[(N): n(r)\text{ is a positive smooth function on } [0, 1] \text{ and}
\]

\[
n''(r) > 0, \quad n(0) > n(1);
\]

\[
n'(r) < 0 \quad \text{on } (0, s), \quad n'(r) > 0 \quad \text{on } (s, 1) \text{ for a fixed } s \in (0, 1). \quad (41)
\]

We note that (41) implies that the surface is “convergent–divergent”.

**Lemma 2** Suppose (N) holds, and the flow depends only on $r$, and $v \equiv 0$. Then for given $p(0) > 0$, $\rho(0) > 0$ (i.e., the pressure and density at the entrance $\{r = 0\}$ of $\mathcal{M}$),

(1) [subsonic flow] there exists a $p_e < p(0)$ depending only on $p(0)$ and $n$ such that if $p(1) \in (p_e, p(0))$, then there exists a unique subsonic flow on $\mathcal{M}$ with the pressure at the exit $\{r = 1\}$ is $p(1)$;

(2) [subsonic–sonic flow] there exists a unique flow which is subsonic on $\{0 < r < s\} \cup \{s < r < 1\}$ and sonic at $\{r = s\}$ with the pressure at the exit $\{r = 1\}$ is $p_1$.

(3) [transonic flow] there exists a unique flow which is subsonic on $\{0 < r < s\}$, sonic at $\{r = s\}$, and supersonic on $\{s < r < 1\}$, with the pressure at the exit $\{r = 1\}$ is $p_1$. Here the number $p_1 < p_e$ depends only on $n(r)$ and $p(0), \rho(0)$;

(4) [transonic shock] there exists a unique flow which is subsonic on $\{0 < r < s\}$, sonic at $\{r = s\}$, and supersonic on $\{s < r < r_s\}$, and subsonic on $\{r_s < r < 1\}$, with $r = r_s$ a shock, such that: the pressure at the exit $\{r = 1\}$ is $p_1$. Here $p_1 \in [p_j, p_e]$ with $p_j > p_t$ being a number depending only on $n(r)$ and $p(0), \rho(0)$. 

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PROOF. 1. We first write down the Euler system in this simplified case. We write \( u_1 = u_1(r), p = p(r) \) etc. in the following. Now (12) is

\[
\begin{align*}
\rho u_1 u_1' + p' &= 0, \\
(n \rho u_1)' &= 0, \\
\frac{1}{2}|u_1|^2 + \frac{a^2}{\gamma - 1} &= c_1.
\end{align*}
\]

Equation (42) may be replaced by \( S' = 0 \) by using (44) and the equation of state (5) if we consider only continuous flows. So

\[
S(r) \equiv S_0 := A^{-1}\left(\frac{p(0)}{(0)^{\gamma}}\right).
\]

Thus to prove claim (1)–(3) of this lemma, we are led to the following two algebraic equations

\[
\begin{align*}
n \rho u_1 &= c_0, \\
\frac{1}{2}|u_1|^2 + \frac{a^2}{\gamma - 1} &= c_1,
\end{align*}
\]

with \( c_0, c_1 \) two positive numbers.

2. For simplicity, we denote \( p(r) = p_r, \rho(r) = \rho_r, a(r) = a_r, n(r) = n_r \) for \( r \in [0, 1] \). By substituting the first equation in (46) to the second one, we see \( c_1 \) is a function of \( c_0^2 \):

\[
c_1 = \frac{1}{2} \frac{c_0^2}{n_0^2 \rho_0^2} + \frac{a_0^2}{\gamma - 1}.
\]

Now define

\[
g(\rho) = c_1 \rho^2 - \frac{\gamma}{\gamma - 1} A(S_0) \rho^{\gamma+1}, \quad \rho > 0.
\]

We need to solve \( \rho(r) \) by

\[
g(\rho(r)) = \frac{1}{2} \frac{c_0^2}{n(r)^2},
\]

thus determine \( p(r), u(r) \) on \( \mathcal{M} \).

3. To guarantee that the function \( g \) to be positive, we see that there should hold
\[ 0 < \rho < \rho^* := \left( \frac{\gamma - 1}{\gamma} c_1 \frac{A(S_0)}{\rho^*} \right)^{\frac{1}{\gamma - 1}}. \] (50)

Direct calculation shows that

\[ g'(\rho) = \rho \left( 2c_1 - \frac{\gamma + 1}{\gamma - 1} a^2 \right) = \rho(u_1^2 - a^2) \begin{cases} > 0 & \text{if } 0 < \rho < \rho^*, \ i.e., \ the \ flow \ is \ supersonic, \\ < 0 & \text{if } \rho > \rho^*, \ i.e., \ the \ flow \ is \ subsonic. \end{cases} \] (51)

Here

\[ \rho_* := \left( \frac{2}{\gamma + 1} \right)^{\frac{1}{\gamma - 1}} \rho^*. \] (52)

Thus the range of \( g \) on \( (0, \rho^*) \) is \( (0, g(\rho_*]) \) with

\[ g(\rho_*) = \frac{\gamma - 1}{\gamma + 1} c_1 \rho_*^2. \] (53)

4. So to solve (49), one requires that for any \( r \in [0, 1] \),

\[ \frac{c_0^2}{2n(r)^2} \leq g(\rho_*), \] (54)

or equivalently,

\[ G(t) := c_2 \left( \frac{t}{2n_0^2\rho_0} + \frac{a_0^2}{\gamma - 1} \right)^{\frac{\gamma - 1}{\gamma + 1}} - t \geq 0, \] (55)

with

\[ c_2 = 2\frac{\gamma - 1}{\gamma + 1} \left( \frac{2(\gamma - 1)}{\gamma(\gamma + 1)A(S_0)} \right)^{\frac{2}{\gamma - 1}} n(s)^2, \]

and \( t \) represents \( c_0^2 \). Note that \( n(s) = \min\{n(r) : r \in [0, 1]\} \) by assumption (N).

Since \( G(0) > 0 \) and \( G''(t) < 0 \) for
\[ 0 \leq t \leq t^* := \frac{a_0^2}{\gamma - 1} \left( \frac{\gamma + 1}{2} \left( \frac{\rho_0}{\rho_s} \right)^{\gamma - 1} - 1 \right), \]  
(56)

while \( G(t^*) < 0 \), there exists uniquely one \( t_* := \sup\{t \in [0, t^*) : G(t) > 0\} \) with \( G(t_*) = 0 \). So we see that if \( c_0 \in [0, \sqrt{t_*}] \), then (49) is solvable. The solution \( (p(r), \rho(r), u(r)) \) depends on \( c_0 \) as it varies. In aerodynamics, when \( c_0 = \sqrt{t_*} \), a de Laval nozzle is then "chocked".

