A Remark on Determination of Transonic Shocks in Divergent Nozzles for Steady Compressible Euler Flows

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Abstract

In this paper we construct a class of transonic shock in a divergent nozzle which is a part of an angular sector (for two-dimensional case) or a cone (for three-dimensional case) which does not contain the vertex. The state of the compressible flow depends only on the distance from the vertex of the angular sector or the cone. It is supersonic at the entrance, while for appropriately given large pressure at the exit, a transonic shock front appears in the nozzle and the flow becomes subsonic after passing it. The position and strength of the shock is automatically adjusted according to the pressure given at the exit. We demonstrate these phenomena by using the two dimensional and three dimensional full steady compressible Euler systems. The idea involved is to solve discontinuous solutions of a class of two-point boundary value problems for systems of ordinary differential equations. Results established in this paper may be used to analyze transonic shocks in general nozzles.

Key words: Euler system, spherical flows, transonic shocks, two-point boundary value problems, nozzles
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1 Introduction

One of the prominent directions in gas dynamics concerning nozzle flows is to thoroughly understand the flow fields in convergent-divergent nozzles (the so-called de Laval nozzles), due to the existence of numerous mathematically challenging problems, and its importance in designing appropriate nozzles which

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meet the needs in building wind tunnels, jet propulsions etc. As illustrated in [10,13,18], if the pressure at the exit of the nozzle lies in a certain interval, subsonic gas flow at the entrance will accelerate to sonic at the throat of the de Laval nozzle, becomes supersonic at the divergent part, and then a normal shock front appears and the flow becomes subsonic behind it, with the pressure increases to the given value at the exit. Such phenomena have been observed in experiments, numerical simulations for numerous times. However, rigorous theoretical analysis are fairly inadequate. Most of them relies upon several simplified models. For the so-called "quasi-one-dimensional model" (see, for example, [18,19]), which totally neglects the motion of the gas perpendicular to the axis of the nozzle, we know the significant works of T.-P. Liu and J. Glimm etc. [11,12,15,16]. For von Kármán equation and Chaplygin equation (which is equivalent to potential flow equation), several interesting results based on perturbation arguments have been established concerning the transition from subsonic to supersonic at the throat, see the monograph [13] by A. G. Kuz'min and references therein. See also [1] for some earlier developments on transonic flows.

In recent years there has been an increasing interest in transonic shocks appearing in nozzle flows with the developments in theory of elliptic boundary value problems on non-smooth domains and the introducing of some new ideas to treat the free boundary, i.e., the shock front (see [2–6,8,20–22] etc.). We note that for supersonic-supersonic shocks, many powerful tools have already been proposed and fruitful results have been established, see, for instance, [7,9,14].

In the study of transonic shocks, one assumes that the flow is supersonic at the entrance of a nozzle, and some additional condition should be given at the exit to induce subsonic flow. (Prescribe receivers' pressure is a physically well-accepted condition [10], although some artificial conditions were also introduced to ensure well-posedness.) Initiated by the work of G.-Q. Chen and M. Feldman [3], where they concerned transonic shock in a finite duct with rectangle section for potential flow equation, later in a series of papers ([4,5] etc.) they established the existence and stability of transonic shock in an infinite cylinder with appropriately given data at infinity. Z. Xin and H. Yin also used potential equation in [20] to study the transonic shock in two-dimensional nozzles which are small perturbations of straight ducts. S. Chen [6] firstly studied the transonic shock problem in two-dimensional straight duct by using full Euler system. H. Yuan generalized these results to flows with cylindrical symmetry in [21] and two-dimensional variable-area ducts in [22] for full Euler system. In [8], S. Chen and H. Yuan also studied transonic shocks in three-dimensional duct by elaborate decomposition of Euler system. Lots of fruitful ideas and tools were used and developed in doing these works, which help us understanding the structure of transonic shock problems in nozzles in depth.

However, it may be surprising that all the results cited above indicate that the transonic shock problem is ill-posed for given pressure at the exit. For example, [22] showed that the pressure at the exit cannot be given arbitrarily: to assure the stability, the pressure at the exit can only be given with freedom one, that is, it contains an unknown constant to be solved. This contradicts to what we know before as illustrated by the experiments and other simplified models. So why this happens?

