双曲型偏微分方程讲义

高维双曲型方程组的初边值问题
数学理论导引

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编 写 说 明

高维双曲型方程组初边值问题的局部经典解和分片光滑解理论是双曲型守恒律偏微分方程和数学流体力学中重要的研究方向，已经建立了相对比较完整的数学基础理论，国内外学者对于相关的非线性自由边界问题等前沿课题也一直在持续研究中，取得了大量重要的成果. 为了方便同学们了解、学习和进入这一研究方向，我们将 2021 年春季学期《双曲型偏微分方程选讲》课程的教学资料整理成这本讲义，供大家学习参考使用.

特别要指出，本讲义内容除最后一讲来自编者的研究工作外,其余都取自 Sylvie Benzoni-Gavage 和 Denis Serre 的专著《Multidimensional hyperbolic partial differential equations. First-order systems and applications》. 该书由 Oxford University Press 在 2007 年出版. 我们挑选了其中部分内容，添补了书中省略的许多细节和知识点，加入了编者的一些学习体会和相关的研究经验. 希望通过本讲义的导引，读者能更好地学习和领会 Sylvie Benzoni-Gavage 和 Denis Serre 这本重要的著作，促进相关研究工作. 讲义正文是英文，部分文字都来自该专著.

讲义共包括七讲，简要介绍了有关高维双曲型偏微分方程组初边值问题数学理论的三方面的内容.

第一部分是第一讲—第四讲，介绍常系数具常重数特征族的双曲型方程组在半空间适定的初边值问题中边界条件的合理提法. 第一讲介绍了 Friedrichs 对称双曲组带耗散边界条件的初边值问题，用算子半群方法给出了 L^2 解的适定性.

第二讲和第三讲分别介绍了 Kreiss-Lopatinskii 条件和一致 Kreiss-Lopatinskii 条件的来源和计算方法. 第四讲介绍非定常非等熵可压缩 Euler 方程组数学上合理的边界条件的提法. 这一部分的结论对于研究工程应用问题和开展数值计算都是非常重要的, 也是每个研究可压缩 Euler 方程组初边值问题的学生都应当掌握的知识点. 这一部分涉及大量有关 Fourier 变换、线性代数和矩阵, 以及复变量函数的知识和方法, 一方面显示了双曲型方程组数学理论的深刻和优美, 一方面也对初学者是很大的挑战. 讲义补充了所需的线性代数和复分析方面的知识及其参考文献, 而对 Fourier 变换, 可以参考我编写的《调和分析与偏微分方程讲义》中前四讲.

第二部分是第五讲和第六讲, 简要介绍了拟微分算子和仿微分算子理论, 帮助读者理解这些理论的意义, 了解其概貌.

第三部分是第七讲, 来自编者的一篇研究论文, 目的是通过介绍研究高维激波稳定性的方法框架, 帮助读者理解非特征非线性自由边界问题的求解.

在实际教学中, 前五讲是仔细讲解的. 第六讲和第七讲内容旨在帮助读者大致了解, 对细节可不必过于纠结. 不过第六讲介绍的有关 Sobolev 函数乘法和复合以及交换子的估计在求解非线性问题中非常重要, 对其结论要熟悉.

讲义虽经备课和课堂反馈做了仔细修改, 但根据以往的经验, 其中必然还会存在大量的瑕疵, 请读者通过邮件不吝赐教指正!

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目 录

LECTURE NOTES 1:
FRIEDRICHS-SYMMETRIC WEAKLY DISSIPATIVE IBVPS

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In this note, we first review semigroup theory, and then applying it to study Initial-Boundary Value Problems (IBVPs) for Friedrichs-symmetric systems. We will focus on the case with constant coefficients, half-space domain, and weakly dissipative boundary conditions. [This lecture is based upon Chapter 3, Section 1 of [1].]

1. Review: Semigroups

Our start point is the following Hill-Yosida Theorem (Theorem 4 in §7.4.2 [2, p.418]):

Theorem 1.1. *Let A be a closed, densely-defined linear operator on a Banach space X. Then A is the generator of a contraction semigroup $\{S(t)\}_{t\geq 0}$ if and only if $(0,\infty) \subset \rho(A)$ and $\|(\lambda I - A)^{-1}\| \leq 1/\lambda$ for $\lambda > 0$.*

For the related definitions and notations, we refer to §7.4 in [2]. Notice that a main issue is to overcome the difficulty that A might NOT be a bounded (continuous) operator on X.

In many applications, it is more convenient to consider maximal monotone operators for verification of the requirements in Hill-Yosida Theorem.

Definition 1.1. Let X be a Hilbert space, $D(A)$ a linear subspace, and $A : D(A) \to X$ a linear operator. A is called **monotone** if $(Au, u)_X \geq 0$ for all $u \in D(A)$, and **maximal monotone** if, moreover, $I + A$ is onto, that is,

$$\forall f \in X,\ \exists u \in D(A) \ \text{ such that } u + Au = f.$$

Theorem 1.2. *Let A be a linear operator defined on a subspace $D(A)$ of Hilbert space X. If A is maximal monotone, then $-A$ is a generator of a contraction semigroup.*

Proof. **1.** We first show the reflexive of X and maximal monotone of A implies A is densely defined: $\overline{D(A)} = X$.

Suppose $\overline{D(A)}$ is a proper subspace of X. Then there is a *nonzero* $u \in X$ such that $(u, w) = 0$ for any $w \in D(A)$. Since $I + A$ is onto, there is a *nonzero* $v \in D(A)$ such that

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$v + Av = u$. Now taking $w = v$, we have $0 = (v + Av, w) = (v + Av, v) = \|v\|^2 + (Av, v) \geq \|v\|^2 > 0$. A contradiction!

2. Next we show A is closed. By monotonicity, for $\lambda > 0$, $((\lambda I + A)u, u) \geq \lambda \|u\|^2$. Hence $\|(\lambda I + A)u\| \geq \lambda \|u\|$ and $\lambda I + A$ is one-to-one. Particularly, for $\lambda = 1$, by maximal monotonicity of A, $I + A : D(A) \to X$ is onto. These facts show that $(I+A)^{-1} : X \to D(A)$ is a bounded linear operator, hence closed. Therefore its inverse $I + A$ is closed. Hence A itself is closed.

3. From step 2, we see that, supposing $\lambda > 0$ lies in $\rho(-A)$, then $\|(\lambda I + A)^{-1}\| \leq 1/\lambda$.

4. Finally we show $(0, \infty) \subset \rho(-A)$. To this end, we only need to show

$$\lambda I + A \text{ is onto for every } \lambda > 0.$$

Method of continuity. Set $\Lambda = \{\lambda > 0 : \lambda I + A \text{ is onto.}\}$ From step 2, we have got $1 \in \Lambda$. Suppose $\mu_0 \in \rho(-A)$. Then

$$\mu I + A \quad = \quad \left[I + (\mu - \mu_0)(\mu_0 I + A)^{-1}\right](\mu_0 I + A).$$

For $\mu - \mu_0$ small, $I + (\mu - \mu_0)(\mu_0 I + A)^{-1}$ is a invertible bounded linear operator on X. Hence $\mu \in \rho(-A)$. This shows that Λ is open.

We then show Λ is closed with respect to $(0, \infty)$. Suppose $\lambda_k \in \Lambda$ and $\lambda_k \to \lambda > 0$ as $k \to \infty$, we need prove for any $f \in X$, there is a $u \in D(A)$ such that $\lambda u + Au = f$.

Since $\lambda_k \in \Lambda$, so there exist $u_k \in D(A)$ with $\lambda_k u_k + Au_k = f$ and $\lambda_k \|u_k\| \leq \|f\|$. Note there holds

$$\lambda_k(u_k - u_m) + A(u_k - u_m) = (\lambda_m - \lambda_k)u_m,$$

which implies

$$\|u_k - u_m\| \leq \frac{|\lambda_m - \lambda_k|}{\lambda_k} \|u_m\| \leq \frac{|\lambda_m - \lambda_k|}{\lambda_k \lambda_m} \|f\|,$$

that is, $\{u_k\}$ is a Cauchy sequence in X. So there is one $u \in X$ such that $u_k \to u$ as $k \to \infty$. Note that $f - \lambda_k u_k \to f - \lambda u$ in X. By closeness of A, $\lambda u + Au = f$ holds. □

Remark 1.1. Written in the form of an abstract Cauchy problem, the above Theorem claims the following: For every $u_0 \in D(A)$, there exists uniquely one $u \in \mathscr{C}([0, \infty); D(A)) \cap \mathscr{C}^1([0, \infty); X)$, such that

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + Au = 0 \quad \text{on} \quad [0, \infty), \\ u(0) = u_0. \end{cases}$$

Moreover, one has

$$\|u(t)\|_X \leq \|u_0\|_X, \qquad \forall t \geq 0.$$

2. Dissipative boundary conditions of Friedrichs-symmetric system

2.1. The equation.

Definition 2.1. A system of partial differential equations

$$Lu = \partial_t u + \sum_{\alpha=1}^{d} A^\alpha \partial_\alpha u = f, \tag{1}$$

with $u \in \mathbf{R}^n$, $A^\alpha \in \mathbf{M}_n(n \times n$ matrices), $\alpha = 1, \cdots, d$, is called Friedrichs-symmetric if all A^α are symmetric matrices.

Example 2.1. *The compressible Euler system is a quasi-linear Friedrichs-symmetric system for $\rho > 0$, with the form $A^0 \partial_t u + \sum_{\alpha=1}^d A^\alpha \partial_\alpha u = C$. Here $u = (v_1, v_2, v_3, p, S)^T$, $d = 3$, $C = (\rho F_1, \rho F_2, \rho F_3, 0, 0)^T$, and*

$$A^0 = \begin{pmatrix} \rho & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \rho^{-1}c^{-2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A^1 = \begin{pmatrix} \rho v_1 & 0 & 0 & 1 & 0 \\ 0 & \rho v_1 & 0 & 0 & 0 \\ 0 & 0 & \rho v_1 & 0 & 0 \\ 1 & 0 & 0 & \rho^{-1}c^{-2}v_1 & 0 \\ 0 & 0 & 0 & 0 & v_1 \end{pmatrix},$$

$$A^2 = \begin{pmatrix} \rho v_2 & 0 & 0 & 0 & 0 \\ 0 & \rho v_2 & 0 & 1 & 0 \\ 0 & 0 & \rho v_2 & 0 & 0 \\ 0 & 1 & 0 & \rho^{-1}c^{-2}v_2 & 0 \\ 0 & 0 & 0 & 0 & v_2 \end{pmatrix}, \quad A^3 = \begin{pmatrix} \rho v_3 & 0 & 0 & 0 & 0 \\ 0 & \rho v_3 & 0 & 0 & 0 \\ 0 & 0 & \rho v_3 & 1 & 0 \\ 0 & 0 & 1 & \rho^{-1}c^{-2}v_3 & 0 \\ 0 & 0 & 0 & 0 & v_3 \end{pmatrix}.$$

Notice that for $\rho > 0$ (no vacuum), A^0 is positive-definite, and A^1, A^2, A^3 are symmetric.

2.2. **The domain.** Let $\Omega \in \mathbf{R}^d$ be a smooth domain, with outward unit normal ν. In the present note, we consider mainly the case

$$\Omega = \{x \in \mathbf{R}^d : x_d > 0\}.$$

We frequently write $x = (y, x_d)$ afterwards, with $y \in \mathbf{R}^{d-1}$. The frequency vectors are also split into $\xi = (\eta, \xi_d)$ with $\eta \in \mathbf{R}^{d-1}$.

2.3. Boundary conditions.

Definition 2.2. A boundary condition

$$Bu = g \quad \text{on} \quad x \in \partial\Omega, \ t > 0, \tag{2}$$

with B a $p \times n$ matrix, is called *dissipative* for the symmetric operator L in (1), if $A(\nu) = \sum_{\alpha=1}^{d} A^\alpha \nu_\alpha$ is non-negative on $\ker(B)$:

$$\boxed{v \in \mathbf{R}^n,\ Bv = 0 \Rightarrow (A(\nu)v, v)_{\mathbb{R}^n} \geq 0.} \tag{3}$$

Boundary condition (2) is called *maximal dissipative*, if it is dissipative, and moreover, $\ker B$ is not a proper subspace of some linear space on which $A(\nu)$ is non-negative.

Remark 2.1. It is natural to assume further that $\operatorname{rank} B = p$. In the homogeneous case $g = 0$, we may drop many redundant boundary conditions; In the inhomogeneous case, this guarantees that (2) is at least solvable at a boundary point on the algebraic level.

Remark 2.2. Note for the case $\Omega = \{x_d > 0\}$, since $\nu = -e_d$, (3) reads

$$\boxed{v \in \mathbf{R}^n,\ Bv = 0 \Rightarrow (A^d v, v) \leq 0.}$$

Remark 2.3. If $u \in C^1 \cap L^2(\Omega)$ is a solution to (1) with homogeneous boundary condition $Bu = 0$ on $(0, \infty) \times \partial\Omega$, by multiplying $u^\top$ from left to the equation (i.e., taking inner product) and integrating in Ω, an integration-by-parts shows that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|^2_{L^2(\Omega)} + \int_{\partial\Omega} \big(A(\nu)u,\ u\big)\,\mathrm{d}S = 2\int_\Omega \big(f,\ u\big)\,\mathrm{d}x.}$$

The dissipative of B then implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|^2_{L^2} \leq 2\,\|f(t)\|_{L^2(\Omega)}\,\|u(t)\|_{L^2(\Omega)}\,,$$

or, provided $u(t) \neq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|_{L^2} \leq \|f(t)\|_{L^2}\,.$$

So we get an energy estimate:

$$\|u(t)\|_{L^2} \leq \|u(0)\|_{L^2} + \int_0^t \|f(s)\|_{L^2}\,\mathrm{d}s.$$

2.4. **Main result.** In this note we will consider the following IBVPs

$$\begin{cases} Lu = f, & x \in \Omega,\ t > 0, \\ Bu = g, & x_d = 0,\ t > 0, \\ u = u_0, & x \in \Omega,\ t = 0, \end{cases} \tag{4}$$

under the assumptions that (a) L is Friedrichs-symmetric; (b) B is maximal dissipative; (c) A^α ($\alpha = 1, \cdots, d$) and B are all constant matrices; (d) $g = 0$ (the case of homogeneous boundary conditions); (e) Ω is the half space $\{x \in \mathbf{R}^d : x_d > 0\}$.

The method to prove existence and uniqueness is semigroup theory reviewed before. The main result is the following theorem.

Theorem 2.1. *Let $L = \partial_t + \sum_1^d A^\alpha \partial_\alpha$ be a symmetric hyperbolic operator, and the boundary matrix $B \in \mathbb{M}_{p\times n}$ be maximal dissipative. Set*

$$\boxed{D(A) = \{u \in L^2(\Omega)^n : \sum_\alpha A^\alpha \partial_\alpha u \in L^2(\Omega)^n \text{ and } Bu = 0 \text{ on } \partial\Omega\}. \qquad (\clubsuit)}$$

Then the homogeneous IBVPs in $\Omega \times \mathbb{R}_t^+$:

$$Lu(x,t) = 0, \quad Bu(y,0,t) = 0, \quad u(x,0) = u_0(x) \tag{5}$$

is L^2 well-posed in the following sense. For every $u_0 \in D(A)$, there exists a unique $u \in \mathscr{C}([0,\infty); D(A)) \cap \mathscr{C}^1([0,\infty); L^2)$ that solves $Lu = 0$ as an ODE in $X = L^2(\Omega)^n$, such that $u(0) = u_0$. Furthermore,

$$t \mapsto \|u(t)\|_{L^2} \tag{6}$$

is non-increasing.

Remark 2.4. It is a basic property of semigroup $S(t)$ that, if $u \in D(A)$, (A is the generator of $S(t)$), then $u(t) = S(t)u \in D(A)$. Hence by definition of $D(A)$, there always holds $Bu(t) = 0$ on $\partial\Omega$. In this sense the boundary condition is satisfied. However, a main difficulty is how to understand the boundary condition $Bu|_{\partial\Omega} = 0$ in ($\clubsuit$), which does not make sense for general functions u in $L^2(\Omega)$. This is to be solved in the following section.

3. Analysis and definition of boundary condition

3.1. **Algebraic level.** We first study linear algebraic property of the boundary matrix B.

Proposition 3.1. *If B is maximal dissipative for L, then $\ker A(\nu) \subset \ker B$. So there is $M \in \mathbb{M}_{p\times n}$ such that $B = MA(\nu)$.*

Proof. **1.** For $u \in \ker A(\nu)$, let $w = u + v$, with $v \in \ker B$. Then

$$(A(\nu)w, w) = (A(\nu)v, u) + (A(\nu)v, v) = (v, A(\nu)u) + (A(\nu)v, v) = (A(\nu)v, v) \geq 0.$$

Maximal dissipative implies that $w \in \ker B$. Hence $u \in \ker B \Rightarrow \ker A(\nu) \subset \ker B$.

2. By linear algebra, suppose $\operatorname{rank} A(\nu) = r$, $\operatorname{rank} B = p$, then $\dim\ker A(\nu) = n - r$, $\dim\ker B = n - p$. So $p \leq r$.

3. Without loss of generality, suppose $A(\nu) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \\ \vdots \\ \alpha_n \end{pmatrix}$, $B = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}$, where α_i, β_j are row vectors in $\mathbf{R}^n$, and $\alpha_1, \cdots, \alpha_r$; $\beta_1, \cdots, \beta_p$ are respectively linearly independent.

Suppose that $h \in \ker A(\nu)$, then obviously $\alpha_i \perp h$. Then

$$\mathrm{span}\,\{\alpha_1, \cdots, \alpha_r\} = (\ker A(\nu))^{\perp}, \quad \mathrm{span}\,\{\beta_1, \cdots, \beta_p\} = (\ker B)^{\perp}.$$

$\ker A(\nu) \subset \ker B$ implies $(\ker B)^{\perp} \subset (\ker A(\nu))^{\perp}$, hence $\beta_j \in \mathrm{span}\,\{\alpha_1, \cdots, \alpha_r\}$. Then we may take $M = (P_{p\times r},\ O_{p\times(n-r)})$, where $P_{p\times r}$ is obtained by representing β_j via $\alpha_1, \cdots, \alpha_r$. □

Proposition 3.2. *If B is maximal dissipative for L, then $\ker B = E_+ \bigoplus^{\perp} E_0$. Here E_+, E_-, E_0 are subspaces in $\mathbb{R}^n$ spanned by eigenvectors of $A(\nu)$ corresponding to positive, zero, and negative eigenvalues.*

Proof. **1.** We first note since $A(\nu)$ is symmetric, hence diagonalizable over $\mathbb{R}$, we have $\mathbb{R}^n = E_- \bigoplus^{\perp} E_+ \bigoplus^{\perp} E_0$, and all of them are invariant subspaces of $A(\nu)$. Obviously on $E_+ \bigoplus^{\perp} E_0$, the bilinear form $(A(\nu)u, u)$ is nonnegative. So by maximality, $E_+ \bigoplus^{\perp} E_0 \subset \ker B$.

2. Now if $u \in \ker B$ but $u \notin E_+ \bigoplus^{\perp} E_0$, then by decomposition $u = u_1 + u'$ with $u_1 \in E_+ \bigoplus^{\perp} E_0, u' \in E_-$, we find $u' \in E_- \cap \ker B$. However, $(A(\nu)u', u') < 0$, contradiction to dissipativeness. The proposition is proved. □

3.2. **Function level.** Now we consider u as a vector field in Ω. We need give a rigorous definition about what $Bu = 0$ on $\partial\Omega$ means in the definition of $D(A)$, when u is merely in L^2.