5. One easily checks that
\[
G(\rho_0^2 a_0^2 n_0^2) = \rho_0^2 a_0^2 (n_s^2 - n_0^2) < 0,
\]
so we have an estimate
\[
t_* < \rho_0^2 a_0^2 n_0^2.
\]
(57)
(58)
This means that when \( c_0 \) varies in \((0, \sqrt{t_*})\), the flow at the entrance is always subsonic. That is, \( \rho_0 > \rho_s \).

Now for any fixed \( c_0 \in (0, \sqrt{t_*}) \), by assumption (N) and the graph of the function \( g(\rho) \), we see \( \rho(r) \) decreases if \( r \in (0, s) \) and attains the minimum \( \rho_{\text{min}} > \rho_s \) at \( r = s \), then increases as \( r \in (s, 1) \). Thus \( \rho(r) > \rho_s \) for \( r \in (0, 1) \) and the flow is purely subsonic. By assumption (N), \( n_1 < n_0 \) implies that \( \rho_1 < \rho_0 \), thus \( p_1 < p_0 \).

6. Now from (48)(49), we see that
\[
c_1(t) \rho_1^2 - \frac{\gamma}{\gamma - 1} A(S_0) \rho_1^{\gamma + 1} = t \quad \frac{2n_1^2}{2n_1^2}.
\]
(59)
As seen in step 4, \( \rho_1 \) is a function of \( t = c_0^2 \). So by differentiating the above equation with respect to \( t \), one obtains that
\[
\frac{d \rho_1(t)}{dt} = \frac{1}{2n_1^2 \rho_1(u_1^2 - a^2)} \left( \frac{n_s^2}{n_1^2} - \frac{\rho_1^2}{\rho_0^2} \right). \]
(60)
By the result of step 5, we know \( u_1^2 - a^2 < 0 \) and \( n_0 > n_1 \), \( \rho_0 > \rho_1 \), so \( \rho_1'(t) < 0 \) for all \( t \in (0, t_*) \). We denote the minimum \( \rho_1(t_*) \) as \( \rho_e \). Thus we obtain \( p_e \) in claim (1) of Lemma 2.

For any \( p_1 \in (p_e, p_0) \) (thus \( \rho_1 \) is known), by the monotonicity of the function \( \rho_1(t) \), we may uniquely obtain a \( t \in (0, t_*) \) with \( \rho_1(t) = \rho_1 \). Then as in step 4 we obtain uniquely one smooth subsonic flow. This proves claim (1).
7. Now taking $c_0 = \sqrt{\tau_*}$ to solve (49), we see $\rho(r)$ monotonically deceases to $\rho_*$ as $r$ increases in $(0, s)$. The flow becomes sonic at $r = s$. Then as $r$ varies in $(s, 1)$, $n(r)$ increases and $\rho(r)$ also increases to $\rho_c$. Thus we proved claim (2).

Another possibility is that, due to the “A”–like shape of the graph of the function $g(\rho)$, $\rho(r)$ may further decreases to the “left branch” for $r \in (s, 1)$. In this case since $\rho(r) < \rho_*$ for $r \in (s, 1)$, the flow becomes supersonic. Then $p_t$ is the smaller root to $g(\rho) = t_*(2n_1^2)$. This proves claim (3).

8. (4) can be proved by analysis similar to those in [28]. ∎

The next lemma concerns regularity of the above constructed flows.

**Lemma 3** Under the assumptions of Lemma 2, the subsonic flow is smooth, the transonic flow is $C^1$, while the subsonic–sonic flow is smooth on $[0, s) \cup (s, 1]$, with jumps of the derivatives of $\rho, u_1$ at $r = s$.

**PROOF.** 1. From (46), we have the following ordinary differential equations satisfied by $\rho, u_1$:

\begin{align}
\rho' &= \frac{n'}{n} - \frac{\rho u_1^2}{n (a^2 - u_1^2)}, \tag{61} \\
u_1' &= -\frac{n'}{n} u_1 a^2 \frac{1}{n (a^2 - u_1^2)}.	ag{62}
\end{align}

So subsonic flow is everywhere smooth, and the subsonic–sonic flow and transonic flow are smooth on $[0, s) \cup (s, 1]$, since $u_1^2 - a^2 \neq 0$ there.

2. We may determine $\rho'(s)$ by L’Hospital principle in calculus for subsonic–sonic flows and transonic flows since $n'(s) = a(s)^2 - u_1(s)^2 = 0$. By (61)(62), we have

\begin{align}
\rho'(s)^2 &= \frac{\rho u_1^2}{n (a^2 - u_1^2)} \frac{n''(s)}{\rho} = \frac{\rho_s^2 n''(s)}{\gamma + 1 n(s)} \neq 0, \tag{63} \\
u_1'(s)^2 &= \frac{u_1(s) n''(s)}{\gamma + 1 n(s)} \neq 0. \tag{64}
\end{align}

For subsonic–sonic flow, whose density decreases on $(0, s)$ and increases on $(s, 1)$, one thus sees that $\rho'(s) = \mp \rho_s \sqrt{\frac{n''(s)}{\gamma + 1 n(s)}}$, so the derivative is not continuous at $r = s$. 

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However, for transonic flow, since $\rho(r)$ always decreases and $u_1$ increases, we see $\rho'(s) = -\rho_s \sqrt{\frac{1}{\gamma+1} \frac{\rho''(s)}{\rho(s)}}$ and $u'_1(s) = \sqrt{\frac{u_1(s)}{\gamma+1} \frac{\rho''(s)}{\rho(s)}}$. This finishes the proof of Lemma 3.

In this paper, we will focus on the stability of the subsonic flow constructed in Lemma 2 under appropriate boundary conditions. Study of the transonic flow and other patterns of flows on $\mathcal{M}$ will be presented in forthcoming papers. For convenience of this paper, we introduce the following definition and notations.

**Definition 4** We call the subsonic flow constructed in Lemma 2 as the background solution determined by $(p_0, \rho_0, p_1)$, and denoted as $U_b = U_b(r) = (p_b(r), \rho_b(r), u_b(r), 0)$. The constant $c_1$ in Bernoulli’s law will be simply denoted as $c_0$ for background solution; The mass flux $\eta_0$ defined by (23) for background solution will be denoted as $\eta_b$.

4 The Perturbed–Subsonic–Flow Problem

We first state the results on stability of subsonic flows which will be proved in this paper.