The main reason is that, all the above mentioned results are based on the following *"background solution"*. Let $U = (u_0, p, \rho)$ represent the one-dimensional motion of gas, with u_0 the velocity, p the scalar pressure, and ρ the density. For polytropic gas we can determine the sound speed $a = \sqrt{\gamma p/\rho}$. Then for the constant vector U_b^- satisfying $(u_0)_b^- > a_b^-$, suppose the shock front is flat, then by Rankine-Hugoniot conditions, we can uniquely determine a constant state U_b^+ which is subsonic ($(u_0)_b^+ < a_b^+$) and physical entropy condition $p_b^- < p_b^+$ also holds (this is exactly Proposition 3 below). We call (U_b^-, U_b^+) together with the flat shock front as "background solution" (see Figure 1).

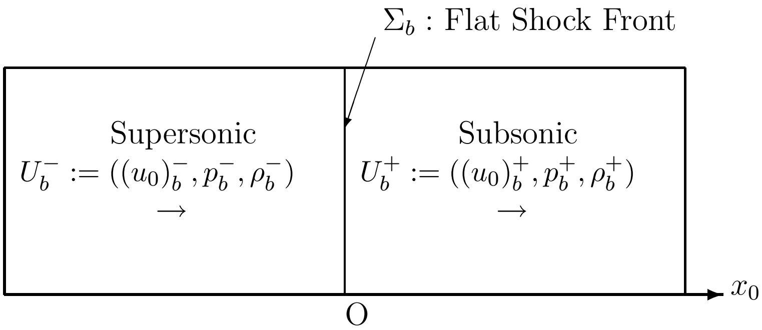

Figure 1. Background Solution

We note here that the position of the flat shock front can be moved along the x_0-axis arbitrarily without changing $U_b^{\pm}$, and U_b^+ (especially, pressure p_b^+) is uniquely determined by U_b^-. This leads to the ill-posedness for given arbitrary pressure at the exit. So we are in a dilemma: on one hand, from the background solution and the small perturbation arguments based on it, one can only know that for given pressure at the exit, the transonic shock may not exist [20,22]; on another hand, this result is not so physical.

The purpose of this paper is to construct another class of special solutions to the transonic shock problem in two–dimensional or three–dimensional divergent nozzles for the full steady compressible Euler system, which may partly dissolve the above dilemma. We assume that the nozzles are either parts of angular sectors or cones (not containing the origin) and the flow is with spherical symmetry. This leads to solve a class of two-point boundary value problems with discontinuous solutions for systems of ordinary differential equations. Our result can be stated in a rough way as (see Theorem 6 and Corollary 8 below for the precise statements):

Theorem. *For any given supersonic state at the entrance of the above divergent nozzle, there exists an interval $I \in \mathbb{R}^+$. If the receiver's pressure $p_1 \in I$, then there exists a unique transonic shock front in the nozzle, such that the pressure of the subsonic flow behind the shock front increases exactly to p_1 at the exit.*

Thus this class of exact solutions of Euler system are more "physical" and should be considered as the first order approximation of transonic flows in divergent nozzles. (For convenience, we call this class of solutions as *"the second class background solution"* and then the background solution in Figure 1 as *"the first class"*.) The relation between the first and second class background solution is an interesting topic needs to be investigated further, and the way we construct the second class of background solutions may be instructive to study transonic shocks in more general nozzles.

We remark that for potential flow equations, analysis of spherical flow has already been carried out in [10]. In that case, one just needs to solve two algebraic equations, namely, (8) below and the Bernoulli's law. Thus the process there is simpler than here.

The rest of the paper, §2, is devoted to the construction of these solutions. We first formulate the transonic shock problem for two dimensional Euler system, and then study some integral curves of the reduced differential equations. Since for spherical flow there is little difference in two dimensional and three dimensional Euler system, as explained in the last subsection, the result obtained before is also valid for three dimensional case.

2 Determination of Transonic Shocks for Steady Euler Systems

2.1 Formulation of the Spherical Transonic Shock Problem for 2-D Euler System

The Euler system describes steady compressible inviscid flow in two dimensional spaces takes the following conservation form:

$$\begin{cases} \nabla \cdot \mathbf{m} = 0, \\ \nabla \cdot \left(\dfrac{\mathbf{m} \otimes \mathbf{m}}{\rho} \right) + \nabla p = 0, \end{cases} \tag{1}$$

together with the Bernoulli's law

$$\frac{1}{2}\mathbf{u}^2 + i = c_0, \tag{2}$$

where c_0 is a constant along the streamline (c_0 may have different values along different streamlines). Here ρ, p, i are the density, scalar pressure, and enthalpy of the fluid, while $\mathbf{u} = (u_1, u_2)$ and $\mathbf{m} = \rho\mathbf{u}$ are velocity and momentum density vector respectively. The first equation in (1) is conservation of mass, the second one is conservation of momentum, and the Bernoulli's law corresponds to the conservation of energy. In the case of polytropic gas we take the equation of state as $p = A(S)\rho^\gamma, \gamma \in (1, \infty)$, with S the entropy. Then (2) takes the form

$$\frac{1}{2}(u_1^2 + u_2^2) + \frac{a^2}{\gamma - 1} = c_0, \tag{3}$$

where $a = \sqrt{\partial p/\partial \rho} = \sqrt{\gamma A(S)\rho^{\gamma-1}} = \sqrt{\gamma p/\rho}$ is the local speed of sound.