3.2.1. *Normal trace of vector field.* We start with a general result.

Theorem 3.1. *Let Ω be a smooth domain in $\mathbb{R}^d$, ν its outward unit normal along $\partial\Omega$, and H the Hilbert space of vector field $q \in L^2(\Omega;\mathbb{R}^d)$ with $\mathrm{div}\, q \in L^2(\Omega)$, endowed with a norm*

$$\|q\|_H = (\|q\|^2_{L^2(\Omega)^d} + \|\mathrm{div}\, q\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

Then

$$\langle \gamma_\nu q, \gamma_0 \phi\rangle = \int_\Omega (q \cdot \nabla\phi + (\mathrm{div}\, q)\phi)\, \mathrm{d}x, \qquad \forall \phi \in H^1(\Omega) \tag{7}$$

defines a bounded linear operator $\gamma_\nu : H \to H^{-\frac{1}{2}}(\partial\Omega)$. Here $\gamma_0 : H^1(\Omega) \to H^{\frac{1}{2}}(\Omega)$ is the standard trace operator on Sobolev functions, and $\langle\cdot,\cdot\rangle$ is the pairing between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$.

Furthermore, if $q \in C^1(\bar{\Omega})$, then

$$\boxed{\gamma_\nu q = q \cdot \nu|_{\partial\Omega}.}$$

Remark 3.1. We note that if $\partial\Omega$ is bounded, it has no boundary ($\partial(\partial\Omega)=\emptyset$), so $H^{\frac12}(\partial\Omega)=H_0^{\frac12}(\partial\Omega)$. If $\partial\Omega=\mathbb{R}^{n-1}$ as used below, it is a basic result on Sobolev spaces that $H^{\frac12}(\partial\Omega)=H_0^{\frac12}(\partial\Omega)$.

The mapping γ_ν is called *normal trace* of vector field q on $\partial\Omega$.

Proof. **1.** By Extension Theorem of Sobolev functions, $\gamma_0 : H^1(\Omega)\to H^{\frac12}(\partial\Omega)$ is onto and there holds $\|\phi\|_{H^1(\Omega)}\le C\|\gamma_0\phi\|_{H^{\frac12}(\partial\Omega)}$. So one easily gets from (7) that

$$\boxed{|\langle\gamma_\nu q,\gamma_0\phi\rangle|\le C\|q\|_H\|\gamma_0\phi\|_{H^{\frac12}} \quad\Rightarrow\quad \|\gamma_\nu q\|_{H^{-\frac12}}\le C\|q\|_H\,.}$$

The linearity of γ_ν is obvious.

2. For $q\in C^1(\bar\Omega)$, by Divergence Theorem, there holds $\int_\Omega(q\cdot\nabla\phi+(\operatorname{div}q)\phi)\,\mathrm{d}x=\int_{\partial\Omega}(q\cdot\nu)\phi\,\mathrm{d}S=\langle q\cdot\nu|_{\partial\Omega},\phi|_{\partial\Omega}\rangle$ for any $\phi\in C^1(\bar\Omega)$. So $\gamma_\nu q=q\cdot\nu|_{\partial\Omega}$. □

3.2.2. *Boundary conditions as normal traces.* Now since $u\in L^2(\Omega;\mathbb{R}^n)$, for any constant matrix $A\in\mathbb{M}_{n\times n}$, one has $Au\in L^2(\Omega)^n$. Consider the $n\times d$ matrix $Q=(A^1u,\cdots,A^du)=\begin{pmatrix}Q^1\\Q^2\\\vdots\\Q^n\end{pmatrix}$. Every row of Q, Q^i $(i=1,\cdots,n)$ belongs to $L^2(\Omega)^d$. In addition, for $u\in D(A)$, there holds $\sum_1^d\partial_\alpha(A^\alpha u)\in L^2$, which means $\operatorname{div}Q^i\in L^2$. So $\gamma_\nu Q^i\in H^{-\frac12}(\partial\Omega)$. We write $\gamma_\nu Q=(\gamma_\nu Q^1,\cdots,\gamma_\nu Q^n)^T$. For $u\in C^1(\bar\Omega)$, note $\nu=-e_d$, we have

$$\gamma_\nu Q=(Q^1\cdot\nu|_{\partial\Omega},\cdots,Q^n\cdot\nu|_{\partial\Omega})^T=-A^du|_{\partial\Omega}.$$

Therefore, for $u\in D(A)$, we may interpret the boundary condition $Bu=MA^du=0$ as

$$\boxed{M\gamma_\nu Q=0.}\tag{8}$$

However, the left-hand side is a functional, which is not easy to handle locally as we do for functions. The new idea is to use Fourier transform, which will uncover a hidden fact that $Bu|_{\partial\Omega}=0$ is meaningful after Fourier transform!

3.3. **Frequency level.** Next we show, by using Fourier transform, it is rather easier to verify (8).

Recall that for a function $f(y,x_d)$ with fine properties with respect to y, we set

$$\boxed{\mathscr{F}_yf(\eta,x_d)=\int_{\mathbb{R}^{d-1}}f(y,x_d)\mathrm{e}^{-\mathrm{i}y\cdot\eta}\,\mathrm{d}y}$$

as its Fourier transform respect to $y\in\mathbb{R}^{d-1}$. Except its tremendous power of reduce PDE to ODE, ODE to algebra, an advantage of Fourier Transform is, sometimes, the

Fourier transform of a function is more smooth than itself, therefore easier to handle. For example, to the present case, the trace $\gamma_\nu Q$ is only a distribution, while its Fourier transform introduced below is a function.

Set $Q_\alpha = A^\alpha u \in L^2(\Omega)^n$. Applying Parseval's Equality, $(\eta, x_d) \mapsto \mathscr{F}_y(Q_\alpha)$ is in $L^2(\Omega)$. $\sum_{\alpha=1}^d \partial_\alpha Q_\alpha \in L^2$ implies $(\eta, x_d) \mapsto \sum_{\alpha=1}^{d-1} \mathrm{i}\eta_\alpha \mathscr{F}_y(Q_\alpha) + \partial_d \mathscr{F}_y(Q_d)$ is also in $L^2(\Omega)$. It follows that for any compact set $K \subset \mathbb{R}^{d-1}$,

$$\int_K \left[\int_{\mathbb{R}^+} \left(|\mathscr{F}_y(Q_d)(\eta, s)|^2 + |\partial_d \mathscr{F}_y(Q_d)(\eta, s)|^2\right) \mathrm{d}s\right] \mathrm{d}\eta < \infty, \tag{9}$$

that is, $\mathscr{F}_y(Q_d) \in L^2_{\mathrm{loc}}(\mathbb{R}^{d-1}; H^1(\mathbb{R}^+))$. By Sobolev Embedding Theorem, we get $\mathscr{F}_y(Q_d) \in L^2_{\mathrm{loc}}(\mathbb{R}^{d-1}; \mathscr{C}(\mathbb{R}^+))$. So for a.e. η and every x_d, $\mathscr{F}_y(Q_d)(\eta, x_d)$ makes sense.

Lemma 3.1. *There holds for a.e. $\eta \in \mathbb{R}^{d-1}$ that*

$$\boxed{\mathscr{F}_y(\gamma_\nu Q) = -\mathscr{F}_y(Q_d)(\eta, 0).}$$

Proof. **1.** Since $\gamma_\nu Q \in H^{-\frac{1}{2}}$, by definition of $H^{-\frac{1}{2}}$, we know $\mathscr{F}_y(\gamma_\nu Q) \in L^2_{\mathrm{loc}}$ and $\int_{\mathbb{R}^{d-1}} |\mathscr{F}_y(\gamma_\nu Q)(\eta)|^2 (1+|\eta|^2)^{-1} \mathrm{d}\eta < \infty$. Hence the left-hand side is also a locally integrable function of η. Therefore, to show the identity of functions, we need to prove that the two functions coincide as distributions.

2. For $\phi(\eta, x_d) \in C_c^\infty(\mathbb{R}^d)$, and $i \in \{1, \cdots, n\}$, suppose that $Q^i \in C^1$, we have

$$\begin{aligned}
&\langle \mathscr{F}_y(\gamma_\nu Q^i)(\cdot), \phi(\cdot, 0)\rangle = \langle (\gamma_\nu Q^i)(\cdot), \mathscr{F}_\eta \phi(\cdot, 0)\rangle \\
= &\int_\Omega [Q^i(y, x_d) \cdot \nabla \mathscr{F}_\eta(\phi)(y, x_d) + (\operatorname{div} Q^i)(y, x_d) \mathscr{F}_\eta(\phi)(y, x_d)] \,\mathrm{d}y \,\mathrm{d}x_d \qquad (10) \\
= &\int_{\mathbb{R}^+} \int_{\mathbb{R}^{d-1}} \{\sum_{\alpha=1}^{d-1} [Q^i_\alpha(y, x_d) \partial_\alpha \mathscr{F}_\eta(\phi)(y, x_d) + (\partial_\alpha Q^i_\alpha) \mathscr{F}_\eta(\phi)(y, x_d)] \\
&\qquad + Q^i_d(y, x_d) \partial_d \mathscr{F}_\eta(\phi)(y, x_d) + (\partial_d Q^i_d) \mathscr{F}_\eta(\phi)(y, x_d)\} \,\mathrm{d}y \,\mathrm{d}x_d.
\end{aligned}$$

For the first term,

$$
\begin{aligned}
&\sum_{\alpha=1}^{d-1}\int_{\mathbb{R}^{d-1}}[Q^i_\alpha(y,x_d)\partial_\alpha\mathscr{F}_\eta(\phi)(y,x_d)+(\partial_\alpha Q^i_\alpha)\mathscr{F}_\eta(\phi)(y,x_d)]\,\mathrm{d}y\\
=&\sum_{\alpha=1}^{d-1}\int_{\mathbb{R}^{d-1}}[Q^i_\alpha(y,x_d)\mathscr{F}_\eta(-\mathrm{i}\eta_\alpha\phi)(y,x_d)+(\partial_\alpha Q^i_\alpha)\mathscr{F}_\eta(\phi)(y,x_d)]\,\mathrm{d}y\\
=&\sum_{\alpha=1}^{d-1}\int_{\mathbb{R}^{d-1}}[\mathscr{F}_y(Q^i_\alpha)(\eta,x_d)(-\mathrm{i}\eta_\alpha\phi)(\eta,x_d)+\mathscr{F}_\eta(\partial_\alpha Q^i_\alpha)\phi(\eta,x_d)]\,\mathrm{d}\eta\\
=&\sum_{\alpha=1}^{d-1}\int_{\mathbb{R}^{d-1}}[-\mathscr{F}_y(\partial_\alpha Q^i_\alpha)(\eta,x_d)\phi(\eta,x_d)+\mathscr{F}_\eta(\partial_\alpha Q^i_\alpha)\phi(\eta,x_d)]\,\mathrm{d}\eta\\
=&\ 0,
\end{aligned}
$$

One may also use directly Divergence Theorem to show this as we assumed that $Q^i_\alpha\in C^1$ and ϕ is compactly supported. We note that the only assumption of $Q^i\in L^2,\sum_\alpha\partial_\alpha Q^i_\alpha\in L^2$ is not enough to ensure $\partial_\alpha Q^i_\alpha\in L^2$ for each α, which is necessary for the above computation to be valid.

For the second term, using Fubini's Theorem, it is

$$
\begin{aligned}
&\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^+}\{Q^i_d(y,x_d)\partial_d\mathscr{F}_\eta(\phi)(y,x_d)+(\partial_d Q^i_d)\mathscr{F}_\eta(\phi)(y,x_d)\}\,\mathrm{d}x_d\,\mathrm{d}y\\
=&\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^+}\partial_d\{Q^i_d(y,x_d)\mathscr{F}_\eta(\phi)(y,x_d)\}\,\mathrm{d}x_d\,\mathrm{d}y\\
=&-\int_{\mathbb{R}^{d-1}}\{Q^i_d(y,0)\mathscr{F}_\eta(\phi)(y,0)\}\,\mathrm{d}y=-\int_{\mathbb{R}^{d-1}}\{\mathscr{F}_y(Q^i_d)(\eta,0)\phi(\eta,0)\}\,\mathrm{d}\eta\\
=&\ \langle-\mathscr{F}_y(Q^i_d)(\eta,0),\phi(\eta,0)\rangle.
\end{aligned}
$$

3. Now for $Q^i\in L^2$ with $\sum_\alpha\partial_\alpha Q^i_\alpha\in L^2$, and any $\phi\in C^\infty_c(\mathbb{R}^d)$, we prove that

$$\langle\mathscr{F}_y(\gamma_\nu Q^i)(\cdot),\phi(\cdot,0)\rangle=\langle-\mathscr{F}_y(Q^i_d)(\eta,0),\phi(\eta,0)\rangle.$$

Suppose that $Q^{(\varepsilon)}\in C^1$ and $Q^{(\varepsilon)}\to Q^i$ in H, then by (10), we have

$$
\begin{aligned}
&\langle\mathscr{F}_y(\gamma_\nu Q^{(\varepsilon)})(\cdot),\phi(\cdot,0)\rangle\\
=&\int_\Omega[Q^{(\varepsilon)}(y,x_d)\cdot\nabla\mathscr{F}_\eta(\phi)(y,x_d)+(\operatorname{div}Q^{(\varepsilon)})(y,x_d)\mathscr{F}_\eta(\phi)(y,x_d)]\,\mathrm{d}y\,\mathrm{d}x_d\\
\to&\int_\Omega[Q^i(y,x_d)\cdot\nabla\mathscr{F}_\eta(\phi)(y,x_d)+(\operatorname{div}Q^i)(y,x_d)\mathscr{F}_\eta(\phi)(y,x_d)]\,\mathrm{d}y\,\mathrm{d}x_d\\
=&\langle\mathscr{F}_y(\gamma_\nu Q^i)(\cdot),\phi(\cdot,0)\rangle.
\end{aligned}
$$

As indicated by (9), one also verifies that

$$\lim_{\varepsilon\to 0}\left|\langle \mathscr{F}_y(Q_d^i - Q_d^{(\varepsilon)})(\eta,0), \phi(\eta,0)\rangle\right| = 0.$$

This completes the proof. □

Remark 3.2. The result of the lemma is no wonder since we know for u smooth we have $\gamma_\nu Q = -A^d u|_{\partial\Omega}$. The point is, the Fourier transform of a bad "function" (even a distribution that is not a function) might be better. Thus by the Lemma, the abstract boundary condition (8), which is an identity of functionals, could be reformulated as the more classical point-wise conditions of a function $M\mathscr{F}_y(A^d u)(\eta,0) = 0$. That is

$$\boxed{B\mathscr{F}_y u(\eta,0) = 0 \quad a.e. \ \ \eta\in\mathbb{R}^{d-1}.} \tag{11}$$

Therefore we proved for $u\in L^2(\Omega)^n$ so that $\sum_\alpha A^\alpha\partial_\alpha u\in L^2(\Omega)^n$, if the boundary matrix B is maximal dissipative, then (11) is meaningful and and the boundary condition $Bu|_{\partial\Omega} = 0$ shall be rigorously defined as (11).

4. The proof of Theorem 2.1

4.1. **Monotonicity.** In the following we write $A = \sum_{\alpha=1}^d A^\alpha\partial_\alpha$ as an operator defined on

$$\boxed{D(A) = \{u\in L^2(\Omega)^n : \sum_\alpha A^\alpha\partial_\alpha u \in L^2(\Omega)^n \text{ and } Bu = 0 \text{ on } \partial\Omega\}. \qquad (\clubsuit)}.$$

To prove our Main Theorem, we only need show A is maximal monotone, with $X = L^2(\Omega)$ and $D(A)$ defined before. As suggested by (11), the strategy is to study the problem after Fourier transform with respect to y variables.

Let $v(\eta,x_d) = \mathscr{F}_y(u)(\eta,x_d)$. The operator A is now $A_F v = A^d\frac{\partial}{\partial x_d}v + iA(\eta)v$, with $A(\eta) = \sum_{\alpha=1}^{d-1}(A^\alpha\eta_\alpha)$. It is defined on

$$\boxed{D(A_F) = \{v\in L^2(\Omega:\mathbb{C}^n) : A_F v\in L^2 \text{ and } Bv = 0 \text{ a.e. on } \partial\Omega\}.}$$

Note the boundary condition is understood in the sense of (11) and well-defined. Also, in this definition, functions are complex-valued.

By Plancherel's Theorem, since u is real,

$$\begin{aligned}(Au,u)_{L^2(\Omega;\mathbb{R}^n)} &= (Au,u)_{L^2(\Omega;\mathbb{C}^n)} = (\mathscr{F}_y(Au), \mathscr{F}_y u)_{L^2(\Omega;\mathbb{C}^n)}\\ &= \mathrm{Re}(\mathscr{F}_y(Au), \mathscr{F}_y u)_{L^2(\Omega;\mathbb{C}^n)} = \mathrm{Re}(A_F v, v)_{L^2(\Omega;\mathbb{C}^n)}.\end{aligned}$$

So the monotonicity is

$$\boxed{\mathrm{Re}(A_F v, v)_{L^2(\Omega;\mathbb{C}^n)} \geq 0, \quad \forall v\in D(A_F).}$$

To verify it, the case that A^d is nonsingular (i.e. $\partial\Omega$ is a non-characteristic boundary) is easier. To overcome the difficulty that $\det A^d = 0$, we use the following technique.

Introducing a matrix $S \in \mathbb{M}_n$, which is the inverse of A^d on $\mathrm{R}(A^d)$. (It's convenience is explained later. Particularly, $S = (A^d)^{-1}$ if $\det A^d \neq 0$.)

Since A^d is symmetric, we see $\boxed{\mathbb{R}^n = \mathrm{R}(A^d) \overset{\perp}{\bigoplus} \ker A^d.}$ For $w \in \ker A^d$, we set $Sw = 0$. For $w \in \mathrm{R}(A^d)$, note $A^d : \mathrm{R}(A^d) = \mathbb{R}^n / \ker A^d \to \mathrm{R}(A^d)$ is a homeomorphism, there is uniquely one $w' \in \mathrm{R}(A^d)$ such that $A^d w' = w$. Hence we define $Sw = w'$. S shares many properties:

(a) S is symmetric.
(b) SA^d is the orthogonal projector onto $\mathrm{R}(A^d)$.
(c) $\boxed{(A^d u, v)_{\mathbb{R}^n} = (SA^d u, A^d v)_{\mathbb{R}^n} \text{ for } u, v \in \mathbb{C}^n.}$

For the proof of (a), let $u = u_r + u_k, v = v_r + v_k$, with $u_r, v_r \in \mathrm{R}(A^d)$, $u_k, v_k \in \ker A^d$. Then $(Su, v)_{\mathbb{R}^n} = (Su_r, v_r + v_k)_{\mathbb{R}^n} = (Su_r, v_r)_{\mathbb{R}^n} = (u_r', v_r)_{\mathbb{R}^n}$, and $(u, Sv)_{\mathbb{R}^n} = (u_r + u_k, Sv_r)_{\mathbb{R}^n} = (u_r, Sv_r)_{\mathbb{R}^n} = (u_r, v_r')_{\mathbb{R}^n} = (A^d u_r', v_r')_{\mathbb{R}^n} = (u_r', A^d v_r')_{\mathbb{R}^n} = (u_r', v_r)_{\mathbb{R}^n} = (Su, v)_{\mathbb{R}^n}$ as desired.

For (b), it is clear that $SA^d u = SA^d u_r = S(A^d u_r) = u_r$. Here we used uniqueness of w'.