**Theorem 5** Let $U_b$ be a background solution determined by $(p_b(0), \rho_b(0), p_b(1))$. There are constants $\varepsilon_0$ and $C_0$ depending on $n(r)$ and $U_b$ such that if:

1. The Bernoulli constant $c$ does not depend on streamlines;
2. On $\{r = 0, \theta \in [0, 2\pi]\}$ the following hold for some $\alpha \in (0, 1)$:
   \[
   \|p_0(\theta) - p_0(0)\|_{C^{2,\alpha}[0,2\pi]} \leq \varepsilon \leq \varepsilon_0, \quad (65)
   \]
   \[
   \|\rho_0(\theta) - \rho_0(0)\|_{C^{2,\alpha}[0,2\pi]} \leq \varepsilon \leq \varepsilon_0; \quad (66)
   \]

then there exists a unique subsonic flow $U = (u, v, p, \rho)$ on $\mathcal{M}$ satisfying $p = p_0(\theta)$, $\rho = \rho_0(\theta)$ at the entry $r = 0$, $p = p_b(1)$ at the exit $r = 1$, and

\[
\int_0^{2\pi} \frac{v(0, \theta)}{u(0, \theta)} \, d\theta = 0 \quad (67)
\]

together with the following estimate

\[
\|U - U_b\|_{C^{1,\alpha}(\mathcal{M}; \mathbb{R}^4)} \leq C_0 \varepsilon. \quad (68)
\]

This theorem may be strengthened a little as:
Theorem 6 For given \( p_b(0), \rho_b(0) \), let \((p_b)_{e}\) be the number determined by \( p_b(0), \rho_b(0) \) and \( n \) as in Lemma 2 (2), and \([p_c, p_d] \subset ((p_b)_{e}, p_b(1)) \). Then there are constants \( \varepsilon_0 \) and \( C_0 \) depending on \( n(r), p_b(0), \rho_b(0) \) and \( p_c, p_d \) such that for background solution \( U_b \) determined by \((p_b(0), \rho_b(0), p_b(1))\) with any fixed \( p_b(1) \in [p_c, p_d] \), if

1. The Bernoulli constant \( c \) does not depend on streamlines;
2. On \( \{r = 0, \theta \in [0, 2\pi)\} \) the following hold for some \( \alpha \in (0, 1) \):

\[
\|p_0(\theta) - p_b(0)\|_{C^2,\alpha[0,2\pi]} \leq \varepsilon \leq \varepsilon_0,
\]

\[
\|\rho_0(\theta) - \rho_b(0)\|_{C^2,\alpha[0,2\pi]} \leq \varepsilon \leq \varepsilon_0;
\]

then there exists a unique subsonic flow \( U = (u, v, \rho) \) on \( M \) satisfying \( p = p_0(\theta), \rho = \rho_0(\theta) \) at the entry \( r = 0 \), \( p = p_b(1) \) at the exit \( r = 1 \), and

\[
\int_0^{2\pi} \frac{v(0, \theta)}{u(0, \theta)} \, d\theta = 0
\]

together with the following estimate

\[
\|U - U_b\|_{C^{1,\alpha}(M; \mathbb{R}^4)} \leq C_0 \varepsilon.
\]

Another result is the following theorem.

Theorem 7 Let \( U_b \) be a background solution determined by \((p_b(0), \rho_b(0), p_b(1))\). There are constants \( \varepsilon_0 \) and \( C_0 \) depending on \( n(r) \) and \( U_b \) such that if:

1. The Bernoulli constant \( c \) is given and does not depend on streamlines;
2. On \( \{r = 0, \theta \in [0, 2\pi)\} \) the following hold for some \( \alpha \in (0, 1) \):

\[
\|p_0(\theta) - p_b(0)\|_{C^2,\alpha[0,2\pi]} \leq \varepsilon \leq \varepsilon_0,
\]

\[
\|\rho_0(\theta) - \rho_b(0)\|_{C^2,\alpha[0,2\pi]} \leq \varepsilon \leq \varepsilon_0;
\]

\[
|c - c_b| \leq \varepsilon \leq \varepsilon_0;
\]

then there exists a unique subsonic flow \( U = (u, v, \rho) \) on \( M \) and a constant \( e \) satisfying \( p = p_0(\theta), \rho = \rho_0(\theta) + e \) at the entry \( r = 0 \), \( p = p_b(1) \) at the exit \( r = 1 \), and

\[
\int_0^{2\pi} \frac{v(0, \theta)}{u(0, \theta)} \, d\theta = 0
\]

together with the following estimate
\[ |c| + \|U - U_0\|_{C^{1,\alpha}(\mathcal{M}; \mathbb{R}^4)} \leq C_0 \varepsilon. \]

A strengthened version of Theorem 7 analogy to Theorem 6 also holds.

**Remark 8** The differences between Theorem 7 and Theorem 5 are that, the Bernoulli constant is given in the former, but unknown in the latter; and the density is given with a constant difference (i.e., containing an unknown constant \(e\) to be solved) in the former, while is totally given in the latter. The occurrence of \(e\) in Theorem 7 may be viewed as that the average of the density at the entrance is not arbitrary.

As mentioned in section 1, it is perplex that what is a physically meaningful boundary condition to study nozzle flows. Although there may be numerous mathematical ways to pose boundary conditions at the entrance and exit (such as the two ways showed above and see also [19]), it seems that (from the author’s opinion) formulate boundary conditions as in Theorem 7 and 5 are more physical, since the pressures at the entrance and exit should be totally given. The methods of this paper can also deal with the case if the Bernoulli constant depends on streamlines (c.f. the proof of Theorem 7).

The condition (67) comes from the periodic assumption on \(\theta\) (as well as \(\eta\)) coordinate and is used to guarantee the uniqueness. For a two-dimensional de Laval nozzle, it should be replaced by the slip condition on walls.

**Remark 9** We will show that for given pressure at the entrance and the exit simultaneously, the problem requires an integral condition to be solvable. Thus there needs a constant \(c\) or \(e\) to adjust the flow so that it is realizable. This reflects the change of total mass flow through a nozzle as pressure varies and is then natural from physical point of view (c.f. the construction of special solutions).

**Remark 10** We may also perturb the metric \(g\) of the surface \(\mathcal{M}\) a little and obtain stability of subsonic flows with the above given boundary conditions.

We now begin to prove Theorem 5. Theorem 7 may be proved in a similar way and we will point out the difference later. Theorem 6 follows from a simple observation on how the constant \(C_0\) and \(\varepsilon_0\) depending on the background solutions.

We first note that \(\eta_0\) is unknown; that is, we encounter a “semi-fixed” boundary value problem. We may use the following transformation \(\Phi : (\xi, \eta) \mapsto (\xi, \overline{\eta}) :\)
\[
\begin{aligned}
\xi &= \xi, \\
\hat{\eta} &= \frac{\eta}{\eta_0} 
\end{aligned}
\] 

(69)

to normalize the domain \( N_L \) to \( \Omega = \{(\xi, \hat{\eta}) \in [0, 1] \times [0, 1]\}. Then we will write out the boundary value problem to be studied in an easy-to-be-linearized form.