In polar coordinates, under the transformation

$$r = \sqrt{x_1^2 + x_2^2}, \qquad \theta = \arctan\frac{x_2}{x_1}$$

and define

$$u_r = u_1\cos\theta + u_2\sin\theta, \qquad u_\theta = -u_1\sin\theta + u_2\cos\theta,$$

then (1)(3) are of the following conservation form:

$$\begin{cases} \partial_r(r\rho u_r) + \partial_\theta(\rho u_\theta) = 0, \\ \partial_r(r\rho u_r^2 + rp) + \partial_\theta(\rho u_\theta u_r) - (\rho u_\theta^2 + p) = 0, \\ \partial_r(r\rho u_r u_\theta) + \partial_\theta(\rho u_\theta^2 + p) + \rho u_r u_\theta = 0, \\ \frac{1}{2}(u_r^2 + u_\theta^2) + \frac{a^2}{\gamma-1} = c_0. \end{cases} \tag{4}$$

For spherical flow, i.e., the flow depends only on the variable r and $u_\theta = 0$, (4) is reduced to the following system of ordinary differential equations (for simplicity, we write u_r as u):

$$\begin{cases} \frac{\mathrm{d}(r\rho u)}{\mathrm{d}r} = 0, \\ \frac{\mathrm{d}(\rho u^2 + p)}{\mathrm{d}r} + \frac{\rho u^2}{r} = 0, \\ \frac{1}{2}u^2 + \frac{a^2}{\gamma-1} = c_0. \end{cases} \tag{5}$$

Denote the nozzle as

$$\mathrm{N} := \{(r, \theta) : 0 \le \theta \le \alpha, r_0 \le r \le r_1\} \tag{6}$$

with $\alpha, r_0 > 0$, and $U := (u, p, \rho)$. Then for spherical flow, the transonic shock problem may be formulated as

$$(\mathbf{TSP}): \begin{cases} (5) & r \in [r_0, r_1], \\ U = U_0 & r = r_0, \\ p = p_1 & r = r_1, \end{cases} \tag{7}$$

where $U_0 := (u_0, p_0, \rho_0)$ is the given supersonic state (i.e., $u_0 > a_0$) at the entrance of the nozzle, and p_1 is a given appropriately large receiver pressure at the exit. (We remark that the condition at $r = r_1$ can also be replaced by either $u = u_1$ or $\rho = \rho_1$ etc.) We always suppose $u_0 > 0$ in this paper, that is, $r = r_0$ is really the entrance of N. We are looking for solutions of (**TSP**) with the following structure: there exists a $r_s \in [r_0, r_1]$ such that the flow is supersonic in $[r_0, r_s)$ and subsonic in $(r_s, r_1]$, while $r = r_s$ is a shock front.

2.2 Analysis of Supersonic Curves, Subsonic Curves and R-H Curves

It is not easy to solve (**TSP**) directly for given $p = p_1$ at $r = r_1$. We would rather like to change our strategy as to find out the relation between the exhaust pressure p_1 and the position of shock front $r = r_s$. To this end, we need analyze in detail the variation of the state for supersonic, subsonic flow, and the transonic shock front separates them along the nozzle.

The first equation in (5) indicates that

$$r\rho u = b_0 := r_0 \rho_0 u_0, \tag{8}$$

while by the initial value $c_0 = \frac{1}{2}u_0^2 + \frac{a_0^2}{\gamma-1}$. Using the fact that for C^1 solutions the entropy S is invariant along streamline (this can be induced from (1)(2), see, for example, §7 of [10]), from (5) we solve

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{-\rho u^2 a^2}{r(u^2 - a^2)}, \tag{9}$$

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{u a^2}{r(u^2 - a^2)}, \tag{10}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{-\rho u^2}{r(u^2 - a^2)}. \tag{11}$$

For Mach number $M = u/a$, using Bernoulli's law and (10) we have

$$\frac{\mathrm{d}M}{\mathrm{d}r} = \frac{M}{r} \cdot \frac{1 + \frac{\gamma-1}{2}M^2}{M^2 - 1}. \tag{12}$$