(c) follows from (a)(b). Indeed, for $u, v \in \mathbb{R}^n$, there holds $(A^d u, v)_{\mathbb{R}^n} = (A^d(u_r + u_k), v_r + v_k)_{\mathbb{R}^n} = (A^d u_r, v_r + v_k)_{\mathbb{R}^n} = (A^d u_r, v_r)_{\mathbb{R}^n} = (A^d u_r, SA^d v_r)_{\mathbb{R}^n} = (A^d u, SA^d v)_{\mathbb{R}^n} = (SA^d u, A^d v)_{\mathbb{R}^n}$. For the case u, v being complex vectors, one may easily verify by separating u, v in real and imaginary parts.

Remark 4.1. From the definition of S, we see it is the inverse of A^d on $\mathrm{R}(A^d)$. From this point of view, (c) is clear: for $v \in \ker A^d$, $(A^d u, v) = 0$, while for $v \in \mathrm{R}(A^d)$, $(A^d u, v) = (A^d u, SA^d v) = (SA^d u, A^d v)$.

We observe the dissipative boundary condition means, for $v \in \ker B$ complex-valued, there holds $(A^d v, \bar{v})_{\mathbb{R}^n} \leq 0$. (This can be checked by writing v in real and imaginary parts.)

Now let $w \in L^2(\mathbb{R}^+)$ be such that $A^d \mathrm{d}w/\mathrm{d}x^d \in L^2(\mathbb{R}^+)$. Then $z = A^d w \in H^1(\mathbb{R}^+)$. For fixed $\eta \in \mathbb{R}^{d-1}$, we compute

$$\begin{aligned}
&\int_0^\infty \mathrm{Re}(A^d \frac{\mathrm{d}w}{\mathrm{d}x_d} + \mathrm{i}A(\eta)w, \bar{w})_{\mathbb{R}^n} \,\mathrm{d}x_d = \int_0^\infty \mathrm{Re}(A^d \frac{\mathrm{d}w}{\mathrm{d}x_d}, \bar{w})_{\mathbb{R}^n} \,\mathrm{d}x_d \\
= &\int_0^\infty \mathrm{Re}(SA^d \frac{\mathrm{d}w}{\mathrm{d}x_d}, A^d\bar{w})_{\mathbb{R}^n} \,\mathrm{d}x_d = \int_0^\infty \mathrm{Re}(\frac{\mathrm{d}z}{\mathrm{d}x_d}, S\bar{z})_{\mathbb{R}^n} \,\mathrm{d}x_d \\
= &\frac{1}{2}\int_0^\infty \mathrm{Re}\frac{\mathrm{d}}{\mathrm{d}x_d}(z, S\bar{z})_{\mathbb{R}^n} \,\mathrm{d}x_d = -\frac{1}{2}\mathrm{Re}(z(0), S\overline{z(0)})_{\mathbb{R}^n} \\
= &-\frac{1}{2}(z(0), S\overline{z(0)})_{\mathbb{R}^n} = -\frac{1}{2}(A^d w(0), \overline{w(0)})_{\mathbb{R}^n} \geq 0.
\end{aligned}$$

Here we used three facts. i) S is a real matrix. ii) Only upon introduction of S, we can infer the nice property that $z \in H^1$ and using Newton-Leibniz Formula. Otherwise, we should explain the identity $\frac{\mathrm{d}}{\mathrm{d}x_d}(A^d w, \bar{w}) = 2\mathrm{Re}(A^d \frac{\mathrm{d}w}{\mathrm{d}x_d}, \bar{w})$, while it looks $(A^d w, w)$ is only in L^2. (The derivative here is merely weak derivative. We can not use difference quotients.) iii) we assumed that $w(0) \in \ker B$ to apply the dissipativeness.

Recall that for $v \in D(A_F)$, we have $v(\eta, \cdot) \in L^2(\mathbb{R}^+)$ and $A^d \mathrm{d}v/\mathrm{d}x_d \in L^2(\mathbb{R}^+)$ for almost every η. Furthermore, by definition of $D(A_F)$, we required that $v(\eta, 0) \in \ker B$. Hence by taking $w(x_d) = v(\eta, x_d)$ in the above, we deduce for non-negative test functions $\phi \in \mathscr{D}(\mathbb{R}^{d-1})$,

$$\boxed{\int_\Omega \phi(\eta)\mathrm{Re}(A_F v, \bar{v})_{\mathbb{R}^n} \,\mathrm{d}x_d \,\mathrm{d}\eta \geq 0.}$$

Finally, let ϕ tend monotonically to 1, the left-hand side then tends to $\mathrm{Re}(A_F v, v)_{L^2(\Omega;\mathbb{C}^n)}$ and this shows A is a monotone operator.

4.2. **Maximality.** For any $f \in L^2$, we need solve a $u \in D(A)$ from the equation $u + Au = f$. Thanks to Fourier Transform with respect to y as used before, this may be transferred to an ODE subjected to boundary conditions, with $\eta \in \mathbb{R}^{d-1}$ being a parameter:

$$\boxed{v + \mathrm{i}A(\eta)v + A^d v' = g(\eta, \cdot), \qquad Bv(\eta, 0) = 0.} \tag{12}$$

Here $g = \mathscr{F}_y f \in L^2(\Omega)$, and v also to be solved in $L^2(\Omega)$.

4.2.1. *Non-characteristic case.* We assume A^d is non-singular. This means the boundary $\partial\Omega$ is non-characteristic. In this case, we introduce

$$\boxed{\mathcal{A}(\tau, \eta) := -(A^d)^{-1}(\tau I_n + \mathrm{i}A(\eta)).}$$

Lemma 4.1. *Suppose the operator $\partial_t + \sum_{\alpha=1}^d A^\alpha \partial_\alpha$ is hyperbolic. That is, $\sum_{\alpha=1}^d A^\alpha \xi_\alpha$ is diagonalizable (with real eigenvalues) uniformly for every $\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d$. Then for $\eta \in \mathbb{R}^{d-1}$ and $\mathrm{Re}\tau > 0$, the matrix $\mathcal{A}(\tau, \eta)$ does not have any pure imaginary eigenvalue.*

The number of stable eigenvalues (eigenvalues with negative real parts), counted with multiplicities, equals p, the number of positive eigenvalues of A^d.

Proof. **1.** Let ω be pure imaginary root of the characteristic polynomial of $\mathcal{A}(\tau,\eta)$:

$$P(X;\tau,\eta) = \det(XI_n - \mathcal{A}(\tau,\eta)).$$

Thus w satisfies

$$\det(\tau I_n + \mathrm{i}A(\eta) + \omega A^d) = 0.$$

Then hyperbolicity implies $\tau \in \mathrm{i}\mathbb{R}$, contradicts to the assumption that $\mathrm{Re}\tau > 0$.

2. Since P depends continuously on τ, η and has a constant degree, we infer the number of roots with positive real part (countered with multiplicity) may not vary locally.

In fact, the n roots of a polynomial of degree n depend continuously on the coefficients of the polynomial: there are n continuous functions $r_1, \cdots, r_n$ depending on the coefficients that parameterize the roots with correct multiplicity (see [4, p.26]). By the continuity, a root with positive real part cannot change to a root with negative real part, as long as $\mathrm{Re}\tau > 0$ (cf. Step 1 above). So the number of root with positive real part is locally a constant.

Then because $\{\mathrm{Re}\tau > 0\} \times \mathbb{R}^{n-1}$ is connected, we see the number of root with positive real part is a constant in $\mathrm{Re}\tau > 0$.

Computing at the point $\tau = 1, \eta = 0$, we see it is just the number of eigenvalues with positive real part of $-(A^d)^{-1}$, which equals the number of eigenvalues with negative real part of A^d. Notice that since A^d is symmetric and nonsingular, all the eigenvalues are real and there is no zero eigenvalue. □

Let $E_-(\tau,\eta)$ and $E_+(\tau,\eta)$ be the stable and unstable subspaces of $\mathcal{A}(\tau,\eta)$ respectively. This lemma implies that we have a decomposition $\mathbb{C}^n = E_-(\tau,\eta) \bigoplus E_+(\tau,\eta)$. However, we cannot infer the dimension of these subspaces (*the algebraic multiplicity of an eigenvalue might not equal its geometric multiplicity, unless the matrix could be diagonalized*). We need the following lemma, which is useful in the proof of Lemma 4.2 later.

Proposition 4.1. *The stable and unstable subspaces $E_\pm(\tau,\eta)$ of $\mathcal{A}(\tau,\eta)$ depend holomorphically on τ, analytically on η. In particular, their dimensions do not depend on τ,η as long as $\eta \in \mathbb{R}^{n-1}$ and $\mathrm{Re}\tau > 0$.*

Proof. **1.** For given (τ,η) with $\mathrm{Re}\tau > 0$, by Dunford-Taylor Formula, we may choose a large enough loop γ in the half-plane $\mathrm{Re}\tau > 0$, enclosing the unstable eigenvalues of

$\mathcal{A}(\tau,\eta)$, then the projector onto $E_+(\tau,\eta)$, along $E_-(\tau,\eta)$, is given by

$$\boxed{\pi_+(\tau,\eta)=\frac{1}{2\mathrm{i}\pi}\oint_\gamma (zI_n-\mathcal{A}(\tau,\eta))^{-1}\,\mathrm{d}z.}$$

A similar formula holds for $\pi_-(\tau,\eta)=I_n-\pi_+(\tau,\eta)$.

Since we may vary slightly τ,η without changing contour (because of continuity of roots of a polynomial), we may infer the projection mappings depend holomorphically on τ, analytically on η, as long as $\mathrm{Re}\tau>0$.

We remark this might not be true if $\mathrm{Re}\tau=0$. (By homogeneity on τ,η, it is not necessary to consider $\mathrm{Re}\tau<0$. Just let $\eta\to-\eta$ in that case.) Indeed, if $\mathrm{Re}\tau=0$, then $\mathcal{A}(\tau,\eta)$ might have pure imaginary eigenvalues. So it may happen we cannot find a fixed loop in $\mathrm{Re}\tau>0$ to contain all the eigenvalues with positive real part, as some of these eigenvalues may tend to pure imaginary numbers as $\mathrm{Re}\tau\to0$. While, if we choose a loop contains all eigenvalues with nonnegative real parts, then by perturbation, this loop may also contain eigenvalues with negative real parts. So the defined projection is no longer on the stable subspace.

2. Then $\dim\mathrm{R}(\pi_+(\tau,\eta))$ is a constant follows from Lemma 4.10 in [3, p.34]. See also [3, p.68]. The proof is copied in the Appendix of this note. □

Remark 4.2. Note the Dunford-Taylor Formula gives the projection map to the **generalized eigenspaces** correspond to the eigenvalues contained in the contour γ. The vector in its image might not be eigenvectors, but must be generalized eigenvectors.

Set $\mathcal{A}(\eta)=\mathcal{A}(1,\eta)$. Equation (12), which we need to solve, reads

$$\boxed{v'=\mathcal{A}(\eta)v+(A^d)^{-1}g.}$$

By the Lemma above, we may decompose $\mathbb{C}^n=E_-(\eta)\bigoplus E_+(\eta)$. Here $E_\pm$ are the stable and unstable subspaces corresponding to $\mathcal{A}(\eta)$. Indeed, E_- (resp. E_+) is the space spanned by those eigenvectors of $\mathcal{A}(\eta)$ corresponding to eigenvalues with negative (resp. positive) real parts. Note $E_\pm$ are invariant subspaces of $\mathcal{A}(\eta)$.

Therefore we decompose

$$v=v_s+v_u,\quad (A^d)^{-1}g=g_s+g_u,\qquad v_s,\ g_s\in E_-,\quad v_u,\ g_u\in E_+.$$

The equation is reduced to

$$v_s'=\mathcal{A}(\eta)v_s+g_s,\qquad v_u'=\mathcal{A}(\eta)v_u+g_u.$$

Let $\{S(z)\}_{z\in\mathbb{R}}$ be the group generated by $\mathcal{A}(\eta)$, that is, $S(z) = \exp(z\mathcal{A}(\eta))$. We look for a solution v of the form:

$$v_s(\eta, x_d) = S(x_d)v_0 + \int_0^{x_d} S(x_d - z)g_s(\eta, z)\,\mathrm{d}z, \tag{13}$$

$$v_u(\eta, x_d) = -\int_{x_d}^{\infty} S(x_d - z)g_u(\eta, z)\,\mathrm{d}z, \quad (\maltese) \tag{14}$$

where $v_0 \in E_-$ is to be chosen. Note that

$$\boxed{v(\eta, 0) = v_0 - \int_0^{\infty} S(-z)g_u(\eta, z)\,\mathrm{d}z. \qquad (\spadesuit)}$$

Obviously v_s, v_u solves the equations. We need to check that, for almost every η, they belong to $L^2(\mathbb{R}^+)$.

For fixed η and $v_0 \in E_-(\eta)$, $S(t)v_0 \in L^2$, since it decays exponentially as $x_d \to \infty$.

Since $g \in L^2(\Omega)$, by Fubini theorem, for almost every η, $g(\eta, \cdot) \in L^2(\mathbb{R}^+)$. Hence $g_s(\eta, \cdot)$ and $g_u(\eta, \cdot)$ are both in $L^2(\mathbb{R}^+)$. (Indeed, consider, for example, the projection mapping $P_s : \mathbb{C}^n \to E_-(\eta)$, along $E_+(\eta)$. It is continuous with respect to η, and is independent of x_d. So it's norm is $M(\eta)$. Hence $|u_s| \le M(\eta)|u|$.) While, the convolution kernel $S(x_d - \cdot)$ are also L^1 integrable. Actually, denoting S_s and S_u as the restriction of $S(t)$ on the invariant subspace $E_-(\eta)$ and $E_+(\eta)$. We know that $S_s(z)$ and $S_u(-z)$ decays exponentially as $z \to +\infty$. Then

$$\int_0^{x_d} S(x_d - z)g_s(\eta, z)\,\mathrm{d}z = \tilde{S}_s * \tilde{g}_s(x_d), \qquad x_d > 0,$$

where $h \mapsto \tilde{h}$ is the extension from $\mathbb{R}^+$ to $\mathbb{R}$ by taking $\tilde{h}(s) = 0$ for $s < 0$. Similar formula holds for convolution in (14). So by Young's Inequality,[1] the convolution products belong to $L^2(\mathbb{R}^+)$.

Now we show we can choose $v_0 \in E_-(\eta)$ such that $Bv = 0$ holds. That is, see ($\spadesuit$),

$$\boxed{Bv_0 = B\int_0^{\infty} S(-z)g_u(\eta, z)\,\mathrm{d}z, \qquad v_0 \in E_-(\eta).} \tag{15}$$

Lemma 4.2. *For L Friedrichs-symmetric, B maximal dissipative, it holds that*

$$\boxed{E_-(\eta) \bigoplus \ker B = \mathbb{C}^n.} \tag{16}$$

Consequently equation (15) admits uniquely one solution v_0.

Remark 4.3. We remind that (16) is a special case of the Kreiss-Lopatinskii condition which is necessary for a hyperbolic IBVP to be stable.

[1] $\|f * g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}$, $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$. Here we take $r = 2, p = 1, q = 2$.

Proof. **1.** We first show $E_-(\eta) \cap \ker B = \{0\}$. For $U_0 \in E_-(\eta)$, we set $U(x_d) = S(x_d)U_0$, which decays exponentially as $x_d \to \infty$, and satisfies the differential equation

$$\boxed{A^d \frac{\mathrm{d}U}{\mathrm{d}x_d} + (I_n + \mathrm{i}A(\eta))U = 0.} \tag{17}$$

Multiplying U^* on both sides of the equation, taking the real part, we get

$$2|U|^2 + \frac{\mathrm{d}}{\mathrm{d}x_d}(U^* A^d U) = 0.$$

Integrating from 0 to ∞, there follows

$$2 \|U\|^2_{L^2(\mathbb{R}^+)} = U^*(0)A^d U(0) = (A^d U(0), \overline{U(0)})_{\mathbb{R}^n} \leq 0.$$

We supposed $U_0 = U(0) \in \ker B$ for the last inequality. So $U = 0$ and especially $U_0 = 0$.

2. Next, to prove (16), we only need show $\dim \ker B = \dim E_+(\eta)$. By virtue of Lemma 4.1, the latter is the number of eigenvalues of A^d with negative real parts. By Proposition 3.2 and the assumption that A^d nonsingular, this is exactly $\dim \ker B$, as B is maximal dissipative.

3. We have shown (16). Which implies that $B : E_-(\eta) \to \mathrm{R}(B)$ is a homeomorphism. So (15) is uniquely solvable. □

We have now solved (12). We see the solution $v(\eta, x_d)$, defined for a.e. $\eta \in \mathbb{R}^{n-1}$, is H^1 with respect to x_d, and also measurable in (η, x_d). For given η, applying the energy estimate, which reads

$$\begin{aligned} 2\int_0^\infty |v(\eta, x_d)|^2 \,\mathrm{d}x_d &= (A^d v(\eta, 0), \overline{v(\eta, 0)}) + 2\mathrm{Re} \int_0^\infty (g, \bar{v})_{\mathbb{R}^n}(\eta, x_d) \,\mathrm{d}x_d \\ &\leq 2\mathrm{Re} \int_0^\infty (g, \bar{v})_{\mathbb{R}^n}(\eta, x_d) \,\mathrm{d}x_d \end{aligned}$$

by using $Bv(\eta, 0) = 0$ and dissipativeness. Applying Cauchy-Schwarz Inequality, there comes

$$\int_0^\infty |v(\eta, x_d)|^2 \,\mathrm{d}x_d \leq \int_0^\infty |g(\eta, x_d)|^2 \,\mathrm{d}x_d.$$

Integrating with respect to η, we get $\|v\|_{L^2(\Omega)} \leq \|g\|_{L^2(\Omega)}$. By Plancherel's Formula, we conclude $u \in L^2(\Omega)$. Also by Fourier Inversion, $u + Au = f$ holds in the sense of distribution. Note we get $Au = f - u \in L^2$. At last, $Bu = 0$ as we have made sure $Bv = 0$, so the boundary condition holds. So $u \in D(A)$, and the maximality of A is proved.

4.2.2. *Characteristic case.* Now we solve (12) in the case A^d is singular. Recall that since A^d is symmetric, there is a decomposition $\mathbb{R}^n = \ker A^d \bigoplus^{\perp} \mathrm{R}(A^d)$. This implies a decomposition $\mathbb{C}^n = \ker A^d \bigoplus^{\perp} \mathrm{R}(A^d)$, for the latter being consider as, for example, with vector like $\ker A^d + \mathrm{i} \ker A^d$, a linear space over $\mathbb{C}$.