### 4.1 Perturbed Equations

Let

\[ w_2 = p(\xi, \hat{\eta}) - p_b(\xi). \] 

(70)

We need to find the equations of \( w \) and \( w_2 \).

1. We begin with (38). Let

\[
\begin{align*}
e_2(\xi) &:= \frac{1 + n'(\xi)^2}{n(\xi)} \cdot \frac{a^2 - |u|^2}{(a^2 - u_1^2)u_1 \eta_0} \bigg|_{U=U_b(\xi)} , \\
d_2(\xi) &:= \left( \frac{n' a^2 - 2u_1^2}{n} - \frac{n'n''}{1+n'^2} \right) \bigg|_{U=U_b(\xi)} ,
\end{align*}
\] 

(71) (72)

and

\[
\tilde{f}_2(U) := \sqrt{1 + n''^2 v_1 p} a^2 \frac{\partial \eta}{\eta_0(a^2 - u_1^2)} \partial_\eta w + \left( e_2(\xi) - \frac{1 + n'(\xi)^2}{n(\xi)} \cdot \frac{a^2 - |u|^2}{(a^2 - u_1^2)u_1 \eta_0} \right) \partial_\eta p \\
+ \frac{n''^2 v_1}{u(a^2 - u_1^2)} \left( d_2(\xi) - \frac{n' a^2 - 2u_1^2}{n} - \frac{n'n''}{1+n'^2} \right) w.
\] 

(73)

Then (38) is

\[ \partial_\xi w + e_2(\xi) \partial_\eta w_2 + d_2(\xi) w = \tilde{f}_2(U). \] 

(74)

By setting

\[
\begin{align*}
D_2(\xi) &:= \exp \int_0^\xi d_2(s) \, ds, \\
f_2(U) &:= D_2(\xi) \tilde{f}_2(U),
\end{align*}
\] 

(75) (76)

the equation (74) may also be written in divergence form as
\[
\partial_\xi(D_2w) + \partial_\eta(D_2v_2w_2) = f_2(U).
\]

(77)

Note that \(f_2\) consists of higher order terms.

2. Now we turn to (39). By (61) one sees that \(p_\eta(\bar{\xi})\) solves

\[
\frac{dp}{d\xi} = \frac{n'}{n} \frac{\rho u_1^2 a^2}{a^2 - u_1^2}.
\]

(78)

Then (39) is equivalent to

\[
\partial_\xi w_2 - \frac{n\rho^2 a^2 u_1^3}{(a^2 - u_1^2)\eta_0} \partial_\eta w - \frac{n'}{n} (H(U) - H(U_b))
= \sqrt{1 + n^2 \rho a^2} \frac{v_1 \partial_\eta p}{(a^2 - u_1^2)\eta_0} + \frac{n'\rho a^2}{a^2 - u_1^2} v_2,
\]

(79)

where

\[
H(U) := \frac{\gamma pu_1^2}{a^2 - u_1^2}.
\]

(80)

Since

\[
u_1^2 = \frac{2c - \frac{2a^2}{1 + n^2\rho w^2}}{1 + \frac{n^2\rho w^2}{1 + n^2\rho w^2}}
\]

(81)

holds by Bernoulli's law, and

\[
a^2 = \gamma \left( \frac{p_0(\bar{\eta})}{\rho_0(\bar{\eta})^{\gamma}} \right) \frac{1}{p^{\frac{\gamma - 1}{\gamma}}}.
\]

(82)

holds by constancy of entropy, and note that \(c\) is also an unknown number, we see \(H\) is actually an analytical function of \(p, w\) and \(c\). So by Taylor expansions, we have

\[
H(U) - H(U_b) = \partial_p H(U_b) w_2 + \partial_w H(U_b) w + \partial_c H(U_b)(c - c_b)
+ O(|U - U_b|^2 + |c - c_b|^2)
= \hat{d}_1(\bar{\xi}) w_2 + \hat{d}_0(\bar{\xi})(c - c_b) + O(|U - U_b|^2 + |c - c_b|^2),
\]

(83)

where
\[
\hat{d}_1(\xi) := \frac{a^4}{(a^2 - u_1^2)} \left( \gamma \left( \frac{u_1}{a} \right)^4 + (1 - \gamma) \left( \frac{u_1}{a} \right)^2 - 2 \right) \bigg|_{U=U_b(\xi)}, \tag{84}
\]
\[
\hat{d}_0(\xi) := \frac{2\gamma p a^2}{(a^2 - u_1^2)^2} \bigg|_{U=U_b(\xi)} > 0. \tag{85}
\]

Let
\[
d_1(\xi) := \frac{n'}{n} \hat{d}_1, \quad d_0(\xi) := \frac{n'}{n} \hat{d}_0, \quad e_1(\xi) := \frac{n' \rho^2 a^2 u_1^3}{(a^2 - u_1^2) \eta_0} \bigg|_{U=U_b(\xi)}, \quad f_1(U, c) := \frac{\sqrt{1 + n' \rho a^2}}{(a^2 - u_1^2) \eta_0} v_1 \partial_p^p + \frac{n' \rho a^2}{a^2 - u_1^2} v^2 + \left( \frac{n' \rho^2 a^2 u_1^3}{(a^2 - u_1^2) \eta_0} - e_1 \right) \partial_\eta w + O(|U - U_b|^2 + |c - c_b|^2).
\]

Then (79) is
\[
\partial^\xi w_2 - e_1 \partial_\eta w - d_1(\xi)w_2 - d_0(\xi)(c - c_b) = \bar{f}_1(U, c). \tag{86}
\]

By introducing an auxiliary function
\[
D_1(\xi) := \exp \left( - \int_0^{\xi} d_1(s) \, ds \right), \tag{87}
\]
and setting
\[
f_1(U, c) = D_1(\xi) \bar{f}_1(U, c), \tag{88}
\]
then (86) in divergence form is
\[
\partial^\xi (D_1 w_2) - \partial_\eta (D_1 e_1 w) - D_1 d_0 \cdot (c - c_b) = f_1(U, c). \tag{89}
\]

Note that \( f_1 \) also consists of higher order terms.