These four differential equations describe the variation of states of subsonic and supersonic flows in the nozzle, so we call their integral curves as either *"subsonic"* or *"supersonic" curves.*

Proposition 1 *[Global Existence and Asymptotic Behavior] For given non-sonic initial data* (u_0, p_0, ρ_0)*, supposing* $u_0 > 0$*, we have:*

(1) If $M_0 > 1$*, then* ρ, p *monotonically decrease to* 0 *as* r *increases, while* M, u *monotonically increase. More precisely,* $\rho \sim O(\frac{1}{r})$*,*$p \sim O(\frac{1}{r^\gamma})$*,* $M \sim O(r^{\frac{\gamma-1}{2}})$*,* $u \to \sqrt{2c_0}$ *as* $r \to \infty$*;*

(2) If $M_0 < 1$*, then* ρ, p *monotonically increase, while* M, u *monotonically decrease to* 0 *as* r *increases. More precisely,* $\rho \to (\frac{(\gamma-1)c_0}{\gamma A(S_0)})^{\frac{1}{\gamma-1}}$*,* $p \to (\frac{\gamma-1}{\gamma}c_0)^{\frac{\gamma}{\gamma-1}} A(S_0)^{\frac{-1}{\gamma-1}}$*,* $M \sim O(\frac{1}{r})$*,* $u \sim O(\frac{1}{r})$ *as* $r \to \infty$*.*

PROOF. (1) For $M_0 > 1$, it is easy to see that M increases, and furthermore the solution of (12) exists on $[r_0, \infty)$ since the right hand side increases linearly on M for large M. Note that $\rho = 0$ is a solution of (11), so by the uniqueness of solutions of Cauchy problem for ordinary differential equations, ρ, well defined on $[r_0, \infty)$, decreases to 0 as $r \to \infty$. Since entropy S is invariant, the same also holds for p. Because the right hand side of (12) is bounded away from zero, M has no upper bound on $[r_0, \infty)$. Thus the asymptotic behavior of M (also ρ) is clear. The others are obtained by the equation of state $p = A(S)\rho^\gamma$ and Bernoulli's law.

(2) can be proved in a similar fashion.

Remark 2 *Using* (8) *we can further determine that (note that* $A(S_0) = p_0/\rho_0^\gamma$*)*

(1) For $M_0 > 1$*,* $\rho \sim \frac{b_0}{\sqrt{2c_0}}\frac{1}{r}$*,* $p \sim A(S_0)(\frac{b_0}{\sqrt{2c_0}})^\gamma \frac{1}{r^\gamma}$*,* $M \sim \frac{(2c_0)^{\frac{\gamma+1}{4}}}{\sqrt{\gamma A(S_0) b_0^{\frac{\gamma-1}{2}}}} \cdot r^{\frac{\gamma-1}{2}}$*, as* $r \to \infty$*;*

(2) For $M_0 < 1$*,* $u \sim b_0(\frac{\gamma A(S_0)}{(\gamma-1)c_0})^{\frac{1}{\gamma-1}} \cdot \frac{1}{r}$*,* $M \sim b_0 \frac{(\gamma A(S_0))^{\frac{1}{\gamma-1}}}{((\gamma-1)c_0)^{\frac{\gamma+1}{2(\gamma-1)}}} \cdot \frac{1}{r}$*, as* $r \to \infty$*.*

Now suppose $r_s \in [r_0, r_1]$ is a discontinuous point of the flow, and denote the left and right limit of the state at r_s as $U_- := (u_-, p_-, \rho_-)$, $U_+ := (u_+, p_+, \rho_+)$ respectively, then the following Rankine-Hugoniot conditions should hold:

$$\rho_+ u_+ = \rho_- u_- = \frac{b_0}{r_s}, \tag{13}$$

$$\rho_+ u_+^2 + p_+ = \rho_- u_-^2 + p_-, \tag{14}$$

$$\frac{1}{2}u_+^2 + \frac{\gamma}{\gamma-1}\frac{p_+}{\rho_+} = \frac{1}{2}u_-^2 + \frac{\gamma}{\gamma-1}\frac{p_-}{\rho_-} = c_0. \tag{15}$$

Proposition 3 *[Solvability of Rankine-Hugoniot Conditions] If $M_0 > 1$, then for any $r_s \in [r_0, r_1]$, there exists a unique $U_+(r_s)$ such that the above Rankine-Hugoniot conditions hold for $(U_-(r_s), U_+(r_s))$. ($U_-(r_s)$ is obtained by solving (9)–(12) with initial data $U = U_0$ at $r = r_0$.) Furthermore, $U_+(r_s)$ is subsonic and the physical entropy condition holds: $p_-(r_s) < p_+(r_s)$.*