We denote by π the projection onto $\ker A^d$, along $\mathrm{R}(A^d)$. Set $\pi v = k, (I-\pi)v = r$, $g_k = \pi g, g_r = (I-\pi)g$, then (12) is decomposed as

$$A^d r' + (I-\pi)(I + \mathrm{i}A(\eta))(r+k) = g_r, \tag{18}$$

$$\pi[(I+\mathrm{i}A(\eta))(r+k)] = g_k. \tag{19}$$

Recall $\ker A^d \subset \ker B$, so the boundary condition is simply

$$Br(\eta, 0) = 0. \tag{20}$$

We first study the algebraic equation (19), that is,

$$k + \pi[\mathrm{i}A(\eta)k] = g_k - \pi[\mathrm{i}A(\eta)r].$$

Considering the mapping $\boxed{\pi A(\eta)\pi : \ker A^d \to \ker A^d,}$ which is symmetric (note that π is an orthogonal projection, so $\pi^* = \pi$). So all of its eigenvalues are real. This means $\pi + \mathrm{i}\pi A(\eta)\pi : \ker A^d \to \ker A^d$ is invertible. Set $M(\eta)$ be the inverse. Hence we may solve

$$k = M(\eta)\big(g_k - \pi[\mathrm{i}A(\eta)r]\big). \tag{21}$$

Substituting k to (18), we have

$$\boxed{A^d r' + \mathcal{B}(\eta) r = G_r, \qquad Br(\eta,0) = 0,} \tag{22}$$

where

$$\mathcal{B}(\eta) = (I-\pi)[I + \mathrm{i}A(\eta)]\big\{I - \pi M(\eta)\pi[\mathrm{i}A(\eta)]\big\}(I-\pi) : \mathrm{R}(A^d) \to \mathrm{R}(A^d),$$

$$G_r = g_r - (I-\pi)[I+\mathrm{i}A(\eta)]M(\eta)g_k \in \mathrm{R}(A^d).$$

To be more explicit, we choose the base of $\mathbb{C}^n$ consists of eigenvectors of A^d, so $A^d = \begin{pmatrix} \mathbf{0_m} & \\ & \Lambda \end{pmatrix}$. Here we assume 0 is an eigenvalue of A^d with multiplicity m, and $\Lambda = \operatorname{diag}\{b_1, \cdots, b_q, a_1, \cdots, a_p\}$ with $\boxed{p+q = n-m}$; all a_i are positive, b_j are negative. Then $E^0(A^d) = \ker A^d = \mathbb{C}^m \times \mathbf{0}_{n-m}, E^-(A^d) = \mathbf{0}_m \times \mathbb{C}^q \times \mathbf{0}_p$, $E^+(A^d) = \mathbf{0}_m \times \mathbf{0}_q \times \mathbb{C}^p$, $\mathrm{R}(A^d) = \mathbf{0}_m \times \mathbb{C}^{n-m}$, and $\pi = \begin{pmatrix} I_m & \\ & \mathbf{0} \end{pmatrix}$. Recall $\ker A^d = E^0(A^d) \subset \ker B$, so we may set

$$B \doteq (0_{p\times m}, B_1), \tag{23}$$

with B_1 a $p \times (n-m)$ matrix.

We set $A(\eta) \doteq \begin{pmatrix} A_m & C^\top \\ C & A_{n-m} \end{pmatrix}$. Then $M(\eta) = \begin{pmatrix} (I_m + \mathrm{i}A_m)^{-1} & \\ & 0 \end{pmatrix}$, and

$$\mathcal{B}(\eta) = \begin{pmatrix} 0 & 0 \\ 0 & [I_{n-m} + \mathrm{i}A_{n-m}] + C[I_m + \mathrm{i}A_m]^{-1}C^\top \end{pmatrix}.$$

So, by considering $r(\eta, x_d) \in \mathbb{C}^{n-m}$, (22) is

$$\boxed{\Lambda r' + ([I_{n-m} + \mathrm{i}A_{n-m}] + C[I_m + \mathrm{i}A_m]^{-1}C^\top)r = G_r, \qquad B_1 r(\eta, 0) = 0.} \tag{24}$$

Lemma 4.3. *Set $\mathcal{A}'(\eta) = -\Lambda^{-1}([I_{n-m} + \mathrm{i}A_{n-m}(\eta)] + C(\eta)[I_m + \mathrm{i}A_m(\eta)]^{-1}C^\top(\eta))$. Then $\mathcal{A}'(\eta)$ has no pure imaginary eigenvalues. In addition, the dimension of stable (resp. unstable) subspace of $\mathcal{A}'(\eta)$ is p (resp. q).*

Proof. **1.** Suppose for $\tau \in \mathbb{R}$, $\mathrm{i}\tau$ is an eigenvalue of $\mathcal{A}'(\eta)$, that means

$$\det(I_{n-m} + \mathrm{i}A_{n-m}(\eta) + \mathrm{i}\tau\Lambda + C(\eta)[I_m + \mathrm{i}A_m(\eta)]^{-1}C^\top(\eta)) = 0.$$

On the other hand, we have

$$\begin{aligned}
&(I_n + \mathrm{i}A(\eta) + \mathrm{i}\tau A^d) \begin{pmatrix} I_m & -\mathrm{i}(I_m + \mathrm{i}A_m)^{-1}C^\top \\ 0 & I_{n-m} \end{pmatrix} \\
= &\begin{pmatrix} I_m + \mathrm{i}A_m & \mathrm{i}C^\top \\ \mathrm{i}C & I_{n-m} + \mathrm{i}A_{n-m} + \mathrm{i}\tau\Lambda \end{pmatrix} \begin{pmatrix} I_m & -\mathrm{i}(I_m + iA_m)^{-1}C^\top \\ 0 & I_{n-m} \end{pmatrix} \\
= &\begin{pmatrix} I_m + iA_m & 0 \\ \mathrm{i}C & I_{n-m} + \mathrm{i}A_{n-m}(\eta) + \mathrm{i}\tau\Lambda + C(\eta)[I_m + \mathrm{i}A_m(\eta)]^{-1}C^\top(\eta) \end{pmatrix},
\end{aligned}$$

so $\det(I_n + \mathrm{i}A(\eta) + \mathrm{i}\tau A^d) = \det(I_m + \mathrm{i}A_m)\det(I_{n-m} + \mathrm{i}A_{n-m}(\eta) + \mathrm{i}\tau\Lambda + C(\eta)[I_m + \mathrm{i}A_m(\eta)]^{-1}C^\top(\eta))$. Note both the left-hand side and the first factor in the right-hand side is nonzero (by hyperbolicity and symmetry of A_m). So we get a contradiction.

2. For the second claim we may use similar arguments as before and hence omit the details. □

As shown by Lemma 4.2, the following Lopatinskii condition guarantees that (24) has a solution $r \in L^2(0, \infty)$ for fixed η, hence by (21), $k \in L^2(0, \infty)$, thus $v = r + k \in L^2(0, \infty)$ as desired.

Lemma 4.4 (Lopatinskii condition)**.** *Let $E^-(\eta)$ be the stable subspace of $\mathcal{A}'(\eta)$. Then for all $\eta \in \mathbb{R}^{d-1}$, it holds*

$$\ker B_1 \bigoplus E^-(\eta) = \mathbb{C}^{n-m}.$$

Proof. **1.** As before (see proof of Lemma 4.2), since $\dim\ker B_1 = n-m-p$, (cf. (23)), $\dim E^-(\eta) = p$, so we only need show $\ker B_1 \cap E^-(\eta) = \{0\}$.

2. As before, we use energy estimate (comparing (17) with (24)) and only need prove, for any $r \in \mathbb{C}^{n-m}$, it holds, for the new extra term that

$$\mathrm{Re}(C(I_m + \mathrm{i}A_m)^{-1}C^\top r, r)_{\mathbb{C}^{n-m}} \geq 0.$$

This is simple. Let $v = C^\top r$. Then

$$\begin{aligned}&\mathrm{Re}(C(I_m + \mathrm{i}A_m)^{-1}C^\top r, r)_{\mathbb{C}^{n-m}} = \mathrm{Re}((I_m + \mathrm{i}A_m)^{-1}v, v)_{\mathbb{C}^m}\\ =&\mathrm{Re}(w, (I_m + \mathrm{i}A_m)w)_{\mathbb{C}^m} = |w|^2 \geq 0.\end{aligned}$$

Here, $w = (I_m + \mathrm{i}A_m)^{-1}v$. ☐

5. Nonhomogeneous Equations

For $f \neq 0$, we may use the semigroup $\{S_t\}$ established in Main Theorem, together with the Duhamel's Formula

$$\boxed{u(t) = S_t u_0 + \int_0^t S_{t-s} f(s)\,\mathrm{d}s}$$

to give a mild solution to the IBVPs in $\Omega \times \mathbb{R}_t^+$:

$$Lu(x,t) = f, \quad Bu(y,0,t) = 0, \quad u(x,0) = u_0(x),$$

provided f is integrable from $(0,T)$ to $X = L^2(\Omega)^n$. This mild solution is a distributional one. If $f \in L^2((0,T)\times\Omega)$, we also have the following fundamental estimate, for arbitrary positive γ :

$$\boxed{\mathrm{e}^{-2\gamma T}\|u(T)\|_{L^2}^2 + \gamma\int_0^T \mathrm{e}^{-2\gamma t}\|u(t)\|_{L^2}^2\,\mathrm{d}t \leq \|u_0\|_{L^2}^2 + \frac{1}{\gamma}\int_0^T \mathrm{e}^{-2\gamma t}\|f(t)\|_{L^2}^2\,\mathrm{d}t.}$$

To prove this, we consider $v = \mathrm{e}^{-\gamma t}u.$ Then v satisfies

$$\frac{\mathrm{d}v}{\mathrm{d}t} + Av + \gamma v = \mathrm{e}^{-\gamma t} f.$$

Taking $L^2(\Omega)^n$ inner product with v, using monotonicity of A, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{L^2}^2 + 2\gamma\|v(t)\|_{L^2}^2 \leq 2(\mathrm{e}^{-\gamma t}f(t), v(t))_{L^2} \leq \gamma\|v\|_{L^2}^2 + \frac{1}{\gamma}\left\|\mathrm{e}^{-\gamma t}f\right\|_{L^2}^2.$$

Integrating this with respect to t from 0 to T and the estimate follows.

6. Appendix: Perturbation Results on Polynomials and Eigenvectors

6.1. The n roots of a polynomial of degree n depend continuously on the coefficients of the polynomial. This means that there are n continuous functions $r_1, \cdots, r_n$ depending on the coefficients that parametrize the roots with correct multiplicity.

This result implies that the eigenvalues of a matrix depend continuously on the matrix. A proof can be found in [4, Theorem 3.9.1, p. 26].

6.2. Let X be a Banach space. A bounded operator P is a projection, if $P^2 = P$.

Lemma 6.1. *Let $P(t)$ be a projection in X depending continuously on a parameter t varying in a connected region of real or complex numbers. Then the range* $\mathrm{R}(P(t))$ *for different t are isomorphic to one another. In particularly,* $\dim \mathrm{R}(P(t))$ *is a constant.*

Proof. **1.** Let P, Q be two projections in X. We prove, if the spectrum radius (or norm) of

$$R = (P - Q)^2 = P + Q - PQ - QP$$

is less than 1, then P and Q are similar to each other.

2. R commutes with P and Q:

$$PR = P + PQ - P^2Q - PQP = P - PQP = RP.$$

Similarly, $(I - P - Q)^2$ commute with P, Q since $I - P$ is a projection. We also have the identities

$$(P - Q)^2 + (I - P - Q)^2 = I, \quad (PQ - QP)^2 = (P - Q)^4 - (P - Q)^2 - R^2 - R.$$

3. Set

$$U' = QP + (1 - Q)(1 - P), \quad V' = PQ + (1 - P)(1 - Q). \tag{25}$$

U' maps $\mathrm{R}(P) = PX$ to QX, $(I - P)X$ to $(I - Q)X$; V' maps QX to PX, $(I - Q)X$ to $(I - P)X$. There also holds

$$V'U' = U'V' = 1 - R.$$

A pair of mutually inverse operators U, V with the mapping properties stated above can be constructed easily, since R commutes with P, Q and therefore with U', V' too. It suffices to set

$$U = U'(1 - R)^{-\frac{1}{2}} = (1 - R)^{-\frac{1}{2}}U', \qquad V = V'(I - R)^{-\frac{1}{2}} = (I - R)^{-\frac{1}{2}}V',$$

provided the inverse square root $(I-R)^{-\frac{1}{2}}$ exists. A natural definition of this operator is given by the binomial series

$$(I-R)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix} (-R)^n.$$

This series is absolutely convergent if $\|R\| < 1$, or, more generally, if its spectral radius $r(R) < 1$. The sum T of the series satisfies $T^2 = I - R$ just as in the numerical binomial series. Thus

$$VU = UV = I, \quad V = U^{-1}, \quad U = V^{-1}.$$

4. Since $U'P = QP = QU'$ and $PV' = PQ = V'Q$ as can be seen from (25), we have $UP = QU, PV = VQ$ by the commutativity of R with all the operators here considered. Thus we have

$$Q = UPU^{-1}, \qquad P = U^{-1}QU.$$

So P and Q are similar to each other.

5. This implies that $\mathrm{R}(P)$ and $\mathrm{R}(Q)$ are isomorphic to each other by U and U^{-1}, as we can see from $UPX = QUX$ and $U^{-1}QX = PU^{-1}X$. So $\dim \mathrm{R}(P) = \dim \mathrm{R}(Q)$. □

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LECTURE NOTES 2: INITIAL-BOUNDARY VALUE PROBLEM IN HALF-SPACE WITH CONSTANT COEFFICIENTS: KREISS-LOPATINSKII CONDITION

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We derive necessary conditions for very weak well-posedness of Initial-Boundary Value Problems (IBVPs) for Hyperbolic Operators. The essential of the necessary conditions is the Kreiss-Lopatinskii condition (KL). We focus on the case of constant-coefficients and half-space domain. This note is based on Chapter 4, Section 1 and 2 of [1].

1. The Problem

1.1. Hyperbolicity.

We first review the definition of hyperbolic operators.

Definition 1.1. A first-order operator $L' = \partial_t + \sum_{\alpha=1}^{d} A^\alpha \partial_\alpha + D$ is called *hyperbolic* if the corresponding symbol $\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d \mapsto A(\xi) = \sum_{\alpha=1}^{d} A^\alpha \xi_\alpha$ satisfies

$$\boxed{\sup_{\xi \in \mathbb{R}^d} \|\exp(\mathrm{i}A(\xi))\| < \infty.}$$

Here A^α, D are real $n \times n$ constant matrices.

Theorem 1.1 (Kreiss' Matrix Theorem). *Let $\xi \mapsto A(\xi)$ be a linear map from $\mathbb{R}^d$ to $\mathbb{M}_n(\mathbb{C})$.* [1] *Then the following properties are equivalent to each other:*

(a) *Every $A(\xi)$ is diagonalizable with pure imaginary eigenvalues, uniformly with respect to ξ. That is,*

$$A(\xi) = P(\xi)^{-1} \mathrm{diag}(\mathrm{i}\rho_1, \cdots, \mathrm{i}\rho_n) P(\xi), \quad \rho_1(\xi), \cdots, \rho_n(\xi) \in \mathbb{R}$$

with

$$\left\|P(\xi)^{-1}\right\| \|P(\xi)\| \le C', \qquad \forall \xi \in \mathbb{R}^d.$$

(b) *There is a constant $C > 0$, so that*

$$\left\|\mathrm{e}^{tA(\xi)}\right\| \le C, \qquad \forall \xi \in \mathbb{R}^d, \quad \forall t \ge 0.$$

(c) *There is a constant $C > 0$, so that*

$$\left\|(zI_n - A(\xi))^{-1}\right\| \le \frac{C}{\mathrm{Re} z}, \qquad \forall \xi \in \mathbb{R}^d, \quad \forall \mathrm{Re} z > 0.$$

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[1] $\mathbb{M}_n(\mathbb{C})$ is the set of $n \times n$ matrices with complex entries.

1.2. **Set-up of IBVP.** Let $L = \partial_t + \sum_{\alpha=1}^{d} A^\alpha \partial_\alpha$ be a hyperbolic operator with $A^\alpha \in \mathbb{M}_n(\mathbb{R})$, and $B \in \mathbb{M}_{q\times n}(\mathbb{R})$, and

$$\Omega = \{x = (y, x_d) : \ y \in \mathbb{R}^{d-1}, \ x_d > 0\}.$$

The general problem we have in mind is

$$(Lu)(x,t) = f(x,t), \quad x_d, \ t > 0, \quad y \in \mathbb{R}^{d-1}, \tag{1}$$

$$Bu(y,0,t) = g(y,t), \quad t > 0, \quad y \in \mathbb{R}^{d-1}, \tag{2}$$

$$u(x,0) = u_0(x), \qquad x_d > 0, \quad y \in \mathbb{R}^{d-1}. \tag{3}$$

1.3. **Strategy on studying IBVP.**

- The concept "well-posedness" depends on specific spaces or requirements (estimates) in mind. We first consider well-posedness in a very weak sense. In the next lecture, for the applications to variable-coefficients problems or nonlinear problems, we will consider strong well-posedness in L^2 (i.e., there are estimates without loss of derivatives in certain time-weighted L^2 spaces).
- For the given definition of "well-posedness", by considering special cases or particular solutions, deriving some necessary conditions for such well-posedness.
- It is optimal that, under these necessary conditions, we construct a so called symbolic dissipative symmetrizer. By multiplying the symmetrizer to the system (in frequency spaces), it becomes symmetric, and the boundary condition becomes dissipative, and then energy estimate can be obtained. By such estimate and functional analysis methods (for example, duality), we prove existence of a solution, and hence show the IBVP is well-posed in the given sense.

2. Necessary condition for existence and uniqueness

2.1. **Number of Scalar Boundary Conditions.** For the boundary condition $Bu = g$ to be solvable, for any $g \in \mathbb{R}^q$, at least in the linear-algebraic lever, it is necessary that $\mathrm{R}(B) = \mathbb{R}^q$, [2] or

$$\boxed{\operatorname{rank} B = q.} \tag{4}$$

Since we may multiply any regular matrix $D \in \mathbb{M}_q(\mathbb{R})$ to the boundary condition, so the matrix B itself is not essential. What counts is $\ker B$. By (4), there holds

$$\boxed{\dim\ker B = n - q.} \tag{5}$$

In the following, we always consider *homogeneous boundary condition ($g = 0$).*

[2] $\mathrm{R}(B)$ is range of the matrix, or operator B, namely $\mathrm{R}(B) = \{Bx : \ x \in \mathbb{R}^n\}$.

2.2. **Necessary condition for existence.** The strategy of deriving necessary condition here is studying simpler *special* IBVPs.

Suppose $f = f(x_d, t)$, $u_0 = u_0(x_d)$. Then by translation along y and the uniqueness as assumed, the solution u also depends only on x_d and t. So problem (1)–(3) is simplified to

$$\partial_t u(x_d, t) + A^d \partial_d u(x_d, t) = f(x_d, t), \qquad Bu(0, t) = 0, \qquad u(x_d, 0) = u_0(x_d).$$

By hyperbolicity, A^d is diagonalizable. Using a change of dependent variables, without loss of generality, the above problem is further simplified as

$$\begin{aligned} &\partial_t u_j(x_d, t) + a_j \partial_d u_j(x_d, t) = f_j(x_d, t), \qquad Bu(0, t) = 0, \\ &u_j(x_d, 0) = u_{0j}(x_d), \qquad x_d > 0, t > 0. \end{aligned} \tag{6}$$

Here a_j are eigenvalues of A^d, and arranged in the decreasing order

$$a_1 \geq a_2 \geq \cdots \geq a_n.$$

Definition 2.1. The number p of positive eigenvalues of A^d: $a_p > 0 \geq a_{p+1}$, is called the number of *incoming characteristics,* for the domain Ω.