\section*{4.2 The Perturbed-Subsonic-Flow Problem}

So far we see that to prove Theorem 5, we need only to solve \( c, w, w_2 \) and \( u, v, \rho, \eta_0 \) from the following nonlinear boundary value problems:
\[(NP1): \begin{cases} 
(77), (89) & \text{in } \Omega, \\
w_2 = \tilde{p}_0(\tilde{\eta}) - p_0(0) & \text{on } \tilde{\xi} = 0, \\
w_2 = 0 & \text{on } \tilde{\xi} = 1, \\
\text{Periodic conditions} & \text{on } \tilde{\eta} = 0, \tilde{\eta} = 1, \\
f_0 w(0, s) \, ds = 0; 
\end{cases} \]

\[(NP2): \begin{cases} 
\rho = \bar{\rho}_0(\tilde{\eta}) \left( \frac{u_2 + p_0}{\bar{p}_0(\tilde{\eta})} \right)^{\frac{1}{7}}, \\
u_1 = \sqrt{\frac{2 - \frac{2(n_0^2)}{1 + n_0^2}}{1 + \frac{n_0^2}{1 + n_0^2} u^2}}, \\
v = \frac{w u_1}{\sqrt{1 + n^2}}; 
\end{cases} \]

\[(NP3): \eta_0 = \frac{n(0) \sqrt{1 + n'(0)^2} \int_0^{2\pi} \rho_0(\theta) \, d\theta}{\int_0^1 \frac{d\tilde{\eta}}{u(0, \tilde{\eta})}}. \]

There are several remarks to explain the above three problems.

Periodic conditions in (NP1) mean that the functions are periodical with respect to \( \tilde{\eta} \) with period 1.

The expressions in (NP2) come from Bernoulli’s law, invariance of entropy and definition of \( w \).

The formula in (NP3) on computing \( \eta_0 \) comes from (17). We know that on \( r = \xi = \tilde{\xi} = 0 \), there should hold

\[ \eta_0 \frac{d\tilde{\eta}}{d\theta} = n(0) \sqrt{1 + n'(0)^2} \rho_0(\theta) u(0, \tilde{\eta}) \] (90)

and

\[ \tilde{\eta}(0) = 0, \quad \tilde{\eta}(2\pi) = 1. \] (91)

This is a two–point boundary value problem of an ordinary differential equation. From (90) we see

\[ \eta_0 \int_0^{\tilde{\eta}} \frac{ds}{u(0, s)} = n(0) \sqrt{1 + n'(0)^2} \int_0^\theta \rho_0(s) \, ds. \] (92)
Thus if (NP3) holds, then this two-point boundary value problem is uniquely solvable since $u(0, s), \rho_0(s)$ are positive and bounded away from zero.

We note that in (NP1) and (NP2),

$$\bar{\rho}_0(\bar{\eta}) := \rho_0(\theta(0, \bar{\eta}_\theta)), \quad \bar{\rho}_0(\bar{\eta}) := \rho_0(\theta(0, \bar{\eta}_\theta))$$

may depend on the solution. But if (65)(66) hold and the solution $U$ is near the background solution, that is, $U \in \mathcal{O}_5$ with

$$\mathcal{O}_5 := \{ U = (u, v, p, \rho) : U \text{ is periodic with respect to } \bar{\eta} \text{ with period } 1, \text{ and } \| U - U_b \|_{C^{1,\alpha}([0,1]^4)} \leq \delta, \| u \|^2/2 + a^2/\gamma - 1 = \text{ const } \in \mathbb{R} \},$$

then the estimate

$$\| \bar{\rho}_0(\bar{\eta}) - \rho_b(0) \|_{C^{1,\alpha}[0,1]} + \| \bar{\rho}_0(\bar{\eta}) - \rho_b(0) \|_{C^{1,\alpha}[0,1]} \leq C\varepsilon$$

is true. Here $C$ is a positive constant depending only on $U_b$ and $\delta$ is chosen so small that $U$ is still subsonic and $\rho u \neq 0$. This fact follows from results like Proposition 2.2 in [29] (or Lemma 4.6 in [4]) concerning $C^{1,\alpha}$ homeomorphisms on different domains.

5 Analysis of Linearized Problems

To solve the three coupled nonlinear problems (NP1)(NP2)(NP3), we will apply Banach fixed point theorem to a nonlinear mapping constructed by appropriately linearized problems.

5.1 Statement of the Linearized Problems

We begin with (NP1). Let $f_1, f_2 \in C^\alpha(\Omega)$, $g \in C^{1,\alpha}[0,1]$ be periodical with respect to $\bar{\eta}$ with period 1. Problem (P1) solves $\bar{w}, \bar{w}_2$ and $\bar{c}$ from the following linear boundary value problem of first order elliptic system:
\[(P1) : \begin{cases} 
\partial \xi(D_1 \ddot{w}_2) - \partial \eta(D_1 \epsilon_1 \ddot{w}) - D_1 d_0 \cdot (\dddot{c} - c_b) = f_1, \\
\partial \xi(D_2 \ddot{w}) + \partial \eta(D_2 \epsilon_2 \ddot{w}_2) = f_2 \\
\ddot{w}_2 = g \\
\ddot{w}_2 = 0 \\
\text{Periodic conditions} \\
\int_0^1 w(0, s) \, ds = 0. 
\end{cases}\]

For given \(\tilde{\rho}_0(\tilde{\eta}), \tilde{\rho}_0(\tilde{\eta})\) and \(\dddot{w}, \dddot{w}_2, \dddot{c}\) obtained from problem (P1), problem (P2) solves \(\dddot{\rho}, \dddot{\rho}u\) and \(\dddot{v}\) by

\[
(P2) : \begin{cases} 
\dddot{\rho} = \tilde{\rho}_0(\tilde{\eta}) \left( \frac{\dddot{w}_2 + \tilde{p}_0}{\tilde{\rho}_0(\tilde{\eta})} \right)^\frac{1}{4}, \\
\dddot{a} = \gamma(\dddot{w}_2 + \tilde{p}_0)/\dddot{\rho}, \\
\dddot{u} = \frac{1}{\sqrt{1 + n(\xi)^2}} \sqrt{\frac{2c_0 - \tilde{p}_0}{1 + \gamma n(\xi) \dddot{w}_2}}, \\
\dddot{v} = \dddot{w} \dddot{u}. 
\end{cases}
\]

Problem (P3) solves \(\dddot{\eta}_0\) by

\[
(P3) : \quad \dddot{\eta}_0 = \frac{n(0) \sqrt{1 + n'(0)^2} \int_0^{2\pi} \tilde{\rho}_0(\theta) \, d\theta}{\int_0^1 \frac{d \dddot{\eta}}{\dddot{w}(0, \theta)}}
\]

with a given \(\rho_0(\theta)\) and \(\dddot{u}\) obtained from (P2).

5.2 Solving Problem (P1)

Since (P2)(P3) are simple algebraic equations, we focus on the solution of problem (P1).