PROOF. 1. For simplicity we denote below $U_\pm(r_s)$ as $U_\pm$. From (13)(15) we obtain

$$\frac{1}{2}\rho_+ u_+^3 + \frac{\gamma}{\gamma-1} p_+ u_+ = \frac{1}{2}\rho_- u_-^3 + \frac{\gamma}{\gamma-1} p_- u_-,$$

while (14) leads to

$$\frac{\gamma}{\gamma-1}\rho_+ u_+^3 + \frac{\gamma}{\gamma-1} p_+ u_+ = \frac{\gamma}{\gamma-1}\rho_- u_-^2 u_+ + \frac{\gamma}{\gamma-1} p_- u_+.$$

Subtracting the above equations and using (13) we get

$$\frac{\gamma+1}{2(\gamma-1)}\rho_- u_- u_+^2 - \frac{\gamma}{\gamma-1}(\rho_- u_-^2 + p_-)u_+ + (\frac{1}{2}\rho_- u_-^3 + \frac{\gamma}{\gamma-1} p_- u_-) = 0.$$

This is a quadratic equation of u_+ and note that u_- is a solution. Thus another solution is

$$\begin{aligned} u_+ &= \frac{(\frac{1}{2}\rho_- u_-^3 + \frac{\gamma}{\gamma-1} p_- u_-)}{\frac{\gamma+1}{2(\gamma-1)}\rho_- u_-}\frac{1}{u_-} \\ &= \frac{c_*}{u_-} \end{aligned} \tag{16}$$

by using Bernoulli's law. Here we have set the constant

$$c_* := \frac{2(\gamma-1)c_0}{\gamma+1}. \tag{17}$$

2. Now from (13)(14) we get

$$\rho_+ = \frac{b_0}{c_*}\frac{u_-}{r_s}, \tag{18}$$

$$p_+ = \frac{b_0 u_-}{r_s} + p_- - \frac{b_0 c_*}{r_s u_-}. \tag{19}$$

3. We show U_+ is subsonic. By Bernoulli's law we have

$$u^2 - a^2 = \frac{\gamma+1}{2}(u^2 - c_*). \tag{20}$$

Thus

$$
\begin{aligned}
u_+^2 - a_+^2 &= \frac{\gamma+1}{2}(u_+^2 - c_*) = \frac{\gamma+1}{2}(u_+^2 - u_+u_-) \\
&= \frac{\gamma+1}{2}\frac{u_+}{u_-}(c_* - u_-^2) = \frac{u_+}{u_-}(a_-^2 - u_-^2) \\
&< 0
\end{aligned}
$$

since U_- is supersonic by Proposition 1.

4. Finally we demonstrate the entropy condition. From (19)(20) one gets

$$
\begin{aligned}
p_+ - p_- &= \frac{b_0}{r_s}(u_- - \frac{c_*}{u_-}) \\
&= \frac{b_0}{r_s u_-}(u_-^2 - c_*) > 0.
\end{aligned}
$$

This finishes the proof of Proposition 3.

Remark 4 *Under the assumption of Proposition 3, by Proposition 1 and* (16) *we see the strength of the shock (measured by $|u_- - u_+|$) increases as r_s increases.*

Remark 5 *By Proposition 3, we can also see that transonic shock of subsonic-supersonic type (i.e., the states ahead and behind the shock front are subsonic and supersonic respectively) is not stable since this violets entropy condition.*

For fixed supersonic initial data U_0, by supersonic curves we get $U_-(r_s)$ for any $r_s \in [r_0, r_1]$. Due to (16)(18)(19) we obtain $U_+(r_s)$. We call these *R-H curves.* (For example, we call $u(r)$ as the R-H curve of u or $u-$R-H curve.) Now fix a $r_s \in [r_0, r_1]$, we may obtain the subsonic curves $U(r) := (u(r), p(r), \rho(r))$ for $r \in [r_s, r_1]$ with initial data $U = U_+(r_s)$ at $r = r_s$. We are interested in the relation between $U(r_1)$ and r_s. We denote the dependence of $U(r_1)$ on r_s as $U_{r_s}(r_1)$.

2.3 Main Result

The main result of this paper is:

Theorem 6 *[Monotonicity of $U(r_1)$ on r_s] For fixed supersonic initial data U_0 with $u_0 > 0$ at $r = r_0$, we have*

(1). $u(r_1), M(r_1)$ are strictly increasing continuous functions of $r_s \in [r_0, r_1]$;

(2). $\rho(r_1), p(r_1)$ are strictly decreasing continuous functions of $r_s \in [r_0, r_1]$.