System (6) consists of decoupled scalar transport equations. For $j = p + 1, \cdots, n$, we can solve u_j just by using the initial data:

$$u_j(x_d, t) = (u_0)_j(x_d - a_j t) + \int_0^t f_j(x_d + a_j(s - t), s)\, \mathrm{d}s. \qquad (\checkmark)$$

Supposing $B = (b_1, \cdots, b_p, \cdots, b_n)$, we get $B(\mathbb{R}^p \times \{0\}_{n-p}) = \operatorname{span}\{b_1, \cdots, b_p\}$. Let $l \in \mathbb{R}^q$ be a row vector perpendicular to the subspace $B(\mathbb{R}^p \times \{0\}_{n-p})$, and $L = (L_1, \cdots, L_n) = lB \in \mathbb{R}^n$, which is also a row vector. Then $L_1 = \cdots = L_p = 0$. The boundary condition $Bu = 0$ implies $Lu = lBu = 0$, hence $\sum_{j=p+1}^n L_j u_j = 0$, or, specifically, from $(\checkmark)$, that

$$\sum_{j=p+1}^n [L_j (u_0)_j (x_d - a_j t) + \int_0^t L_j f_j(x_d + a_j(s - t), s)\, \mathrm{d}s]\Big|_{x_d = 0} = 0.$$

This is a nontrivial compatibility condition for the nonhomogeneous term f and initial data, if $L \neq 0$. While for general well-posedness (existence), there should be no such compatibility condition. So to guarantee existence, it is necessary that $L = lB = 0$. By (4), this implies $l = 0$. Hence we get

$$B(\mathbb{R}^p \times \{0\}_{n-p}) = \mathbb{R}^q, \tag{7}$$

and particularly

$$q \leq p. \tag{8}$$

2.3. **Necessary condition for uniqueness.** Next we consider the implication of uniqueness. Considering the homogeneous IBVP (6) ($f = 0, u_0 = 0$). From the formula obtained above for u_j, we see $u_j = 0$ for $j = p+1, \cdots, n$.

Now let R be a vector in $\mathbb{R}^p$ such that $R_j = (R, 0)^\top \in \ker B$. We also choose a smooth function v of one-variable that vanishes on $[0, \infty)$, and set

$$u_j(x_d, t) = v\left(\frac{x_d}{a_j} - t\right) R_j, \qquad j = 1, \cdots, p.$$

We easily check that such obtained $u = (u_1, \cdots, u_p, 0, \cdots, 0)^\top$ is a nontrivial solution to (6). So to guarantee uniqueness, we require

$$\ker B \cap (\mathbb{R}^p \times \{0\}_{n-p}) = \{0\}. \tag{9}$$

Since $\dim \ker B = n - q$, this implies [3]

$$p \leq n - (n - q) = q. \tag{10}$$

2.4. **Conclusion.** So to ensure well-posedness, by (8) and (10), it is necessary that $p = q$, or

> *The number of scalar boundary conditions equals the number of incoming characteristics.*

From this, (9) can also be written as

$$\boxed{\ker B \bigoplus (\mathbb{R}^p \times \{0\}) = \mathbb{R}^n.}$$

On the contrary, we note that this implies $p = q$ and (7).

Going back to a general matrix A^d, we prove that

Proposition 2.1. *For the IBVP* (1)–(3) *to be well-posed (uniqueness and existence), it is necessary that* [4]

$$\boxed{\mathbb{R}^n = \ker B \bigoplus E^u(A^d).} \tag{11}$$

[3]Suppose that V_1 and V_2 are linear subspaces of a linear space V, and $\dim V_1 + \dim V_2 > \dim V$, then $V_1 \cap V_2$ is a nontrivial linear subspace of V. For a proof, let $\alpha_1, \cdots, \alpha_p$ be a basis of V_1, and $\beta_1, \cdots, \beta_q$ be a basis of V_2, then the vectors $\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q$ are linearly dependent in V, hence there are numbers a_i, b_j so that $\sum_{i=1}^p a_i \alpha_i = \sum_{j=1}^q b_j \beta_j$. Note that not all a_i shall be zero, and not all b_j shall be zero. Hence $0 \neq \sum_{i=1}^p a_i \alpha_i \in V_1 \cap V_2$.

[4]$E^u(A^d)$ is the unstable subspace of A^d, which is spanned by the eigenvectors associated with positive eigenvalues of A^d. Recall that by hyperbolicity, all eigenvalues of A^d are real.

Proof. **1.** For the system $\partial_t u + A^d \partial_d u = f$, if $P^{-1}A^dP = \operatorname{diag}(a_1, \cdots, a_n)$, then by the transform $u = Pv$, in the v-coordinates, it takes the form (6) (with u_j replaced by v_j). B is replaced by BP. So we actually get, in the v-coordinates,

$$\ker(BP) \bigoplus (\mathbb{R}^p \times \{0\}) = \mathbb{R}^n.$$

2. Now return to the u-coordinates, $\ker BP$ should be $\ker B$, and $(\mathbb{R}^p \times \{0\})$ is replaced by $P(\mathbb{R}^p \times \{0\}) = \operatorname{span}(Pe_1, \cdots, Pe_p)$, with $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)^\top$ the standard unit vector in $\mathbb{R}^n$. However, $P^{-1}A^dP = \operatorname{diag}(a_1, \cdots, a_n)$ implies that Pe_k is the eigenvector of A^d corresponding to eigenvalue a_k. So we see $P(\mathbb{R}^p \times \{0\})$ is the subspace spanned by those eigenvectors of A^d, associated to positive eigenvalues. By definition, it is $E^u(A^d)$. □

Definition 2.2. We say the IBVP (1)–(3) is *normal*, if

(a) $B \in \mathbb{M}_{p\times n}$, and $\operatorname{rank} B = p$ ($p = \dim E^u(A^d)$);

(b) $\ker A^d \subset \ker B$;

(c) property (11) holds true: $\boxed{\mathbb{R}^n = \ker B \bigoplus E^u(A^d).}$

Remark 2.1. (a) and (c) imply p is the number of positive eigenvalues of A^d.

(b) is required for the definition of trace of Bu, so the boundary condition makes sense. (Recall we defined $A^d u|_{\partial\Omega}$ by using the normal trace of vector field, while (b) implies $B = MA^d$).

3. Necessary condition for stability: Kreiss-Lopatinskii condition

In the following we deduce a condition that turns out to be necessary even for very weak well-posedness of (1)–(3) when $g = 0$. The strategy is to consider special solutions of the form (*normal mode analysis*)

$$\boxed{u(x,t) = \mathrm{e}^{\tau t + \mathrm{i} y\cdot\eta} U(x_d),} \tag{12}$$

with $\eta \in \mathbb{R}^{d-1}$ and $\tau \in \mathbb{C}$. *Our aim is to find necessary conditions so that those solutions of the form* (12) *that could contradict well-posedness — that is, those grow rapidly as time increases, while being temperate in space — cannot exist.* **To this end, we need restrict ourselves to complex numbers τ of positive real part:** $\operatorname{Re}\tau > 0$**.**

A field u defined by (12) solves $Lu = 0$ if and only if

$$A^d \frac{\mathrm{d}U}{\mathrm{d}x_d} + (\tau I_n + \mathrm{i}A(\eta))U = 0, \tag{13}$$

where $A(\eta) = \sum_{\alpha=1}^{d-1} A^\alpha \eta_\alpha$. [5] There are two cases, depending on wether A^d is singular.

[5]Note there also appears $A(\xi) = \sum_{\alpha=1}^{d} A^\alpha \xi_\alpha$ before. To tell what $A(\zeta)$ means, just take care whether ζ is in $\mathbb{R}^d$, or $\mathbb{R}^{d-1}$.

3.1. **Kreiss-Lopatinskii condition: non-characteristic case.** We first assume A^d is nonsingular — In other words, the boundary $\partial\Omega$ is non-characteristic. We introduce

$$\mathcal{A}(\tau,\eta) \doteq -(A^d)^{-1}(\tau I_n + \mathrm{i}A(\eta)).$$

So (13) may be recast as an ODE in $\mathbb{C}^n$, with parameters τ,η:

$$\frac{\mathrm{d}U}{\mathrm{d}x_d} = \mathcal{A}(\tau,\eta)U. \tag{14}$$

3.1.1. *Construction of an exponentially-grow-in-time, temperate-in-space solution.* The following Lemma has been proved.

Lemma 3.1. *Suppose the operator $\partial_t + \sum_{\alpha=1}^d A^\alpha\partial_\alpha$ is hyperbolic. Then for $\eta\in\mathbb{R}^{d-1}$ and* $\mathrm{Re}\tau > 0$*, the matrix $\mathcal{A}(\tau,\eta)$ does not have any pure imaginary eigenvalue. The number of stable eigenvalues (eigenvalues with negative real parts), counted with multiplicities, equals p, the number of positive eigenvalues of A^d.*

By this Lemma, we have a decomposition $\mathbb{C}^n = E_-(\tau,\eta)\bigoplus E_+(\tau,\eta)$, with $E_\pm(\tau,\eta)$ being the unstable/stable subspace of $\mathcal{A}(\tau,\eta)$. Set $\pi_\pm$ be the projection of $\mathbb{C}^n$ to $E_\pm(\tau,\eta)$, and $U_\pm(x_d) = \pi_\pm U(x_d)$. Note that $E_\pm$ are invariant subspaces of $\mathcal{A}(\tau,\eta)$, so (14) is also decomposed into $\frac{\mathrm{d}U_\pm}{\mathrm{d}x_d} = \mathcal{A}(\tau,\eta)U_\pm$ and the solutions are

$$U_\pm(x_d) = \exp(x_d\mathcal{A}(\tau,\eta))(U_0)_\pm.$$

The matrix $\exp(x_d\mathcal{A}(\tau,\eta))|_{E_-}$ decays exponentially as $x_d\to\infty$, while the inverse of $\exp(x_d\mathcal{A}(\tau,\eta))|_{E_+}$ decays exponentially as $x_d\to\infty$. Therefore, in order that $U(x_d)$ to be a tempered distribution on $\mathbb{R}^+$, it is necessary that $(U_0)_+ = 0$, or in other words,

$$U(0)\in E_-(\tau,\eta).$$

If this holds, U actually decays exponentially and hence square-integrable.

For this reason, we admit only those solutions of (14) for which $U(0)\in E_-(\tau,\eta)$. They take their values in $E_-(\tau,\eta)$. For such a solution U and corresponding u, which is a solution of $Lu = 0$, if there also holds $BU(0) = 0$, then $Bu(y,0,t) = 0$. At $t = 0$, the initial data

$$u(y,x_d,0) = \mathrm{e}^{\mathrm{i}y\cdot\eta}U(x_d)$$

belongs to any Hölder space $\mathscr{C}^{k,\alpha}(\Omega)$, while the norm $\|u(\cdot,t)\|_{\mathscr{C}^{k,\alpha}(\Omega)}$ grows exponentially fast (like $\mathrm{e}^{t\mathrm{Re}\tau}$) as t increases, provided $U(0)\neq 0$. However, this is not enough to show ill-posedness. We need the following ideas of scaling to demonstrate instability.

3.1.2. *Hadamard instability by scaling.* Now, scaling both space and time variables yields the parameterized solution of the homogeneous IBVP:

$$u^\lambda(x,t) = u(\lambda x, \lambda t), \qquad \lambda > 0.$$

As $\lambda \to \infty$, the initial data $u^\lambda(x,0) = \mathrm{e}^{\mathrm{i}\lambda\eta\cdot y}U(\lambda x_d)$ grows at most polynomially in Hölder space with resect to λ, while $u^\lambda(x,t)$ grows (with respect to λ) always exponentially fast for any given positive time. This shows the mapping

$$u(\cdot,0) \mapsto u(\cdot,t), \qquad t > 0,$$

if ever defined, may not be continuous between Hölder spaces, **even at the price of loss of derivatives** (which means we only control $\mathscr{C}^{l,\beta}$ norm of $u(\cdot,t)$ for $l \le k,\ \beta < \alpha$, by using $\mathscr{C}^{k,\alpha}$ norm of $u(\cdot,0)$).

3.1.3. *Conclusion.* This shows for well-posedness (stability) in Hölder spaces, a necessary condition is

$$\boxed{E_-(\tau,\eta) \cap \ker B = \{0\} \qquad \text{for every } \eta \in \mathbb{R}^{d-1},\ \mathrm{Re}\tau > 0.} \tag{15}$$

Definition 3.1. We say the hyperbolic IBVP (1)–(3) satisfies the *Kreiss-Lopatinskii condition* (KL), if (15) holds true.

Remark 3.1. We have shown that the Lopatinskii condition is necessary for the well-posedness of IBVPs in Hölder spaces. When it fails, no estimate can hold in such norms, even at the price of loss of derivatives. Later we will use Lopatinskii determinant to show the same result for Sobolev spaces.

Lemma 3.2. *For a non-characteristic IBVP* (L,B)*, if* $\mathrm{rank}\, B$ *equals the number of positive eigenvalues of* A^d*, then the Kreiss-Lopatinskii condition is equivalent to*

$$\mathbb{C}^n = E_-(\tau,\eta) \bigoplus \ker B \qquad \forall \eta \in \mathbb{R}^{d-1},\ \forall \mathrm{Re}\tau > 0. \tag{16}$$

Remark 3.2. As $E_-(1,0) = E^u(A^d)$, we see (11) for normal IBVP is a special case of (16). Also note that (16) implies that[6] $p = q$. As $\{0\} = \ker A^d \subset \ker B$ holds trivially (for the non-characteristic case), we see, (16) itself represents all the necessary conditions of well-posedness we obtained, for the non-characteristic case.

Proof. **1.** We first note the following fact:

> *If V is a linear subspace of $\mathbb{R}^n$ with $\dim V = p$, then being considered as a linear subspace of $\mathbb{C}^n$ on $\mathbb{C}$, it's dimension is still p.*

[6]Recall that q is the number of boundary conditions, and p is the number of incoming characteristics.

To show this, let $\alpha_1, \cdots, \alpha_p \in \mathbb{R}^n$ be a basis of V. Then clearly they are also linearly independent on $\mathbb{C}$: If $\sum_{j=1}^p (a_j + \mathrm{i}b_j)\alpha_j = 0$ and $a_j,\ b_j \in \mathbb{R}$, then $a_j = b_j = 0$. Also, V, as a linear space in $\mathbb{C}$, every element is of the form $\sum_{j=1}^p c_j\alpha_j$ and $c_j \in \mathbb{C}$. So the claim holds.

2. We have $\dim\ker B = n - p$, where p is the number of positive eigenvalues of A^d. So by (15), to show (16), we only need prove $\dim E_-(\tau,\eta) = p$. This follows from Proposition 3.1 below, which has been proved before. □

Proposition 3.1. *The stable and unstable subspaces $E_\pm(\tau,\eta)$ depend holomorphically on τ, analytically on η. In particular, their dimensions do not depend on τ, η as long as $\eta \in \mathbb{R}^{n-1}$ and* $\mathrm{Re}\tau > 0$.

3.2. **Kreiss-Lopatinskii condition: characteristic case.** We now consider (13) when A^d is singular.

3.2.1. *Decomposition.* Since A^d is diagonalizable, we have a decomposition

$$\mathbb{C}^n = \mathrm{R}(A^d) \bigoplus \ker(A^d). \tag{17}$$

In fact, choosing a basis consists of eigenvectors of A^d, then there is a decomposition

$$\mathbb{R}^n = E^u(A^d) \bigoplus E^s(A^d) \bigoplus E^c(A^d).$$

We note $E^c(A^d) = \ker A^d$, and $E^u(A^d) \bigoplus E^s(A^d) = \mathrm{R}(A^d)$. So there holds

$$\mathbb{R}^n = \mathrm{R}(A^d) \bigoplus \ker(A)^d.$$

Now by considering $\mathbb{R}^n, \mathrm{R}(A^d)$ and $\ker(A^d)$ all as spaces on $\mathbb{C}$, we have (17).

Denoting by π the projection onto $\ker A^d$, along $\mathrm{R}(A^d)$, we decompose $U = r + k$, with $k = \pi U,\ r = (I_n - \pi)U$. So (13) is equivalent to

$$\begin{aligned} A^d \frac{\mathrm{d}r}{\mathrm{d}x_d} + (I_n - \pi)(\tau I_n + \mathrm{i}A(\eta))(r+k) &= 0, && (18)\\ \pi(\tau I_n + \mathrm{i}A(\eta))(r+k) &= 0. && (19) \end{aligned}$$

3.2.2. *Hyperbolicity of projected subsystem.* We claim the endomorphism

$$\pi A(\eta)\pi : \ker A^d \to \ker A^d$$

has only real spectrum. Hence one could solve k as a function of r from (19), and then (18) becomes a closed system for the unknown r. This follows from the theorem below, by taking $\xi_0 = (0, \cdots, 0, 1)$ and $\lambda_0 = 0$.

Theorem 3.1. *Let $L = \partial_t + \sum_{\alpha=1}^d A^\alpha \partial_\alpha$ be a hyperbolic $n \times n$ operator and $\xi_0 \in \mathbf{S}^{d-1}$. Given an eigenvalue λ_0 of $A(\xi_0) = \sum_{\alpha=1}^d A^\alpha \xi_\alpha$, denote by π the projection onto the eigenspace $F(\lambda_0) = \ker(A(\xi_0) - \lambda_0 I_n)$.*

Then the operator

$$L' = \pi\partial_t + \sum_{\alpha=1}^d \pi A^\alpha \pi \partial_\alpha,$$

acting on functions valued in $F(\lambda_0)$ (thus it is an $m \times m$ operator, m being the multiplicity of λ_0) is hyperbolic.

Proof. **1.** By hyperbolicity of L, using a linear transform of the unknown, which amounts to conjugating the matrices A^α, we may assume $A(\xi_0)$ is diagonal:

$$A(\xi_0) = \begin{pmatrix} \lambda_0 I_m & 0 \\ 0 & D_0 \end{pmatrix},$$

where $D_0 - \lambda_0 I_{n-m}$, of size $n - m$, is invertible. So $F(\lambda_0) = \mathbb{R}^m \times \{0\}_{n-m}$, and

$$\pi(u_1, \cdots, u_n)^\top = (u_1, \cdots, u_m, 0, \cdots, 0)^\top.$$

We decompose vectors and matrices accordingly:

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A^\alpha = \begin{pmatrix} C^\alpha & F^\alpha \\ E^\alpha & D^\alpha \end{pmatrix}.$$

Here $x \in \mathbb{R}^m,\ y \in \mathbb{R}^{n-m}$, and $C^\alpha \in \mathbb{M}_m,\ D^\alpha \in \mathbb{M}_{n-m}$. Then $\pi A^\alpha \pi : F(\lambda_0) \to F(\lambda_0)$ is given by the matrix $C^\alpha :\ \mathbb{R}^m \to \mathbb{R}^m$, and

$$L' = \partial_t + \sum_{\alpha=1}^d C^\alpha \partial_\alpha.$$

We shall prove this $m \times m$ operator is hyperbolic.

2. For $\xi = \xi_0$, we see $A(\xi_0)$ has invariant subspaces $\mathbb{R}^m \times \{0_{n-m}\}$ and $\{0_m\} \times \mathbb{R}^{n-m}$. These two invariant subspaces correspond to disjoint parts of spectrum, and their direct sum is $\mathbb{R}^n$.

By standard results on perturbations of linear operators, the invariant subspace depends analytically on the parameter ξ near ξ_0. More precisely, there exists a neighborhood $\mathcal{V}$ of ξ_0 and an analytical map $\xi \mapsto K(\xi)$ from $\mathcal{V}$ to $\mathbb{M}_{(n-m)\times m}(\mathbb{R})$ such that [7]

[7] A proof of this claim is given here. Let $\Lambda(\xi)$ (i.e., $N(\xi)$ in the claim) be the eigenspace associated with the eigenvalue $\lambda(\xi)$ of $A(\xi)$, and $\lambda(\xi_0) = \lambda_0$. Hence $\Lambda(\xi_0) = \mathbb{R}^m \times \{0\}_{n-m}$. By the assumption, at ξ_0, the eigenvalues of D_0 are distinct from λ_0, so in a small neighborhood of ξ_0, $\lambda(\xi)$, and the eigenspace $\Lambda(\xi)$, are analytic. By Kato's method [2, p.100] (we will give details of this method in the following lecture of Lopatinskii determinant), we may construct a basis $\{e_j(\xi)\}_{j=1}^m$ of $\Lambda(\xi)$, and $e_j(\xi_0) = e_j$, with e_j the standard basis of $\mathbb{R}^n$ (the j-th argument is 1; the others are zero). Also, $e_j(\xi)$

- $K(\xi_0) = 0$;
- The subspace $N(\xi) \doteq \left\{ \begin{pmatrix} x \\ K(\xi)x \end{pmatrix} : \ x \in \mathbb{R}^m \right\}$ is invariant under $A(\xi)$.