Lemma 11 Let \(f_1, f_2 \in C^\alpha(\Omega), g \in C^{1,\alpha}[0, 1]\) be periodical with respect to \(\tilde{\eta}\) with period 1. Then problem (P1) has a unique solution \(\dddot{w}, \dddot{w}_2\) and \(\dddot{c}\). In addition, the following estimate holds:

\[
\|w_2\|_{C^{1,\alpha}(\Omega)} + \|w\|_{C^{1,\alpha}(\Omega)} + |\dddot{c} - c_b| \\
\leq C \left( ||f_1||_{C^\alpha(\Omega)} + ||f_2||_{C^\alpha(\Omega)} + ||g||_{C^{1,\alpha}[0,1]} \right). \tag{95}
\]

Proof. 1. By linearity of (P1), we first separate it as the following two problems:
\[
\begin{align*}
\left( \partial_{\xi}(D_1 \bar{w}_2^{(1)}) - \partial_{\eta}(D_1 e_1 \bar{w}^{(1)}) - D_1 d_0 \cdot (\bar{c} - c_b) \right) &= f_1, \\
\left( \partial_{\xi}(D_2 \bar{w}^{(1)}) + \partial_{\eta}(D_2 e_2 \bar{w}_2^{(1)}) \right) &= 0 \\
\bar{w}_2^{(1)} &= g \\
\bar{w}_2^{(1)} &= 0 \\
\text{Periodic conditions} \\
\int_0^1 w^{(1)}(0, \bar{\eta}) \, d\bar{\eta} &= 0;
\end{align*}
\]

in \( \Omega \); on \( \bar{\xi} = 0 \), \( \bar{\xi} = 1 \), (96)

\[
\begin{align*}
\left( \partial_{\xi}(D_1 \bar{w}_2^{(2)}) - \partial_{\eta}(D_1 e_1 \bar{w}^{(2)}) \right) &= 0, \\
\left( \partial_{\xi}(D_2 \bar{w}^{(2)}) + \partial_{\eta}(D_2 e_2 \bar{w}_2^{(2)}) \right) &= f_2 \ 	ext{in} \ \Omega; \\
\bar{w}_2^{(2)} &= 0 \ \text{on} \ \bar{\xi} = 0, \\
\bar{w}_2^{(2)} &= 0 \ \text{on} \ \bar{\xi} = 1, \\
\text{Periodic conditions} \ &\text{on} \ \bar{\eta} = 0, \ \bar{\eta} = 1, \\
\int_0^1 w^{(2)}(0, \bar{\eta}) \, d\bar{\eta} &= 0.
\end{align*}
\]

Then clearly

\[
\begin{align*}
\bar{w} &= \bar{w}^{(1)} + \bar{w}^{(2)}, \\
\bar{w}_2 &= \bar{w}_2^{(1)} + \bar{w}_2^{(2)},
\end{align*}
\]

and \( \bar{c} \) solve (P1).

2. By the second equation in (96), we may introduce a potential function \( \Phi^{(1)} \) on the simply connected domain \( \Omega = [0, 1] \times [0, 1] \) such that

\[
\begin{align*}
\partial_{\xi} \frac{\partial \Phi^{(1)}}{D_2} &= \partial_{\eta} \frac{\partial \Phi^{(1)}}{D_2 e_2}.
\end{align*}
\]

If \( w^{(1)} \) and \( w_2^{(1)} \) satisfy the periodic conditions and \( \int_0^1 w(0, s) \, ds = 0 \), note that \( D_2(0) = 1 \), we see that \( \Phi^{(1)} \) also satisfies the periodic conditions. The reverse is also true. So we may formulate a Neumann problem:

\[
\begin{align*}
\left( \partial_{\xi} \left( \frac{D_2}{D_2 e_2} \partial_{\xi} \Phi^{(1)} \right) + \partial_{\eta} \left( \frac{D_2}{D_2 e_2} \partial_{\eta} \Phi^{(1)} \right) \right) &= D_1 d_0 \cdot (\bar{c} - c_b) = f_1 \ (\bar{\xi} = 0), \\
\partial_{\xi} \Phi^{(1)} &= e_2(0) g \ (\bar{\xi} = 1), \\
\text{Periodic conditions} \ &\text{on} \ \bar{\eta} = 0, \ \bar{\eta} = 1, \\
\Phi^{(1)}(0, 0) &= 0.
\end{align*}
\]

Let

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\[
\ddot{c} - c_b = -\frac{\int_\Omega f_1(\xi, \tilde{\eta}) \, d\xi \, d\tilde{\eta} + \int_0^1 g(\tilde{\eta}) \, d\tilde{\eta}}{\int_0^1 D_1(\xi) d_0(\xi) \, d\xi},
\] (101)

and note that by (N) the denominator is nonzero; see step 3 below. Then it is well known that (100) is uniquely solvable and the estimate

\[
\|\Phi^{(1)}\|_{C^{2,\alpha}(\Omega)} \leq C \left( \|f_1\|_{C^\alpha(\Omega)} + \|g\|_{C^{1,\alpha}[0,1]} \right)
\] (102)

holds [8].

3. We claim that if the background solution is subsonic and \((u_1)_b \neq 0\), then \(\int_0^1 D_1(\xi) d_0(\xi) \, d\xi < 0\); so it is also bounded away from zero for a compact family of background subsonic flows (as those in the assumption of Theorem 6).

Indeed, if we set

\[
h(\xi) := \frac{(u_1)_b^2}{a_b^2} \in (0, 1),
\]

then by Bernoulli’s law (44) and (78)(61), we have

\[
a_b^2(h) = \frac{c_b}{\gamma h + \frac{1}{\gamma-1}},
\] (103)

\[
\rho_b(h) = \left( \frac{c_b}{\gamma A(S_b)\left(\frac{1}{\gamma} h + \frac{1}{\gamma-1}\right)} \right)^\frac{1}{\gamma-1} > 0,
\] (104)

\[
\frac{n'}{n} d\xi = \frac{-(1-h)dh}{((\gamma-1)h + 2)h},
\] (105)

\[
d_1(\xi) = \frac{n' \gamma h^2 + (1-\gamma)h - 2}{n(1-h)^2},
\] (106)

\[
d_0(\xi) = \frac{2\rho_b(h) \cdot n'}{(1-h)^2 n}.
\] (107)

So, by noting that the change of variables (105) is valid if \(h \in (0, 1)\), then

\[
- \int_0^\xi d_1(\xi) \, d\xi = G(h(\xi))
\]

\[
:= \int_{h(0)}^{h(\xi)} \frac{\gamma h^2 + (1-\gamma)h - 2}{h(1-h)((\gamma-1)h + 2)} \, dh \leq 0
\] (108)

and 27
\[ D_1(\tilde{\xi}) = \exp(G(h(\tilde{\xi}))) \in (0, 1]. \] (109)

Hence we get

\[
\int_0^1 D_1(\tilde{\xi}) d\tilde{\xi} = -\int_{h(0)}^{h(1)} \frac{2\rho_b(h) \exp(G(h))}{h(1-h)((\gamma-1)h+2)} dh.
\] (110)

Note that by (N) \((n(1) < n(0))\), there must hold \(h(1) > h(0)\), so the above integral is negative.