PROOF. 1. We first prove (1) for $u(r_1)$. Note that (2) cannot be induced from (1) by using Bernoulli's law directly, since now entropy is not invariant: it changes with r_s.

2. *$u(r_1)$ is a continuous function of r_s.* By continuously dependence of initial values of solutions of ordinary differential equations, $u(r_1)$ is a continuous function of $u_+(r_s)$. The latter, by (16), is continuous on $u_-(r_s)$, which is C^1 on r_s. Thus our claim is true.

3. Next we show $u(r_1)$ *is a strictly monotonic function of* r_s. We prove this by contradiction.

Suppose there is a $u(r_1)$ such that there exist $r_{s_1}, r_{s_2} \in [r_0, r_1]$ with $r_{s_1} < r_{s_2}$, and the subsonic curves of u passing the two points $(r_{s_1}, u_+(r_{s_1}))$, $(r_{s_2}, u_+(r_{s_2}))$ on the R-H curve of u coincide at $r = r_1$.

4. We may rewrite (10) as

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{u}{r}\frac{c_* - \frac{\gamma-1}{\gamma+1}u^2}{u^2 - c_*} \tag{21}$$

by (20) and Bernoulli's law. Thus $(r_{s_1}, u_+(r_{s_1}))$, $(r_{s_2}, u_+(r_{s_2}))$ must be on the same subsonic curve passing $(r_1, u(r_1))$ by uniqueness of solutions of Cauchy problems of ordinary differential equations. In the following we denote this $u-$subsonic curve as Υ.

5. Next we show *each subsonic curve of u passing a point χ on the R-H curve of u must lie below the R-H curve of u if it is in the right hand side of χ.*

Fix any $r_s \in [r_0, r_1]$, we obtain from (16)(10) the following differential equation for $u-$ R-H curve (note $U_- = U_-(r_s)$ is the supersonic curve passing U_0):

$$\frac{\mathrm{d}u_+}{\mathrm{d}r_s} = -\frac{c_*}{r_s u_-}\frac{a_-^2}{u_-^2 - a_-^2}. \tag{22}$$

While, the u-subsonic curve passing $(r_s, u_+(r_s))$ satisfying

$$\left.\frac{\mathrm{d}u}{\mathrm{d}r}\right|_{r=r_s} = \frac{u_+ a_+^2}{r_s(u_+^2 - a_+^2)}. \tag{23}$$

Thus at $r = r_s$ we get

$$
\begin{aligned}
\frac{\mathrm{d}u_+}{\mathrm{d}r_s} - \frac{\mathrm{d}u}{\mathrm{d}r}\bigg|_{r=r_s} &= -\frac{u_+}{r_s}\left(\frac{a_-^2}{u_-^2 - a_-^2} + \frac{a_+^2}{u_+^2 - a_+^2}\right) \\
&= -\frac{2(\gamma-1)}{\gamma+1}\frac{u_+}{r_s}\left(\frac{c_0 - \frac{1}{2}u_-^2}{u_-^2 - c_*} - \frac{c_0 u_-^2 - \frac{1}{2}c_*^2}{c_*(u_-^2 - c_*)}\right) \\
&= -\frac{2(\gamma-1)}{\gamma+1}\left(-\frac{\gamma}{\gamma+1}\right)\frac{u_+}{r_s} \\
&> 0
\end{aligned}
\tag{24}
$$

by using (16) and Bernoulli's law. This proves the claim.

6. By (24), without loss of generality, we may assume there is no any other common point of the $u-$R-H curve and Υ between r_{s_1}, r_{s_2}. Now we consider the domain Ψ bounded by $u-$R-H curve and Υ. Due to our construction, all the u-subsonic curves issuing from the $u-$R-H curve between r_{s_1}, r_{s_2} must flow in Ψ as r increases. This contradicts the uniqueness of solutions of Cauchy problems for ordinary differential equations since any solution can be extended out a compact set on plane. Thus we proved the claim in **3.**

7. We prove $u(r_1)$ is an increasing function of r_s. To this end, we only need to show that $u_{r_0}(r_1) < u_{r_1}(r_1)$. From **5** we see that the $u-$subsonic curve issuing from $(r_s, u_+(r_s))$ lies always below the $u-$R-H curve for $r > r_s$. Thus $u_{r_0}(r_1) < u_+(r_1) = u_{r_1}(r_1)$. This finishes the proof of (1) for $u(r_1)$.

8. By Bernoulli's law we have

$$
M^2 = \frac{1}{(\gamma-1)(\frac{c_0}{u^2} - \frac{1}{2})}.
$$

Thus it is clear that $M(r_1)$ is a continuous increasing function of r_s.