3. Hence, $N(\xi)$ is invariant under the flow of ODE $\dot{X} = A(\xi)X$, namely, for $x_0 \in \mathbb{R}^m$, we always have

$$\exp(tA(\xi)) \begin{pmatrix} x_0 \\ K(\xi)x_0 \end{pmatrix} \in N(\xi).$$

On the other hand, the flow is defined by $\dot{x} = Q(\xi)x, \ y = K(\xi)x$, where $Q(\xi) = C(\xi) + F(\xi)K(\xi)$, with $C(\xi) = \sum_{\alpha=1}^d C^\alpha \xi_\alpha$, and $F(\xi) = \sum_{\alpha=1}^d F^\alpha \xi_\alpha$ defined similarly. Therefore, for $t > 0$, we should have

$$\exp(tA(\xi)) \begin{pmatrix} x_0 \\ K(\xi)x_0 \end{pmatrix} = \begin{pmatrix} \exp(tQ(\xi))x_0 \\ K(\xi)\exp(tQ(\xi))x_0 \end{pmatrix}.$$

Note that for $t \in \mathbb{C}$, both sides are also well-defined and holomorphic with respect to t. So the above equality also holds if we replace t by $\mathrm{i}t$ for $t > 0$:

$$\exp(\mathrm{i}tA(\xi)) \begin{pmatrix} x_0 \\ K(\xi)x_0 \end{pmatrix} = \begin{pmatrix} \exp(\mathrm{i}tQ(\xi))x_0 \\ K(\xi)\exp(\mathrm{i}tQ(\xi))x_0 \end{pmatrix}.$$

3. We note $\left\| \begin{pmatrix} x \\ K(\xi)x \end{pmatrix} \right\|^2 = \|x\|^2 + \|K(\xi)x\|^2$ for $x \in \mathbb{R}^m$, so there holds $\|x\| \le \left\| \begin{pmatrix} x \\ K(\xi)x \end{pmatrix} \right\| \le \|x\| + \|K(\xi)x\|$. Let us define

$$M \doteq \sup_{\xi \in \mathbb{R}^d} \|\exp(\mathrm{i}tA(\xi))\|,$$

which is finite by our assumption of hyperbolicity. So we have

$$\|\exp(\mathrm{i}tQ(\xi))\| \le M(1 + \|K(\xi)\|).$$

Let $\eta \in \mathbb{R}^d$ be given. One applies the above estimate to the vector $\xi = \xi_0 + s\eta$, for s small enough so that $\xi \in \mathcal{V}$ and $t = 1/s$. (Recall $\mathcal{V}$ is open as assumed.) Since $C(\xi)$ and $F(\xi)$ are linear on ξ, and $C(\xi_0) = \lambda_0 I_m, \ F(\xi_0) = 0$, one gets

$$Q(\xi) = \lambda_0 I_m + sC(\eta) + sF(\eta)K(\xi).$$

---

are analytic. Let $E(\xi) = (e_1(\xi), \cdots e_m(\xi))$, which is a $n \times m$ matrix. We also write it block-wise as $E(\xi) = \begin{pmatrix} (E_1(\xi))_m \\ (E_2(\xi))_{(n-m)\times m} \end{pmatrix}$. Then as $\det(E_1(\xi))$ equals 1 at ξ_0, in a neighborhood of ξ_0, $E_1(\xi)$ is invertible. Note $\Lambda(\xi) = \{E(\xi)y : y \in \mathbb{R}^m\}$. So for any $x \in \mathbb{R}^m$, we may solve $y = E_1(\xi)^{-1}x$, and then $\Lambda(\xi)$ maybe expressed as $\left\{ \begin{pmatrix} x \\ K(\xi)x \end{pmatrix} : x \in \mathbb{R}^m \right\}$, with $K(\xi) = E_2(\xi)E_1(\xi)^{-1}$. (This is similar to represent a linear subspace by the graph of the mapping $K(\xi)$.)

Therefore, as $t = 1/s$,

$$\exp(\mathrm{i}tQ(\xi)) = \mathrm{e}^{\mathrm{i}t\lambda_0}\exp(\mathrm{i}(C(\eta) + F(\eta)K(\xi))).$$

By taking $s \to 0$, note that $K(\xi) \to K(\xi_0) = 0$, we have the desired estimate

$$\|\exp(\mathrm{i}C(\eta))\| \leq M$$

with M independent of η. □

Now we come back to (19). We see, as $\mathrm{Re}\tau > 0$, $\pi(\tau I_n + \mathrm{i}A(\eta))|_{\ker A^d}$ is nonsingular. So we may solve k from (19) as $k = M(\tau,\eta)r \doteq -\big(\pi(\tau I_n + \mathrm{i}A(\eta))\pi\big)^{-1}\pi(\tau I_n + \mathrm{i}A(\eta))r$, with $M(\tau,\eta) \in \mathscr{L}(\mathrm{R}(A^d); \ker A^d)$. Then (18) becomes an ODE:

$$\frac{\mathrm{d}r}{\mathrm{d}x_d} = \mathcal{B}(\tau,\eta)r. \tag{20}$$

Given an initial data r_0, it admits a unique solution $r = r(x_d)$, and hence k is solved by $k = k(x_d) \doteq M(\tau,\eta)r(x_d)$.

Lemma 3.3. *For* $\mathrm{Re}\tau > 0$ *and* $\eta \in \mathbb{R}^{d-1}$*, the matrix* $\mathcal{B}(\tau,\eta)$ *does not have pure imaginary eigenvalues. Consequently, the number of eigenvalues of positive (resp. negative) real part does not depend on* (τ,η)*. It equals the number of negative (resp. positive) eigenvalues of* A^d*.*

Proof. **1.** For λ an eigenvalue of $\mathcal{B}(\tau,\eta)$, it is necessary that there exists $r \in \mathrm{R}(A^d)$ that is nonzero, and a $k \in \ker A^d$, such that

$$\lambda A^d r + (\tau I_n + \mathrm{i}A(\eta))(r + k) = 0.$$

This can be seen by using (18)–(20) (replacing $\frac{\mathrm{d}}{\mathrm{d}x_d}$ by λ). This is also sufficient, as can be seen by using decomposition and noting that A^d is nonsingular on $\mathrm{R}(A^d)$.

2. The equation above is equivalent to

$$(\lambda A^d + \tau I_n + \mathrm{i}A(\eta))(r + k) = 0.$$

If λ is purely imaginary, then by hyperbolicity, there should hold $\mathrm{Re}\tau = 0$, a contradiction!

3. The rest proof is similar to that of Lemma 3.1. Since $\mathcal{B}(\tau,\eta)$ is holomorphic to τ, analytical to η, their eigenvalues also share these properties and hence the number of eigenvalues with positive (resp. negative) real part should be the same as $\mathcal{B}(1,0)$. However, in this case, $M(\tau,\eta) = 0$ and (18) becomes $(A^d\frac{\mathrm{d}r}{\mathrm{d}x_d} + r)|_{\mathrm{R}(A^d)} = 0$, hence $\mathcal{B}(1,0) = -(A^d|_{\mathrm{R}(A^d)})^{-1}$. □

We concluded from this Lemma that bounded solution of (20) decays exponentially as $x_d \to \infty$, and form the stable subspace of $\mathcal{B}(\tau, \eta)$, with dimension p. Hence the solutions of (13) decay to zero as $x_d \to \infty$ take value in a p-dimensional vector space $E_-(\tau, \eta)$, again called the stable subspace of (13). **The $E_-(\tau, \eta)$ is made of sums $r + M(\tau, \eta)r$, with r in the stable subspace $E^s(\mathcal{B}(\tau, \eta))$.**

3.2.3. *Conclusion.* Mimicking the argument for the non-characteristic case (using scaling), we see that a necessary condition for well-posedness in Hölder spaces is again the Kreiss-Lopatinskii condition $E_-(\tau, \eta) \cap \ker B = \{0\}$, or, by assuming $p = q$ (cf. (16)),

$$\boxed{\mathbb{C}^n = E_-(\tau, \eta) \bigoplus \ker B, \qquad \forall \eta \in \mathbb{R}^{d-1}, \quad \mathrm{Re}\tau > 0.}$$

On the contrary, this decomposition implies $p = q$, and, as $E_-(1, 0) = E^u(A^d)$ still holds, implies also (11). However, we cannot deduce from it the requirement $\ker A^d \subset \ker B$ for a normal IBVP.

3.2.4. *Failure of Kreiss-Lopatinskii condition cannot come from the characteristic nature of the boundary.*

Proposition 3.2. *For any $\eta \in \mathbb{R}^{d-1}$ and $\mathrm{Re}\tau > 0$, it holds that*

$$E_-(\tau, \eta) \cap \ker A^d = \{0\}.$$

Proof. Let $u = r + k$ belong to $E_-(\tau, \eta)$. Then $k = M(\tau, \eta)r$. If $u \in \ker A^d$, then $r = u - k \in \ker A^d$, hence $r \in \ker A^d \cap \mathrm{R}(A^d)$, thus $r = 0$, and therefore $k = 0$. Hence $u = 0$ too. $\square$

4. Further remarks on Kreiss-Lopatinskii condition

4.1. **Conclusion.** For IBVP (1)–(3) to be well-posed in a very weak sense, it is necessary to hold the following Kreiss-Lopatinskii condition:

$$\mathbb{C}^n = E_-(\tau, \eta) \bigoplus \ker B, \qquad \forall \eta \in \mathbb{R}^{d-1}, \quad \mathrm{Re}\tau > 0.$$

It contains all the necessary conditions we have derived for existence, uniqueness and stability.

For characteristic case, in order that the boundary condition $Bu = g$ make sense when the solution u is not continuous (for example, in L^2), we also need $\ker A^d \subset \ker B$ to formulate an IBVP.

Violation of KL at a frequency point $(\tau, \eta) \neq 0$ with $\mathrm{Re}\tau > 0, \eta \in \mathbb{R}^{d-1}$ will lead to an Hadamard instability.

4.2. Unlike hyperbolicity, Kreiss-Lopatinskii condition, and the normal boundary condition (11), are not invariant under time reversing. Breaking symmetry in space (introducing boundary conditions) also breaks symmetry in time. [8]

4.3. Note that in Kreiss-Lopatinskii condition, the direct sum is not an orthogonal sum, so for different (τ, η), $E_-(\tau, \eta)$ may actually be different subspace of $\mathbb{C}^n$. As $E_-(\tau, \eta)$ has the same dimension, it may be regarded as a point on the Grassmannian manifold $\mathbf{G}(n, p)$, i.e., the Riemannian manifold consists of p-dimensional subspaces in $\mathbb{C}^n$.

We also note that since $E_-(\mu\tau, \mu\eta) = E_-(\tau, \eta)$ for all $\mu > 0$, KL is positive homogeneous of degree zero. Then $E_-(\tau, \eta)$ may be considered as a map from the hemisphere

$$\mathrm{Re}\tau > 0, \quad |\tau|^2 + |\eta|^2 = 1$$

to $\mathbf{G}(n, p)$. We have shown it is well-defined and smooth.

4.4. By the Kreiss-Lopatinskii condition, we have $B : E_-(\eta, \tau) \to \mathrm{R}(B) = \mathbb{R}^p$ is an isomorphism. So there are constants $C(\tau, \eta) > 0$ so that

$$|V| \le C(\tau, \eta)|BV|, \qquad \forall V \in E_-(\tau, \eta). \tag{21}$$

We remark that the constant may depend on $\mathrm{Re}\tau > 0$ and $\eta \in \mathbb{R}^{d-1}$. On the contrary, if (21) and $\mathrm{R}(B) = \mathbb{R}^p$ hold, with p the number of incoming characteristics, then obviously the Kreiss-Lopatinskii condition holds.

If the constant C does not depend on (τ, η), then we obtain the Uniform Kreiss-Lopatinskii condition (UKL), which turns out to be necessary for L^2 strong well-posedness that will be studied in the next lecture. It turns out that UKL is closely connected to the continuous extension of the map $(\tau, \eta) \mapsto E_-(\tau, \eta)$ to the closed hemisphere $\mathrm{Re}\tau \ge 0, \quad |\tau|^2 + |\eta|^2 = 1$.

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[8]The incoming characteristics become outcome when time is reversed. The stable subspace becomes to be the unstable one.

LECTURE NOTES 3: INITIAL-BOUNDARY VALUE PROBLEM IN HALF-SPACE WITH CONSTANT COEFFICIENTS: UNIFORM KREISS-LOPATINSKII CONDITION, LOPATINSKII DETERMINANT AND EXAMPLES

HAIRONG YUAN

To study variable-coefficient problems or nonlinear problems, one requires the linear initial-boundary value problem (IBVP) with constant-coefficient should be well-posed in a strong sense, therefore robust for small perturbation. Motivated by the estimate obtained for symmetric hyperbolic systems with strongly dissipative boundary conditions, [1] we give the definition of L^2 strong-well-posedness. Then we derive a necessary condition for such well-posedness, namely, uniformly Kreiss-Lopatinskii condition (UKL). It turns out, for constantly hyperbolic operators, and non-characteristic boundary, this is also sufficient for the IBVP to be L^2 strong-well-posed.

We also introduce a somewhat practical tool called Lopatinskii determinant to check wether (uniform or non-uniform) Kreiss-Lopatinskii condition holds.

Computation of Lopatinskii determinant and checking KL (UKL) is usually a bothering job. We give some examples at the end of the note.

This note is based on parts of Sections 3 and 6 in Chapter 4 of [1]. It is only used for teaching.

1. Uniform Lopatinskii Condition: The Non-characteristic Case

1.1. The estimate and L^2 well-posedness.

Definition 1.1. Consider a non-characteristic hyperbolic IBVP

$$Lu = \partial_t u + \sum_{\alpha=1}^d A^\alpha \partial_\alpha u = f, \quad x = (y, x_d):\ y \in \mathbb{R}^{d-1},\ x_d > 0,\ t > 0, \tag{1}$$

$$Bu = g, \quad x = (y, x_d):\ y \in \mathbb{R}^{d-1},\ x_d = 0,\ t > 0, \tag{2}$$

$$u = u_0, \quad x = (y, x_d):\ y \in \mathbb{R}^{d-1},\ x_d > 0,\ t = 0 \tag{3}$$

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[1]We skipped this topic in this series of lectures. However, It will be exhibited later.

in the domain $\{x_d > 0,\ t > 0\}$. We say this IBVP is *strongly well-posed in* L^2 if the inequality

$$\begin{aligned} & \mathrm{e}^{-2\gamma T}\,\|u(T)\|_{L^2}^2 + \gamma\int_0^T \mathrm{e}^{-2\gamma t}\int_\Omega |u(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \mathrm{e}^{-2\gamma t}\int_{\partial\Omega} |(\gamma_0 u)(y,t)|^2\,\mathrm{d}y\mathrm{d}t \\ \le\ & C\left(\|u_0\|_{L^2}^2 + \int_0^T \mathrm{e}^{-2\gamma t}\left(\frac{1}{\gamma}\,\|Lu(t)\|_{L^2}^2 + \|\gamma_0 Bu(t)\|_{L^2}^2\right)\,\mathrm{d}t\right). \end{aligned} \tag{4}$$

holds for every smooth, rapidly decaying (in x) function $u(x,t)$, and every value of $\gamma,\ T > 0$, with a fixed constant C that is independent of γ, T and u. [2]

Recall that $(\gamma_0 u)(y,t)$ is the trace of $u(y,x_d,t)$ on the boundary $\{x_d = 0\}$. Thus $(\gamma_0 u)(y,t) = u(y,0,t)$ for u continuous up to the boundary.

1.2. **A necessary condition for L^2 well-posedness.** Set $v(\eta,x_d,t) = \mathscr{F}_y(u(y,x_d,t))$. Then by Parseval's Formula, (4) is equivalent to

$$\begin{aligned} & \mathrm{e}^{-2\gamma T}\int_{\mathbb{R}^{d-1}}\int_0^\infty |v(\eta,x_d,T)|^2\,\mathrm{d}x_d\mathrm{d}\eta \\ & +\gamma\int_0^T \mathrm{e}^{-2\gamma t}\int_{\mathbb{R}^{d-1}}\int_0^\infty |v(\eta,x_d,t)|^2\,\mathrm{d}x_d\,\mathrm{d}\eta\,\mathrm{d}t + \int_0^T \mathrm{e}^{-2\gamma t}\int_{\mathbb{R}^{d-1}} |v(\eta,0,t)|^2\,\mathrm{d}\eta\mathrm{d}t \\ \le\ & C\int_{\mathbb{R}^{d-1}}\int_0^\infty |v_0(\eta,x_d)|^2\,\mathrm{d}x_d\mathrm{d}\eta \\ & +C\int_0^T \mathrm{e}^{-2\gamma t}\left(\frac{1}{\gamma}\int_{\mathbb{R}^{d-1}}\int_0^\infty |\hat{L}v(\eta,x_d,t)|^2\,\mathrm{d}x_d\mathrm{d}\eta + \int_{\mathbb{R}^{d-1}} |Bv(\eta,0,t)|^2\,\mathrm{d}\eta\right)\,\mathrm{d}t. \end{aligned} \tag{5}$$

Here $\hat{L} = \partial_t + \mathrm{i}A(\eta) + A^d\partial_d$. Now for any nonzero $\phi(\eta) \in \mathscr{D}(\mathbb{R}^{d-1})$, we set

$$w(\eta,x_d,t) = \mathrm{e}^{t\tau}\phi(\eta)\exp(x_d\mathcal{A}(\tau,\eta))V(\tau,\eta),$$

with $\mathrm{Re}\tau > 0$, and $V = V(\tau,\eta) \in E_-(\tau,\eta)$ depending smoothly on (τ,η). [3] We also may suppose $|V(\tau,\eta)| = 1$. Then one easily checks that w solves $\hat{L}w = 0$. Note w decays exponentially as $x_d \to \infty$. So we may apply (5) to w, dropping the first two positive terms

[2] This is very important!

[3] For example, we may take $V(\tau,\eta) = \pi_-(\tau,\eta)V_0$, with V_0 an arbitrary vector in $\mathbb{C}^n$ and $\pi_-(\tau,\eta)$ the eigenprojection onto $E_-(\tau,\eta)$, along $E_+(\tau,\eta)$.

in the left-hand side, [4] then obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} |\phi(\eta)V(\tau,\eta)|^2 \,\mathrm{d}\eta \int_0^T \mathrm{e}^{2t(\mathrm{Re}\tau-\gamma)}\,\mathrm{d}t \\
\leq\ & C\int_{\mathbb{R}^{d-1}}\int_0^\infty |\phi(\eta)|^2|\exp(x_d\mathcal{A}(\tau,\eta))V|^2\,\mathrm{d}x_d\mathrm{d}\eta \\
&+C\int_0^T \mathrm{e}^{2t(\mathrm{Re}\tau-\gamma)}\,\mathrm{d}t \int_{\mathbb{R}^{d-1}} |\phi(\eta)BV(\tau,\eta)|^2\,\mathrm{d}\eta.
\end{aligned}$$

Note for τ fixed, η in a compact set, there is a negative upper bound of real part of eigenvalues $\lambda(\tau,\eta)$ of $\mathcal{A}(\tau,\eta)$ with negative real part. So by $V\in E_-(\tau,\eta)$ and $|V(\tau,\eta)|=1$, we have, for the first term in the right-hand side,

$$\int_{\mathbb{R}^{d-1}}\int_0^\infty |\phi(\eta)|^2|\exp(x_d\mathcal{A}(\tau,\eta))V|^2\,\mathrm{d}x_d\,\mathrm{d}\eta \leq C'\,\|\phi\|_{L^2}^2.$$

Note the constant C' depends only on τ and $\operatorname{supp}\phi$.