4. For (97), similar to step 2, we may introduce a potential \(\Phi^{(2)}\) such that

\[
\tilde{w}^{(2)} = \frac{\partial\tilde{\xi}}{D_1 \tilde{e}_1}, \quad \tilde{w}_2^{(2)} = \frac{\partial\tilde{\eta}}{D_1}. \] (111)

The condition \(\tilde{w}_2^{(2)} = 0\) on \(\tilde{\xi} = 0\) and \(D_2(0) = 1\) guarantee that \(\Phi^{(2)}\) satisfies the periodic conditions on \(\tilde{\eta} = 0, \tilde{\eta} = 1\). We need only to solve the following equi-valued surface problem to determine \(\tilde{w}^{(2)}, \tilde{w}_2^{(2)}\):

\[
\begin{cases}
\partial_{\tilde{\xi}} \left( \frac{D_2}{D_1 \tilde{e}_1} \partial_{\tilde{\xi}} \Phi^{(2)} \right) + \partial_{\tilde{\eta}} \left( \frac{D_{2\tilde{\xi}\tilde{\eta}}}{D_1} \partial_{\tilde{\eta}} \Phi^{(2)} \right) = f_2 \quad \text{in } \Omega; \\
\Phi^{(2)} = 0 \quad \text{on } \tilde{\xi} = 0, \\
\Phi^{(2)} = \tilde{e} \quad \text{on } \tilde{\xi} = 1, \\
\text{Periodic conditions} \quad \text{on } \tilde{\eta} = 0, \tilde{\eta} = 1, \\
\int_0^1 \partial_{\tilde{\xi}} \Phi^{(2)}(0, \tilde{\eta}) d\tilde{\eta} = 0.
\end{cases}
\] (112)

Here \(\tilde{e}\) is also a number to solve.

By standard theory of equi-valued surface problem for elliptic equations (see, for example, [16]), (112) is uniquely solvable, and the following estimate holds:

\[
\| \Phi^{(2)} \|_{C^{3,\alpha}(\Omega)} + |\tilde{e}| \leq C \| f_2 \|_{C^\alpha(\Omega)}. \] (113)

5. The estimate (95) is then easily obtained, and Lemma 11 is proved.

6 Existence of Perturbed Subsonic Flows

In this section we prove Theorem 5, 6 and 7 by Banach fixed point theorem.
6.1 Construction of a Nonlinear Mapping

Let $\mathcal{O}_\delta$ be defined as in (93). For any $U \in \mathcal{O}_\delta$, we will construct a nonlinear mapping $T$ on $\mathcal{O}_\delta$ which maps $U$ to $\tilde{U}$ in the following several steps by using problem (P1)(P2)(P3), and one easily sees that the fixed point of this mapping $T$ is exactly a solution to problems (NP1)(NP2)(NP3).

1. It is easy to verify that $\mathcal{O}_\delta$ is a closed subset of the Banach space $C^{1,\alpha}(\Omega; \mathbb{R}^4)$.

2. For any fixed $U \in \mathcal{O}_\delta$, we denote

$$c = |u|^2/2 + a^2/(\gamma - 1).$$

Then there holds

$$|c - c_\delta| \leq C\delta.$$  \hspace{1cm} (115)

By (P3) we obtain a constant $\eta_0$, and with this $\eta_0$, due to (92), we have a strictly monotonic $C^{2,\alpha}$ function $\theta = \theta(\tilde{\eta})$ with $\theta(0) = 0, \theta(1) = 2\pi$. Let

$$\tilde{p}_0(\tilde{\eta}) := p_0(\theta(\tilde{\eta})), \quad \tilde{p}_0(\tilde{\eta}) := p_0(\theta(\tilde{\eta})).$$

Then (94) holds as explained in section 4.

3. Now, recalling (76)(88), we will take

$$g = \bar{p}_0(\tilde{\eta}) - p_0(0), \quad f_1 = f_1(U, c), \quad f_2 = f_2(U)$$

as nonhomogeneous terms in (P1). Direct calculation and (94) show that

$$\|f_1\|_{C^\alpha(\Omega)} \leq C\delta^2;$$

$$\|f_2\|_{C^\alpha(\Omega)} \leq C\delta^2;$$

$$\|g\|_{C^{1,\alpha}[0,1]} \leq C\varepsilon$$

with a positive constant $C$ depending only on the background solution. Thus by Lemma 11 we infer not only unique existence of $(\bar{w}, \bar{w}_2, \bar{c})$, but also the estimate

$$\|\bar{w}\|_{C^{1,\alpha}(\Omega)} + \|\bar{w}_2\|_{C^{1,\alpha}(\Omega)} + |\bar{c} - c_\delta| \leq C(\delta^2 + \varepsilon).$$

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4. Now by (P2), we have $\tilde{u}, \tilde{v}, \tilde{\rho}$. Since these expressions are analytical, we easily get the following estimates by differential mean value theorem and (94)(121):

$$
\|\tilde{\rho} - \rho_b\|_{C^1(\Omega)} + \|\tilde{u} - u_b\|_{C^1(\Omega)} + \|\tilde{v}\|_{C^1(\Omega)} \leq C(\delta^2 + \varepsilon).
$$

(122)

Therefore we have obtained $\tilde{U}$ and it also satisfies

$$
\|\tilde{U} - U_b\|_{C^1(\Omega; \mathbb{R}^4)} \leq C(\delta^2 + \varepsilon).
$$

(123)

By choosing

$$
\delta = 2C\varepsilon, \quad \varepsilon_0 \leq 1/(2C)^2,
$$

(124)

then the mapping $T : U \mapsto \tilde{U}$ is well defined on $\mathcal{O}_\delta$ and maps $\mathcal{O}_\delta$ into $\mathcal{O}_\delta$.

6.2 Contraction of the Nonlinear Mapping

To apply Banach fixed point theorem, for any $U^{(i)} \in \mathcal{O}_\delta (i = 1, 2)$, we need to show that

$$
\|\tilde{U}^{(1)} - \tilde{U}^{(2)}\|_{C^1(\Omega)} \leq \frac{1}{2} \|U^{(1)} - U^{(2)}\|_{C^1(\Omega)}.
$$

(125)

1. We denote $c^{(i)}$ to be the Bernoulli constant corresponding to $U^{(i)}$. Since $U^{(i)} \in \mathcal{O}_\delta$, one easily has

$$
|c^{(1)} - c^{(2)}| \leq C' \|U^{(1)} - U^{(2)}\|_{C(\Omega)}.
$$

(126)

Let $\eta_0^{(i)}$ and $\theta^{(i)} = \theta^{(i)}(\tilde{\eta})$ be the number and function determined by $U^{(i)}$ according to (P3) and (92) respectively. Then there holds

$$
|\eta_0^{(1)} - \eta_0^{(2)}| \leq C' \|U^{(1)} - U^{(2)}\|_{C(\Omega)}.
$$

(127)

