The following is a complete proof of estimate (8.9).

First, by results in Section 7 of [16] (since the domains we considered are both rectangles, they admit a uniform exterior cone), the solution is in $H_{2+\alpha}^{(-\beta)}$ for $0 < \beta \le \gamma_0$ ($\gamma_0 \in (0,1)$ a universal constant) and there holds

$$\|\phi_2\|_{2+\alpha}^{(-\beta)} \le C \|f_2\|_{1+\alpha}^{(1-\beta)}.$$

Then, it is easy to check that $u = \partial_{\bar\xi}\phi_2$ satisfies the following mixed boundary value problem:

$$\begin{cases} \frac{1}{\lambda_1}\partial_{\bar\xi}^2 u + \lambda_2 \partial_{\bar\eta}^2 u = \partial_{\bar\xi} f_2 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \bar\eta = 0, \quad \bar\eta = 1, \\ \partial_{\bar\xi} u = \lambda_1 f_2 & \text{on} \quad \bar\xi = 0, \quad \bar\xi = 1. \end{cases}$$

By Theorem 2 in [L] of Prof. Lieberman and Fredholm alternative indicated after Theorem 1 and Theorem 2 in [L], $u \in H_{2+\alpha}^{(-\beta)}$ for $0 < \beta \le \gamma_1$ with $\gamma_1 \in (0,1)$ a universal constant. There also holds

$$\|u\|_{2+\alpha}^{(-\beta)} \le C \|f_2\|_{1+\alpha}^{(1-\beta)}.$$

Similarly, $v = \partial_{\bar\eta}\phi_2 \in H_{2+\alpha}^{(-\beta)}$ for $0 < \beta \le \gamma_2$ and $\gamma_2 \in (0,1)$ a universal constant and an estimate like that for u holds.

Therefore the solution ϕ_2 is in $H_{3+\alpha}^{(-1-\beta)}$ for $0 < \beta \le \beta_1$ with $\beta_1 = \min\{\gamma_1, \gamma_2\}$ as claimed by (8.9).

I thank sincerely a reviewer who points out this problem, and shows me that the result in [A] of Prof. Azzam can be applied directly to problem (8.8) and then one obtains (8.9).

[A] A. Azzam, On Dirichlet's problem for elliptic equations in section smooth n-dimensional domains II, SIAM J. Math. Anal. 12 (1981), 242.

[L] G. Lieberman, Mixed boundary value problem for elliptic and parabolic differential equations of second order, J. Math. Anal. Appl. 113(1986), 422–440.

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