We now choose γ so that $\gamma<\mathrm{Re}\tau,$ and set $E(T)\doteq\int_0^T \mathrm{e}^{2t(\mathrm{Re}\tau-\gamma)}\,\mathrm{d}t$. Then we get

$$\int_{\mathbb{R}^{d-1}} |\phi(\eta)|^2|V(\tau,\eta)|^2\,\mathrm{d}\eta \leq CC'\frac{\|\phi\|_{L^2}^2}{E(T)}+C\int_{\mathbb{R}^{d-1}}|\phi(\eta)|^2|BV(\tau,\eta)|^2\,\mathrm{d}\eta.$$

Let $T\to\infty$, and note $E(T)\to\infty$, there comes

$$\int_{\mathbb{R}^{d-1}} |\phi(\eta)|^2(|V(\tau,\eta)|^2-C|BV(\tau,\eta)|^2)\,\mathrm{d}\eta\leq 0,\qquad \forall\phi(\eta)\in\mathscr{D}(\mathbb{R}^{d-1}).$$

This implies (by homogeneity, we drop the assumption that $|V(\tau,\eta)|=1$)

$$\boxed{|V|^2\leq C|BV|^2,\quad \forall\mathrm{Re}\tau>0,\ \ \eta\in\mathbb{R}^{d-1},\ V\in E_-(\tau,\eta).} \tag{6}$$

Note C here, is the same one as in (4), does not depend on τ,η.

Definition 1.2. Let L be hyperbolic, A^d be invertible. Given $B\in\mathbb{M}_{p\times n}(\mathbb{R})$, we say the IBVP (1)–(3) satisfies the *uniform Kreiss-Lopatinskii condition* (UKL) in the domain $x_d>0,\ t>0$, if

- p equals the number of positive eigenvalues of A^d;
- there is a positive number $C>0$ independent of $\mathrm{Re}\tau>0$ and $\eta\in\mathbb{R}^{d-1}$ so that (6) holds.

[4]The first two terms already appear in the study of Cauchy problems. To study boundary conditions, one naturally focuses on the third term.

2. Uniform Kreiss-Lopatinskii Condition: Characteristic case

2.1. Estimate and L^2 strong-well-posedness. For the characteristic case $\det A^d = 0$, recall we always assume $\ker A^d \subset \ker B$ (see the definition of normal IBVP). This implies it is no longer reasonable to have a control of the boundary value of u in (4). [5] Instead, we can only control the normal trace $\gamma_0 A^d u$:

Definition 2.1. Consider a (possibly characteristic) hyperbolic IBVP (1)–(3) in the domain $\{x_d > 0,\ t > 0\}$. We say that this IBVP is *strongly L^2 well-posed* if $\ker A^d \subset \ker B$ and if, more over, the quantity

$$\mathrm{e}^{-2\gamma T}\left\|u(T)\right\|_{L^2}^2 + \int_0^T \mathrm{e}^{-2\gamma t}\,\mathrm{d}t\left(\int_{\partial\Omega} |(\gamma_0 A^d u)(y,t)|^2\,\mathrm{d}y + \gamma\int_\Omega |u(x,t)|^2\,\mathrm{d}x\right)$$

is bounded from above by

$$C\left(\|u_0\|_{L^2}^2 + \int_0^T \mathrm{e}^{-2\gamma t}\left(\frac{1}{\gamma}\left\|(Lu)(t)\right\|_{L^2}^2 + \|\gamma_0 Bu(t)\|_{L^2(\partial\Omega)^2}\right)\,\mathrm{d}t\right),$$

for every smooth, rapidly decaying (in x) function $u(x,t)$, and every positive γ, T, for a fixed constant $C > 0$ independent of γ, T and u.

Remark 2.1. Let's look at a simple example: The system

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} = 0$$

in $x > 0, t > 0$, with boundary condition $v = 0$ on $x = 0$. So $B = (0,1)$, $A^d = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and as $\det A^d = 0$, the boundary $\{x = 0\}$ is characteristic. Note that $\ker A^d = \ker B = \{(u,0)^\top : u \in \mathbb{R}\}$.

The boundary value of the solution $u|_{x=0}$ depends only on the initial value u_0 at the boundary. So since $\|u_0\|_{L^2(0,\infty)}$ bears no any information on its trace on the boundary, we cannot count $\|u_0\|_{L^2}$ to control it. Also, as $\ker A^d = \ker B$, it is hopeless to bound $(u,0)^\top \in \ker A^d$ by any norm of $B(u,0)^\top = 0$. This explains why it is only reasonable to control $\gamma_0 v = \gamma_0 A^d (u,v)^\top$, rather than $\gamma_0 (u,v)^\top$ in the L^2 well-posedness.

We note, by the above example, one may control u in $\ker A^d$ by employ higher order norms of the data u_0, since, for example, $\|u_0\|_{H^1}$ sure takes information on $\gamma_0 u_0$. However, such estimate involves loss of derivatives. [6]

[5] For $0 \neq u \in \ker A^d$, we always have $Bu = 0$, so one cannot use Bu to control such u.

[6] It is interesting to derive such estimates!

2.2. **Necessary condition of L^2 strong-well-posedness.** Following the procedure before for non-characteristic case, we may derive a necessary condition for L^2 well-posedness (UKL) in the form

$$\boxed{\exists C > 0, \forall \eta \in \mathbb{R}^{d-1}, \forall \mathrm{Re}\tau > 0, \forall V \in E_-(\tau, \eta), \quad \text{there holds} \quad |A^d V| \leq C|BV|.} \tag{7}$$

Remark 2.2. Note (7) implies $\ker B \cap E_-(\tau, \eta) \subset \ker A^d$. We have proved before that $E_-(\tau, \eta) \cap \ker A^d = \{0\}$,[7] so we get the Kreiss-Lopatinskii condition (KL) $\ker B \cap E_-(\tau, \eta) = \{0\}$ as well. Comparing to (KL), (7) is uniform modulo $\ker A^d$. We note also (7) reduced to (6) if $\det A^d \neq 0$.

3. An equivalent formulation of (UKL) for case of constantly hyperbolic operators and non-characteristic boundary

3.1. **Continuous extension of stable subspaces to the frequency boundary.** The disadvantage of (UKL) of the form (7) is that it is hard to calculate $C(\tau, \eta)$, namely the upper bound of $|A^d V|/|BV|$ for $\mathrm{Re}\tau > 0, \eta \in \mathbb{R}^{d-1}$ and $V \in E_-(\tau, \eta)$. It turns out that there is a much more explicit way to check (UKL) condition, in the case of IBVPs for constantly hyperbolic operators and non-characteristic boundaries. The idea is to reformulate the analytical expression (7) to a geometrical condition, namely (8) in page 42, which is then transferred to an algebraic problem, i.e., non-vanishing of the Lopatinskii determinant on a connected compact set $\{(\tau, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1} : \mathrm{Re}\tau \geq 0, \ |\tau|^2 + |\eta|^2 = 1\}$.

Definition 3.1. An operator $L = \partial_t + \sum_{\alpha=1}^d A^\alpha \partial_\alpha$ is said to be constantly hyperbolic if the matrices $A(\xi) = \sum_{\alpha=1}^d A^\alpha \xi_\alpha$ are diagonalizable with real eigenvalues and, moreover, as ξ ranges along $\mathbf{S}^{d-1}$ (the unit sphere in $\mathbb{R}^d$), the multiplicities of eigenvalues remain constant. In the special case where all eigenvalues are real and simple for every $\xi \in \mathbf{S}^{d-1}$, we say the operator is strictly hyperbolic.

We also recall the Grassmannian manifold $\mathbf{G}(n, p)$, the set of p-dimensional subspaces of $\mathbb{C}^n$, is a compact and connected differentiable manifold. (See [2] or [3] which are available on internet for elementary introductions.) The topology of Grassmannian $\mathbf{G}(n, p)$ is defined as follows. Let $\mathbb{M}_{n\times p}(\mathbb{C})^\circ$ be the set of $n \times p$ matrices with rank p, which is identified as an open subset of $\mathbb{C}^{np}$. (We may consider the p column vectors of a matrix in $\mathbb{M}_{n\times p}(\mathbb{C})^\circ$ as a basis of a p-dimensional subspace of $\mathbb{C}^n$.) Two matrices $A, B \in \mathbb{M}_{n\times p}(\mathbb{C})^\circ$ are called equivalent, if there is an invertible $p \times p$ matrix $P \in \mathbf{GL}_p(\mathbb{C})$ so that $A = BP$, and is denoted as $A \sim B$. Then $\mathbf{G}(n, p)$ is the quotient space $\mathbb{M}_{n\times p}(\mathbb{C})^\circ / \sim$, with the

[7] Failure of KL cannot come from the characteristic nature of the boundary, see the previous lecture.

quotient topology. We note that in the following, continuity should be understood with respect to such a canonical topology of $\mathbf{G}(n,p)$.

Lemma 3.1. *Assume that the operator is constantly hyperbolic, and the boundary is non-characteristic. Then the map* $(\tau,\eta)\mapsto E_-(\tau,\eta)$*, already defined for* $\mathrm{Re}\tau>0$ *and* $\eta\in\mathbb{R}^{d-1}$*, valued in* $\mathbf{G}(n,p)$*, admits a unique limit at every boundary point* $(\mathrm{i}\rho,\eta)$ *(with* $\rho\in\mathbb{R},\eta\in\mathbb{R}^{d-1}$*), with the exception of the origin (*$\rho=0,\eta=0$*).*

Proof of this lemma is quite technical, see [1, p.139], and we omit it here.

We use $E_-(\mathrm{i}\rho,\eta)$ to denote the limit. We infer that $E_-(\mathrm{i}\rho,\eta)$ contains the stable subspace of $A^dU'+\mathrm{i}(\rho I_n+A(\eta))U=0$, but might not be the same. In fact, by completeness of the manifold $\mathbf{G}(n,p)$, $E_-(\mathrm{i}\rho,\eta)\in\mathbf{G}(n,p)$, so $\dim E_-(\mathrm{i}\rho,\eta)=p$, while since many eigenvalues with positive/negative real parts of $\mathcal{A}(\tau,\eta)$ may become purely imaginary as $\mathrm{Re}\tau\to0$, the number of eigenvalues of $-\mathrm{i}(A^d)^{-1}(\rho I_n+A(\eta))$ with negative real parts should be less or equal p.

3.2. **An equivalent form of UKL.** We have the direct sum $\mathbb{C}^n=E_-(\tau,\eta)\bigoplus^{\perp}(E_-(\tau,\eta))^{\perp}$ and $\mathbb{C}^n=\ker B\bigoplus E_-(\tau,\eta)$, and the latter is the Kreiss-Lopatinskii condition. Let $P(\tau,\eta)$ be the orthogonal projection $\mathbb{C}^n\to E_-(\tau,\eta)$, and $\pi(\tau,\eta)$ the projection of $\mathbb{C}^n$ onto $E_-(\tau,\eta)$ along $\ker B$. [8] We have the technical result.

Lemma 3.2. $P(\tau,\eta)$ *is analytical (continuous) with respect to* $\eta\in\mathbb{R}^{d-1}$*, holomorphic (continuous) with respect to* $\tau\in\mathbb{C}$*, if and only if* $\pi(\tau,\eta)$ *is. Also,* $P(\tau,\eta)$ *and* $\pi(\tau,\eta)$ *are homogeneous of degree zero.*

Proof. **1.** Note that $E_-(k\tau,k\eta)=E_-(\tau,\eta),\forall k>0$, hence $P(k\tau,k\eta)=P(\tau,\eta)$ and $\pi(k\tau,k\eta)=\pi(\tau,\eta)$.

In the following, we set $q\doteq n-p$. Recall that p is the number of positive eigenvalues of A^d, and $\dim\ker B=q$, $\dim E_-(\tau,\eta)=p$.

2. Let $\alpha_1,\cdots,\alpha_q$ be a basis of $\ker B$, and $\alpha_{q+1},\cdots,\alpha_n$ a basis of $(\ker B)^{\perp}$. Here $(\ker B)^{\perp}$ is the orthogonal complement of $\ker B$ in $\mathbb{C}^n$, which is uniquely determined by $\ker B$.

---

[8] Let V be a p-dimensional subspace of $\mathbb{C}^n$, we calculate here the matrix representing the orthogonal projection $P:\mathbb{C}^n\to V$.

Let $A\subset\mathbb{M}_{n\times p}(\mathbb{C})^{\circ}$ (thus $\mathrm{rank}\,A=p$) whose column vectors consist a basis of V. Then $V=\mathrm{R}(A)$. For any $b\in\mathbb{C}^n$, the least-square solution $\tilde{x}$ of the over-determined system $Ax=b$ gives the orthogonal projection of b onto $\mathrm{R}(A)$, namely $A\tilde{x}$. Recall that from linear algebra, we have $\tilde{x}=(A^{\top}A)^{-1}A^{\top}b$. Therefore $Pb=A\tilde{x}=A(A^{\top}A)^{-1}A^{\top}b$. Thus we see $\boxed{P=A(A^{\top}A)^{-1}A^{\top}}$. It is easy to check that $P^2=P$.

3. By definition, we have $\{P(\tau,\eta)\alpha_j\}_{j=1}^n \subset E_-(\tau,\eta)$, and

$$\operatorname{rank}\left(P(\tau,\eta)\alpha_1,\cdots,P(\tau,\eta)\alpha_n\right)_{n\times n} = p.$$

For fixed (τ,η), suppose $P(\tau,\eta)\alpha_{j_k}$ $(k=1,\cdots,p)$ are linearly independent and thus consist a basis of $E_-(\tau,\eta)$. Then in a neighborhood of (τ,η), this property also holds true by continuity, and as $\ker B \bigoplus E_-(\tau,\eta) = \mathbb{C}^n$, $P(\tau,\eta)\alpha_{j_1},\cdots,P(\tau,\eta)\alpha_{j_p},\alpha_1,\cdots,\alpha_q$ spans $\mathbb{C}^n$. For any $m \in \{q+1,\cdots,n\}$, we have

$$\alpha_m = \sum_{k=1}^q a_k\alpha_k + \sum_{k=1}^p b_k P(\tau,\eta)\alpha_{j_k},$$

or, with $A = (\alpha_1,\cdots,\alpha_q)$, $B = (P(\tau,\eta)\alpha_{j_1},\cdots,P(\tau,\eta)\alpha_{j_p})$, we have

$$(A,B)(a_1,\cdots,a_q,b_1,\cdots,b_p)^\top = \alpha_m.$$

Note that (A,B) is an $n\times n$ invertible matrix and $\alpha_m \in \mathbb{C}^n$ is a fixed vector, by Crammer's Rule, the solution $(a_1,\cdots,a_q,b_1,\cdots,b_p)^\top$ also depends well on (τ,η).

Now observe that

$$\pi(\tau,\eta)\alpha_k = B(b_1,\cdots,b_p)^\top, \quad k = q+1,\cdots,n, \text{ and } \pi(\tau,\eta)\alpha_j = 0 \quad \text{for } j = 1,\cdots,q,$$

we infer that, written as a matrix, $\pi(\tau,\eta)$ also depends well on (τ,η).

4. Conversely, suppose that $\pi(\tau,\eta)$ is analytical with respect to $\eta \in \mathbb{R}^{d-1}$, holomorphic with respect to $\tau \in \mathbb{C}$, we prove the same property for $P(\tau,\eta)$. Because $\pi(\tau,\eta)\alpha_k = 0$ for $k = 1,\cdots,q$, we see $\pi(\tau,\eta)\alpha_{q+1},\cdots,\pi(\tau,\eta)\alpha_n$ consist a basis of $E_-(\tau,\eta)$. By Gram–Schmidt Orthogonalization, we may obtain an orthonormal basis $\beta_{q+1}(\tau,\eta),\cdots,\beta_n(\tau,\eta)$ of $E_-(\tau,\eta)$, and these vectors are also analytical with respect to $\eta \in \mathbb{R}^{d-1}$, holomorphic with respect to $\tau \in \mathbb{C}$.

Then, we may further applying Gram–Schmidt Orthogonalization to $\alpha_1,\cdots,\alpha_q$ to obtain an orthonormal base $\beta_1(\tau,\eta),\cdots,\beta_q(\tau,\eta)$ of $(E_-(\tau,\eta))^\perp$, and these vectors are also analytical with respect to $\eta \in \mathbb{R}^{d-1}$, holomorphic with respect to $\tau \in \mathbb{C}$.

Then for any α_k $(k=1,\cdots,n)$, we solve $l_k(\tau,\eta) \in \mathbb{C}^n$ from $(\beta_1(\tau,\eta),\cdots,\beta_n(\tau,\eta))l_k = \alpha_k$ and by Crammer's Rule, $l_k(\tau,\eta)$ is analytical with respect to $\eta \in \mathbb{R}^{d-1}$, holomorphic with respect to $\tau \in \mathbb{C}$. Hence $P(\tau,\eta)\alpha_k = \sum_{j=q+1}^n l_k^j(\tau,\eta)\beta_j(\tau,\eta)$ This finishes the proof. $\square$

Now assume the IBVP defined by the pair (L,B) satisfies (UKL). Then (6) can be rewritten as

$$|P(\tau,\eta)V| \le C|BP(\tau,\eta)V|, \qquad \forall \mathrm{Re}\tau > 0, \forall \eta \in \mathbb{R}^{d-1},\ \forall V \in \mathbb{C}^n.$$

Bearing in mind that $E_-(\tau,\eta) \to E_-(\mathrm{i}\rho,\eta)$ implies that $P(\tau,\eta) \to P(\mathrm{i}\rho,\eta)$ in the operator norm, so by continuity (6) still holds for $\mathrm{Re}\tau = 0$, which means that

$$E_-(\tau,\rho) \cap \ker B = \{0\}, \quad \forall\ \mathrm{Re}\tau \geq 0,\ \eta \in \mathbb{R}^{d-1}. \tag{8}$$

Conversely, assume that $E_-(\tau,\rho) \cap \ker B = \{0\}$ for every $\mathrm{Re}\tau \geq 0,\ \eta \in \mathbb{R}^{d-1}$, then there holds $E_-(\tau,\eta) \bigoplus \ker B = \mathbb{C}^n$ for such (τ,η). By Isomorphism Theorem of Algebra, both $B : (\ker B)^\perp \to \mathrm{R}(B) = \mathbb{C}^p$ and $B : E_-(\tau,\eta) \to \mathbb{C}^p$ are invertible. Observing that $\left\|(B|_{(\ker B)^\perp})^{-1}\right\|$ is finite and independent of (τ,η). For the mapping $T(\tau,\eta) = (B|_{E_-(\tau,\eta)})^{-1} :\ \mathrm{R}(B) \to E_-(\tau,\eta)$, there holds

$$\boxed{T(\tau,\eta) = \pi(\tau,\eta)(B|_{(\ker B)^\perp})^{-1}.}$$

This is true because, as $\ker\pi = \ker B$, $\pi : (\ker B)^\perp \to E_-(\tau,\eta)$ is an isomorphism.

$$\begin{tikzcd}
\mathrm{R}(B) \arrow[r, "T"] \arrow[d, "(B|_{(\ker B)^\perp})^{-1}"'] & E_-(\tau,\eta) \\
(\ker B)^\perp \arrow[ur, "\pi"'] &
\end{tikzcd}$$

Then for each (τ,η) with $\mathrm{Re}\tau \geq 0, \eta \in \mathbb{R}^{d-1}$, and $|\tau|^2 + |\eta|^2 \neq 0$, the number

$$\begin{aligned}
c(\tau,\eta) &\doteq \sup\left\{\frac{|V|}{|BV|} :\ V \in E_-(\tau,\eta), V \neq 0\right\} \\
&= \sup\left\{\frac{|TW|}{|W|} :\ W \in \mathrm{R}(B), W \neq 0\right\} = \|T(\tau,\eta)\|
\end{aligned}$$

is not only finite, but the function $(\tau,\eta) \mapsto c(\tau,\eta)$ is also continuous and homogeneous of degree zero, that is, $c(\lambda\tau,\lambda\eta) = c(\tau,\eta)$ for all $\lambda > 0$. Since the hemisphere defined by $\mathrm{Re}\tau \geq 0, \eta \in \mathbb{R}^{d-1}$ and $|\tau|^2 + |\eta|^2 = 1$ is compact, we infer that there is an upper bound. Hence, the IBVP satisfies (UKL). We then have

Corollary 3.1. *Let L be constantly hyperbolic and the boundary be non-characteristic. Then IBVP* (1)–(3) *satisfies (UKL) if and only if $E_-(\tau,\eta) \cap \ker B = \{0\}$ for every nonzero pair (τ,η) with $\mathrm{Re}\tau \geq 0$ and $\eta \in \mathbb{R}^{d-1}$.*

Remark 3.1. This Corollary provides a practical way to check (UKL). The main difficulty during calculation would be the computation of $E_-(\tau,\eta)$ when $\mathrm{Re}\tau = 0$, since it is there the uniformity may fail.