Since

$$
\frac{d\theta^{(i)}(\tilde{\eta})}{d\tilde{\eta}} = \frac{\eta_0^{(i)}}{n(0)\sqrt{1 + n'(0)^2 \rho_0(\theta) u^{(i)}(0, \tilde{\eta})}}
$$

(128)

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and \( \rho_0(\theta) \) is bounded away from zero, there holds

\[
\| \theta^{(1)}(\tilde{\eta}) - \theta^{(2)}(\tilde{\eta}) \|_{C^{2,\alpha}[0,1]} \leq C' \| U^{(1)} - U^{(2)} \|_{C^{1,\alpha}(\Omega)}. \tag{129}
\]

So for \( \tilde{p}_0^{(i)}(\tilde{\eta}) = p_0(\theta^{(i)}(\tilde{\eta})), \) \( \tilde{\rho}_0^{(i)}(\tilde{\eta}) = \rho_0(\theta^{(i)}(\tilde{\eta})) \), and let

\[ g^{(i)} = \tilde{p}_0^{(i)}(\tilde{\eta}) - p_0(0), \tag{130} \]

we have

\[
\| g^{(1)} - g^{(2)} \|_{C^{1,\alpha}[0,1]} \leq C' \| U^{(1)} - U^{(2)} \|_{C^{1,\alpha}(\Omega)}, \tag{131}
\]

\[
\| \tilde{p}_0^{(1)} - \tilde{\rho}_0^{(2)} \|_{C^{1,\alpha}[0,1]} \leq C' \| U^{(1)} - U^{(2)} \|_{C^{1,\alpha}(\Omega)}, \tag{132}
\]

2. Now let

\[
f_1^{(i)} = f_1(U^{(i)}, c^{(i)}), \quad f_2^{(i)} = f_2(U^{(i)}), \quad i = 1, 2. \tag{133}
\]

Straightforward computation shows that

\[
\| f_{j}^{(1)} - f_{j}^{(2)} \|_{C^{\alpha}(\Omega)} \leq C' \| U^{(1)} - U^{(2)} \|_{C^{1,\alpha}(\Omega)} \quad j = 1, 2. \tag{134}
\]

By Lemma 11, we then see

\[
\| w^{(1)} - w^{(2)} \|_{C^{\alpha}(\Omega)} + \| w_2^{(1)} - w_2^{(2)} \|_{C^{1,\alpha}(\Omega)} \leq C' \| U^{(1)} - U^{(2)} \|_{C^{1,\alpha}(\Omega)}. \tag{135}
\]

3. Now using mean value theorem to (P2), one then easily gets the estimate (125) by choosing \( \varepsilon_0 \) further small. This finishes the proof of Theorem 5.

**Proof of Theorem 4.2.** We only need to note that, more specifically, the constant \( C, C' \) depend only on the distance of the background solution \( U_b \) to the sonic surface \( \{ U : |u| = a \} \) and the stagnation–vacuum surface \( \{ U : \rho u = 0 \} \) in the phase space to guarantee uniform ellipticity and positiveness of \( \rho, u. \)  \( \square \)
6.3 Proof of Theorem 7

We now sketch out the proof of Theorem 7.

1. The first difference is the equation (89). Now, for the Bernoulli constant given, the function \( H(U) \) in (80) is actually depending on \( p, w \) and \( \rho_0 \) and we have

\[
\partial_{\rho_0} H(U_b) = \tilde{d}_0(\tilde{\xi}) := \frac{2\gamma c \rho_0 a_5^2}{(a_5^2 - (u_1)^2)\rho_0(0)} \neq 0. \tag{136}
\]

Then by setting \( \tilde{d}_0 := n'\tilde{d}_0/n \), and using the boundary condition of \( \rho \) at \( \tilde{\xi} = 0 \), (39) is

\[
\partial_{\tilde{\xi}}(D_1 w_2) - \partial_{\tilde{\eta}}(D_1 e_1 w) - D_1 \tilde{d}_0 \cdot (\rho(\tilde{\eta}) - \rho_b(0) + e) = \tilde{f}_1(U, e),
\]

or

\[
\partial_{\tilde{\xi}}(D_1 w_2) - \partial_{\tilde{\eta}}(D_1 e_1 w) - D_1 \tilde{d}_0 e = f_1(U, e) \tag{137}
\]

with \( f_1 = \tilde{f}_1 + D_1 \tilde{d}_0 \cdot (\rho(\tilde{\eta}) - \rho_b(0)) \).

We may then formulate an elliptic problem like (NP1) and solve its linearized version by adjusting \( e \) as done before.

2. To solve the nonlinear problem, the construction of a nonlinear mapping should be modified as follows.

Now define

\[
O_\delta := \left\{ U = (u, v, p, \rho)^t : U \text{ is periodic with respect to } \tilde{\eta} \text{ with period } 1, \right.
\]

\[
\left. \|U - U_b\|_{C^{1,0}(\Omega; \mathbb{R}^4)} \leq \delta, \right\}
\]

and

\[
K_\delta = \{ e \in \mathbb{R} : |e| \leq \delta \}. \tag{138}
\]

For any \( U \in O_\delta \) and \( e \in K_\delta \), we first use (P3) to solve \( \tilde{\eta}_0 \):

\[
\tilde{\eta}_0 = \frac{n(0) \sqrt{1 + n'(0)^2} \int_0^{2\pi} (\rho_0(\theta) + e) \, d\theta}{\int_0^1 \frac{\partial \tilde{\eta}}{u(\theta, \tilde{\eta})}}. \tag{140}
\]

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then determine $\theta = \theta(\tilde{\eta})$ by using (90) with $\eta_0$ there replaced by $\tilde{\eta}_0$. So we get $\tilde{p}_0(\tilde{\eta})$ and $\tilde{\rho}_0(\tilde{\eta})$.

Now we can solve corresponding problem (P1) to obtain $\tilde{w}, \tilde{w}_2$ and $\tilde{\epsilon}$.

Then in problem (P2), by setting

$$\tilde{\rho} = (\tilde{\rho}_0(\tilde{\eta}) + \tilde{\epsilon}) \left( \frac{\tilde{w}_2 + p_b}{\tilde{\rho}_0(\tilde{\eta})} \right)^{\frac{\gamma}{\gamma - 1}}$$

and substituting it in the other expressions in (P2), we may obtain further $\tilde{u}$ and $\tilde{v}$.

One may show that the above procedure established a contractive mapping on $\mathcal{O}_\delta \times \mathcal{K}_\delta$ for some $\delta = C\varepsilon$ if $\varepsilon < \varepsilon_0$ is sufficiently small.

This finishes the proof of Theorem 7.

References