9. In a similar fashion we can prove (2) for $\rho(r_1)$. We first show that any $\rho-$subsonic curve issuing from $\rho-$R-H curve lies above the latter. Indeed, from (18) we get

$$
\frac{\mathrm{d}\rho_+}{\mathrm{d}r_s} = \frac{b_0 u_-}{c_* r_s^2}\frac{2a_-^2 - u_-^2}{u_-^2 - a_-^2}. \tag{25}
$$

Thus by direct calculations

$$
\begin{aligned}
\frac{\mathrm{d}\rho_+}{\mathrm{d}r_s} - \frac{\mathrm{d}\rho}{\mathrm{d}r}\bigg|_{r=r_s} &= \frac{b_0}{c_* u_- r_s^2}\left(-\frac{2\gamma u_-^2}{\gamma+1}\right) \\
&< 0,
\end{aligned}
\tag{26}
$$

here $\frac{\mathrm{d}\rho}{\mathrm{d}r}\Big|_{r=r_s}$ is the tangent of the $\rho-$subsonic curve passing $(r_s, \rho_+(r_s))$ at $r = r_s$. Thus for $r > r_s$ the $\rho-$subsonic curve lies above the $\rho-$R-H curve.

Now by a similar argument as for u, one knows that $\rho_{r_s}(r_1)$ is a continuous, decreasing function of r_s.

10. Finally we prove Theorem 6 for p. Now we can use Bernoulli's law to obtain that

$$p_{r_s}(r_1) = \frac{\gamma - 1}{\gamma}\rho_{r_s}(r_1)(c_0 - \frac{1}{2}u_{r_s}(r_1)^2). \tag{27}$$

Since $u_{r_s}(r_1)$ is increasing on r_s, and $\rho_{r_s}(r_1)$ is decreasing on r_s, we see that $p_{r_s}(r_1)$ is a decreasing function of r_s.

This finishes the proof of Theorem 6.

Remark 7 *We see actually that all the supersonic curves, subsonic curves and R-H curves are analytical curves of* $r \in [r_0, \infty)$*. Thus* $U_{r_s}(r_1)$ *is also an analytical function of* r_s*.*

Corollary 8 *[Solutions of (***TSP***)] For given supersonic state* U_0 *at* $r = r_0$ *with* $u_0 > 0$*, there exists* $0 < p_{min} < p_{max}$ *such that for* $p_1 \in [p_{min}, p_{max}]$*, there exists a unique transonic shock solution* $(U_-, U_+; r_s)$ *of* (**TSP**)*. Here* $r = r_s \in [r_0, r_1]$ *is the transonic shock front separating the supersonic flow* U_- *in* $[r_0, r_s]$ *and the subsonic flow* U_+ *in* $[r_s, r_1]$ *with physical entropy condition holds.*

Similar results also hold for one of u, ρ, M *was given at the exit* $r = r_1$*.*

PROOF. Set $p_{min} = p_+(r_1)$ and $p_{max} = p_{r_0}(r_1)$, then by the continuity and monotonicity of $p_{r_s}(r_1)$ we get the unique r_s with $p_{r_s}(r_1) = p_1 \in [p_{min}, p_{max}]$. U_- is solved by the supersonic curves passing U_0 on $[r_0, r_s]$, while U_+ obtained by the subsonic curves passing $U_+(r_s)$, which is obtained by the R-H curves. By Proposition 3, the physical entropy condition holds.

The following result based on extension of solutions of ODE is obvious:

Proposition 9 *For given supersonic state* U_0 *at* $r = r_0$ *with* $u_0 > 0$*, there exists a positive* h_0 *such that for any* $r_s \in (h_0, r_1]$*, the subsonic curves passing the point on the R-H curves (which corresponds to* U_0*) with* $r = r_s$ *can be extended to left in* $[r_s - h_0, r_s]$*.*

2.4 (**TSP**) *for 3-D Euler System with Spherical Symmetry*

The above analysis and results are also valid for steady spherical flows governed by three dimensional Euler system. Indeed, the Euler system is reduced to the following conservation form

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^2\rho u) = 0, \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^2(\rho u^2 + p)) - 2rp = 0, \tag{29}$$

$$\frac{1}{2}u^2 + \frac{a^2}{\gamma - 1} = c_0 \tag{30}$$

in this case (see §17 of [10], note that (30) is equivalent to $\frac{\mathrm{d}S}{\mathrm{d}r} = 0$ for smooth flow). Thus the Rankine-Hugoniot conditions are the same as (13)–(15). (28)–(29) can also be written as

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{2ua^2}{r(u^2 - a^2)}, \tag{31}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{-2\rho u^2}{r(u^2 - a^2)} \tag{32}$$

for C^1 flow. Compare these with (10)(11) we see the only difference is the rate of decay on asymptotic behavior, which is not used in proving Theorem 6 and Corollary 8. Thus we conclude that Theorem 6 and Corollary 8 also hold for three dimensional spherical flows, with the nozzle replaced by a part of a cone (not containing the vertex) $\mathrm{N} := \{(r,\theta) : 0 < r_0 < r < r_1, \theta \in \Sigma \subset \mathbf{S}^2\}$, where $\mathbf{S}^2$ is the unit sphere in $\mathbb{R}^3$, and Σ may be arbitrary.

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Notes added in proof. See [17] for further progresses on the stability of the second class of transonic shocks constructed in this paper.

References

[1] L. Bers, *Mathematical Aspects of Subsonic and Transonic Gas Dynamics*, Surveys in Applied Mathematics, 3., Wiley, New York, 1958.

[2] S. Čanić, B. L. Keyfitz, G. M. Lieberman, *A proof of existence of perturbed steady transonic shocks via a free boundary problem.* Comm. Pure Appl. Math. **53**(2000), 484–511.

[3] G.-Q. Chen and M. Feldman, *Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type*, J. Amer. Math. Soc. **16**(2003), 461–494.

[4] G.-Q. Chen, M. Feldman, *Steady Transonic Shocks and Free Boundary Problems in Infinite Cylinders for the Euler Equations,* Comm. Pure Appl. Math. **57**(2004), 310–356.

[5] G.-Q. Chen and M. Feldman, *Existence and stability of multi-dimensional transonic flows through an infinite nozzle of arbitrary cross-sections*, Arch. Ration. Mech. Anal., to appear.

[6] S. Chen, *Stability of transonic shock fronts in two-dimensional Euler systems*, Tran. Amer. Math. Soc. **357**(2005), 287–308.

[7] S. Chen, *Existence of Local Solution to Supersonic Flow Past a Three Dimensional Wing*, Adv. in Appl. Math. 13(1992) 273–304.

[8] S. Chen, H. Yuan, *Transonic Shocks in Compressible Flow Passing a Duct for Three-Dimensional Euler Systems*, (2005) submitted.

[9] S. Chen, Z. Xin, H. Yin, *Global shock waves for the supersonic flow past a perturbed cone*, Comm. Math. Phys. **228** (2002), 47–84.

[10] R. Courant and K.O. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience Publishers Inc., New York, 1948.

[11] H. M. Glaz, T.-P. Liu, *The Asymptotic Analysis of Wave Interactions and Numerical Calculations of Tansonic Nozzle Flow,* Adv. in Appl. Math. **5** (1984), 111–146.

[12] J. Glimm, G. Marshall, B. Plohr, *A generalized Riemann problem for quasi-one-dimensioanl gas flows,* Adv. in Appl. Math. **5** (1984), 1–30.

[13] A. G. Kuz'min, *Boundary-Value Problems for Transonic Flow*, John Wiley & Sons, 2002.

[14] T.-T. Li, W.-C. Yu, *Boundary value problems for quasilinear hyperbolic systems*, Duke University Mathematics Series 5, Duke University, Mathematics Department, Durham, N.C., 1985.

[15] T.-P. Liu, *Transonic gas flows in a variable area duct,* Arch. Rat. Mech. Anal. **80** (1982), 1–18.

[16] T.-P. Liu, *Nonlinear stability and instability of transonic gas flow through a nozzle,* Comm. Math. Phys. **83** (1982), 243–260.

[17] L. Liu, H. Yuan, *Stability of Cylindrical Transonic Shocks for Two-Dimensional Steady Compressible Euler Flows*, (2006) submitted.

[18] H. Ockendon, J.R. Ockendon, *Waves and Compressible Flows,* Springer-Verlag, New York, 2004.

[19] B. Whitham, *Linear and nonlinear waves,* John Wiley, New York, 1974.

[20] Z. Xin, H. Yin, *Transonic Shock in a Nozzle I: Two Dimensional Case,* Comm. Pure Appl. Math. **58**(2005), 999–1050.

[21] H. Yuan, *Transonic shocks for steady Euler flows with cylindrical symmetry,* Nonlinear Analysis T.M.A., (2006) doi:10.1016/j.na.2006.02.045.

[22] H.R. Yuan, *On transonic shocks in two dimensional variable-area ducts for steady Euler system*, SIAM J. Math. Anal,, to appear.