4. Lopatinskii Determinant

4.1. The Lopatinskii determinant is a (somewhat) practical method to verify the Kreiss-Lopatinskii condition. It is a function $(\tau, \eta) \mapsto \Delta(\tau, \eta)$ with the following properties:

a) It is well-defined for $\mathrm{Re}\tau > 0,\ \eta \in \mathbb{R}^{d-1}$;
b) It is jointly analytical to (τ, η), hence holomorphic to τ;
c) It vanishes exactly at the point where Kreiss-Lopatinskii condition fails.

4.2. To fulfill these properties, we construct a basis

$$\beta(\tau, \eta) = \{X_1(\tau, \eta), \cdots, X_p(\tau, \eta)\}$$

of $E_-(\tau, \eta)$, which satisfies a) and b). Then we define the Lopatinskii determinant as

$$\boxed{\Delta(\tau, \eta) \doteq \det(BX_1(\tau, \eta), \cdots, BX_p(\tau, \eta)).} \tag{9}$$

Then it satisfies a) and b).

Also, if $\Delta(\tau_0, \eta_0) = 0$, then there is a nontrivial linear combination

$$0 \neq X(\tau_0, \eta_0) = \sum_{k=1}^{p} C_k X_k(\tau_0, \eta_0)$$

so that $BX(\tau_0, \eta_0) = \mathbf{0}$. This shows that $X(\tau_0, \eta_0) \in \ker B \cap E_-(\tau_0, \eta_0)$ and hence (KL) fails at (τ_0, η_0).

On the other hand, if (KL) fails at a point (τ_0, η_0), then there is a nonzero $X(\tau_0, \eta_0) \in E_-(\tau, \eta) \cap \ker B$, so X maybe expressed as a linear combination as before and $BX(\tau_0, \eta_0) = 0$, hence $\Delta(\tau_0, \eta_0) = 0$.

So the key point of writing down Lopatinskii determinant is to construct the basis $\beta(\tau, \eta)$. In the following, we present three ways: i) a general theory provided by Kato; ii) a special result valid for the Friedrichs symmetric system; iii) Examples for which the construction is totally explicit and straightforward.

4.3. Kato's method: single variable case [4, p.100]**.** Let $z \mapsto P(z)$ be a holomorphic operator-valued function with $P(z)$ being projections, i.e., $P^2(z) = P(z)$, defined on a simply connected domain D in the complex plane. We already know that $\dim \mathrm{R}(P(z))$ is a constant.

Taking derivatives with respect to z, we get from $P(z)^2 = P(z)$ that

$$PP' + P'P = P'.$$

Multiplying P to this identity (from left or right), we get

$$PP'P = 0.$$

Now define

$$Q = [P', P] = P'P - PP',$$

which is also analytical to z, we easily check that

$$P' = [Q, P].$$

Considering the Cauchy problem of linear ODE

$$M' = QM, \qquad M(z_0) = I_n,$$

we claim:

a) which has a unique solution $M(z)$ holomorphic for $z \in D$;

b) the solution $M(z)$, as matrix or operator, is invertible;

c) it holds the formula

$$M(z)^{-1}P(z)M(z) = P(z_0).$$

Note here the independent variable is $z \in \mathbb{C}$ (rather than $t \in \mathbb{R}$), so we cannot apply directly the well-known results on ODE. The above claim can be proved by using successive approximation: $M_0(z) = M(z_0)$, $M_n(z) = M(z_0) + \int_{z_0}^{z} Q(z)M_{n-1}(z)\,\mathrm{d}z$. By Cauchy Formula of holomorphic functions, $M_n(z)$ does not depend on curve of integration and hence is well-defined, also holomorphic in D (here we need D to be simply connected). Then we may prove this approximate sequence converges uniformly in each compact subset of D, and the claim a) then follows.

To show b), we consider another Cauchy problem

$$N' = -NQ, \qquad N(z_0) = I_n.$$

As shown above, this problem also has uniquely one solution $N(z)$ holomorphic in D. Then

$$(N(z)M(z))' = N'M + NM' = -NQM + NQM = \mathbf{0},$$

and $N(z_0)M(z_0) = I_n$. So we get $N(z)M(z) = I_n$ for $z \in D$. Since M, N here are matrices, this is enough to conclude that $N = M^{-1}$. b) is proved.

To show c), using the identity $M^{-1}M = I_n$, we get

$$(M^{-1})' = -M^{-1}M'M^{-1},$$

and hence

$$\begin{aligned}(M(z)^{-1}P(z)M(z))' &= -M^{-1}M'M^{-1}PM + M^{-1}P'M + M^{-1}PM' \\ &= M^{-1}(P' - [Q, P])M = \mathbf{0}.\end{aligned}$$

This shows $M(z)^{-1}P(z)M(z) = M(z_0)^{-1}P(z_0)M_{z_0} = P(z_0)$.

Now given a basis β_0 of the range of $P(z_0)$, then

$$\beta(z) = M(z)\beta_0$$

would be a basis of $\mathrm{R}(P(z))$, and obviously it is holomorphic in D.

Indeed, since $M(z)$ is invertible, so $\mathrm{rank}\,(\beta) = p = \mathrm{rank}\,(P(z_0))$. Since $\dim \mathrm{R}(P(z)) = \dim \mathrm{R}(P(z_0)) = p$, we only need show each vector of β belong to $\mathrm{R}(P(z))$. This follows from fact c). Indeed, it follows that $P(z)\beta = P(z)M(z)\beta_0 = M(z)P(z_0)\beta_0 = M(z)\beta_0 = \beta$, so $\beta \in \mathrm{R}(P(z))$.

4.4. For P depends on several variables, the above Kato's procedure cannot be done simultaneously in general. If $Q_j \doteq [\partial P/\partial z_j, P]$, simultaneity requires the compatibility condition

$$\frac{\partial Q_j}{\partial z_k} - \frac{\partial Q_k}{\partial z_j} = -[Q_j, Q_k].$$

Example 4.1. As an example on the compatibility condition, suppose P depends analytically on z_1, z_2, and we find $M(z_1, z_2)$, which is solved by

$$\frac{\partial M}{\partial z_1} = Q_1 M, \qquad \frac{\partial M}{\partial z_2} = Q_2 M, \qquad M(z_1^0, z_2^0) = I_n.$$

Then it is necessary that $\partial Q_1/\partial z_2 M + Q_1 \partial M/\partial z_2 = \partial Q_2/\partial z_1 M + Q_2 \partial M/\partial z_1$, or $(\partial Q_1/\partial z_2 - \partial Q_2/\partial z_1)M = (Q_2Q_1 - Q_1Q_2)M$. Since M should be invertible, we need the compatibility condition $\partial Q_1/\partial z_2 - \partial Q_2/\partial z_1 = -[Q_1, Q_2]$.

However direct computation shows we have $\frac{\partial Q_j}{\partial z_k} - \frac{\partial Q_k}{\partial z_j} = 2[P_j, P_k]$, and (for simplicity, we write $P_j = \frac{\partial P}{\partial z_j}$ etc. here and below), using $P_kP + PP_k = P_k$,

$$\begin{aligned} -[Q_j, Q_k] &= PP_jP_kP - PP_kP_jP + P_jPP_k - P_kPP_j \\ &= [P_j, P_k], \end{aligned}$$

so there holds in practice oddly

$$\frac{\partial Q_j}{\partial z_k} - \frac{\partial Q_k}{\partial z_j} = -2[Q_j, Q_k].$$

Therefore, to construct $\beta(\tau, \eta)$, or, equivalently, $M(\tau, \eta)$, we may only apply Kato's procedure successively to each of the arguments, provided at each step, the Cauchy problem is posed in a simply connected domain. For example, we first solve

$$\frac{\partial \hat{M}(z_1, z_2^0)}{\partial z_1} = Q_1(z_1, z_2^0)\hat{M}(z_1, z_2^0), \qquad \hat{M}(z_1^0, z_2^0) = I_n.$$

Then, we solve

$$\frac{\partial M(z_1, z_2)}{\partial z_2} = Q_1(z_1, z_2)M(z_1, z_2), \qquad M(z_1, z_2 = z_2^0) = \hat{M}(z_1, z_2^0).$$

Because for ODE with analytical coefficients, if the initial data is analytical to parameters, the resulting solution will jointly analytical in the variable for differentiation and the parameters (Cauchy–Kowalevski Theorem), so the resulting matrix M is jointly analytical in its arguments. The inelegant fact is that the result depends on the order in which we solve the ODEs, because the lack of compatibility.

4.5. Applying these ideas to the eigen-projectors $\pi_-(\tau,\eta)$, which are jointly analytical for $\mathrm{Re}\tau>0$ and $\eta\in\mathbb{R}^{d-1}$, we have

Lemma 4.1. *For $\eta\in\mathbb{R}^{d-1}$ and $\mathrm{Re}\tau>0$, the space $E_-(\tau,\eta)$ admits a basis $\beta(\tau,\eta)$, which is jointly analytic in (τ,η) and thus holomorphic in τ.*

4.6. The symmetric case. We restrict ourselves to the case of Friedrichs symmetric operators and non-characteristic boundary. When L is symmetric, that is, $A(\xi)=A(\xi)^\top$ for every $\xi\in\mathbb{R}^d$, and $\det(A^d)\neq 0$, an alternative construction can be done, with the help of the following. [9]

Lemma 4.2. *In the symmetric case with a non-characteristic boundary, one has for every $\eta\in\mathbb{R}^{d-1}$ and $\mathrm{Re}\tau>0$,*

$$E^u(A^d)\cap E_+(\tau,\eta)=\{0\}, \tag{10}$$

where $E^u(A^d)$ stands for the unstable invariant subspace of A^d.

Consequently, there holds

$$E^u(A^d)\bigoplus E_+(\tau,\eta)=\mathbb{C}^n.$$

Remark 4.1. Question: Whether (10) holds under the weaker assumption of hyperbolicity, instead of symmetry?

Proof. **1.** Let $u_0\in E_+(\tau,\eta)$. Then the unique solution

$$A^d u'+(\tau I_n+\mathrm{i}A(\eta))u=0,\qquad u(0)=u_0$$

decays exponentially fast as $x_d\to-\infty$. Multiplying the equation by u^* and integrating on $(-\infty,0)$, we obtain

$$(A^d u_0,u_0)_{\mathbb{C}^n}=-2\mathrm{Re}\tau\int_{-\infty}^0|u|^2\,\mathrm{d}x_d\leq 0.$$

2. If, moreover, $u_0\in E^u(A^d)$, the unique solution of

$$v'=A^d v,\qquad v(0)=u_0$$

[9]Note this method is different from Kato's. The symmetry of $A(\xi)$ is used to obtain energy estimates of ODE in the proof.

decays exponentially fast at $-\infty$. Multiplying the equation by $v^* A^d$ an integrating, we obtain

$$(A^d u_0, u_0)_{\mathbb{C}^n} = \int_{-\infty}^0 |A^d v|^2 \,\mathrm{d}x_d \geq 0.$$

Therefore we conclude $(A^d u_0, u_0) = 0$. This implies $u \equiv 0$, so $u_0 = u(0) = 0$.

3. The last conclusion holds for $\dim E^u(A^d)$ equals the number of positive eigenvalues of A^d, while $\dim E_+(\tau,\eta)$ equals the number of negative eigenvalues of A^d, and since A^d non-singular, their sum is n. □

Thanks to the Lemma, together with $E_-(\tau,\eta) \bigoplus E_+(\tau,\eta) = \mathbb{C}^n$, as $\ker \pi_-(\tau,\eta) = E_+(\tau,\eta)$, [10] the map $\pi_-(\tau,\eta) : E^u(A^d) \to E_-(\tau,\eta)$ is bijective. Now, giving a basis $\mathbb{b}_0$ of $E^u(A^d)$, we obtain a basis $\mathbb{b}(\tau,\eta) = \pi_-(\tau,\eta)\mathbb{b}_0$ of $E_-(\tau,\eta)$, which is obviously jointly analytic.

5. Examples

In the following specific problems, we could construct the Lopatinskii determinant by straightforward computation of eigenvectors, without appealing to the general theory. A trick here is, rather than calculating the roots of Lopatinskii determinant directly, one usually firstly derive some polynomials from the Lopatinskii determinant, and check if the roots of the polynomials are roots of the Lopatinskii determinant, since the latter is generally irrational and involving multi-valued complex functions.

5.1. Example 1. Consider the system $\partial_t u + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x u + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_y u = 0$. Hence $d = n = 2$, and $A(\xi) = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & -\xi_1 \end{pmatrix}$. The spectrum of $A(\xi)$ consists in $\pm|\xi|$. This is a symmetric, as well as constantly (strictly) hyperbolic system. We note each component u_1, u_2 of u satisfies the wave equation $\partial_t^2 w - \Delta_x w = 0$ if it is C^2.

Since $\det A^2 = -1$, the boundary $x_2 = 0$ is non-characteristic. Also, the spectrum of A^2 is ± 1, so $p = 1$ and the boundary condition is scalar: $Bu = b_1 u_1 + b_2 u_2$, $B = (b_1, b_2) \neq 0$.

We also compute $\mathcal{A}(\tau,\eta) = \begin{pmatrix} 0 & -\tau + \mathrm{i}\eta \\ -\tau - \mathrm{i}\eta & 0 \end{pmatrix}$. Its eigenvalue μ satisfies

$$\mu^2 = \tau^2 + \eta^2.$$

A typical eigenvector associated with μ is $R(\tau,\eta) = (\mathrm{i}\eta - \tau, \mu)^\top$. The only exception is the point given by $\tau = \mathrm{i}\eta$ (some frequency boundary points), where $\mu = 0$. Near such a point, a convenient choice would be $R'(\tau,\mu) = (-\mu, \tau + \mathrm{i}\eta)^\top$.

[10] Clearly $\pi_-(\tau,\eta)$ here is projection onto $E_-(\tau,\eta)$, along $E_+(\tau,\eta)$.

The Lopatinskii determinant is $\Delta(\tau,\eta)=b_1(\mathrm{i}\eta-\tau)+b_2\mu$, valid for $\mathrm{Re}\tau>0$. **Always bearing in mind that we should choose the eigenvalue μ corresponding to negative real part here.**

Proposition 5.1. *KL holds if and only if $b_1+b_2\neq 0$.*

Proof. Eliminating μ between $\Delta=0$ and $\tau^2+\eta^2=\mu^2$, we have

$$b_1^2(\mathrm{i}\eta-\tau)^2=b_2^2(\tau^2+\eta^2).$$

Hence, with $z=\mathrm{i}\tau$, we define

$$\mathbf{Lop}\,(z,\eta)\doteq b_1^2(\eta+z)^2+b_2^2(\eta^2-z^2)=(\eta+z)(b_1^2(\eta+z)+b_2^2(\eta-z)).$$

The fact that $\mathbf{Lop}$ vanishes at the point $z=-\eta$, regardless of the value of B, reflects the fact that R does not span an eigenspace at the point (actually it is zero there). A computation using instead with R' leads to $(\eta-z)(b_1^2(\eta+z)+b_2^2(\eta-z))$, this factor is thus irrelevant, and the vanishing of Lopatinskii determinant must imply that of the simpler polynomial

$$\mathbf{Lop}_0(z,\eta)=b_1^2(\eta+z)+b_2^2(\eta-z).$$

From this we can see the IBVP satisfies the Lopatinskii condition if $b_1\neq\pm b_2$. In fact, $\mathrm{Re}\tau>0$ implies $\mathrm{Im}z>0$, so $\mathrm{Im}\mathbf{Lop}_0(z,\eta)=\mathrm{Im}(b_1^2+b_2^2)\eta+\mathrm{Im}(b_1^2-b_2^2)z=(b_1^2-b_2^2)\mathrm{Im}z\neq 0$. Recall here η and B are real.

For $b_1=\pm b_2$, we get $\mathbf{Lop}_0(z,\eta)=(b_1^2+b_2^2)\eta$, and it vanishes at $(z,0)$, while we cannot draw conclusion that IBVP does not satisfies Lopatinskii condition.

We return to $\Delta(\tau,\eta)$ with $\eta=0$. Since $\mathrm{Re}\tau>0$, *the eigenvalue with negative real part should be $\mu=-\tau$.* So we have $\Delta(\tau,0)=-(b_1+b_2)\tau$. So if $b_1=b_2$, Lopatinskii condition still holds (keep in mind $\mathrm{Re}\tau>0$), while for $b_1=-b_2$, it fails at $(\tau,0)$ ($\mathrm{Re}\tau>0$). ☐

For UKL, we only need consider the case $b_1\neq -b_2$ and $\tau=\mathrm{i}\rho$, with $\rho\in\mathbb{R}$ and $\eta\in\mathbb{R}$. Now since $z=-\rho$, we get $\mathbf{Lop}_0(-\rho,\eta)=(b_1^2+b_2^2)\eta-(b_1^2-b_2^2)\rho$. For $b_1=b_2$, there holds $\mathbf{Lop}_0(-\rho,\eta)=2b_1^2\eta=0$ only for $\eta=0$. For $\eta=0$, **we solve $\mu=-\mathrm{i}\rho$,** so $\Delta(\mathrm{i}\rho,0)=-\mathrm{i}b_1\rho-b_1(\mathrm{i}\rho)=-2\mathrm{i}b_1\rho$. So for $b_1=b_2$, Δ does not vanish at the points $(\mathrm{i}\rho,0)$ $(\rho\neq 0)$. Hence (UKL) holds.

For $b_1^2-b_2^2\neq 0$, $\mathbf{Lop}_0(-\rho,\eta)$ vanishes at $\{\rho=\frac{b_1^2+b_2^2}{b_1^2-b_2^2}\eta\}$, we need, again, return to the Lopatinskii determinant itself to check. There are the following cases, where we take $\eta\in\mathbb{R}\setminus\{0\}$.

- $b_1=0$. For this case $\rho=-\eta$, hence $\mu=0$, and

$$\Delta(\mathrm{i}\rho,\eta)=\mathrm{i}b_1(\eta-\rho)+b_2\mu=0.$$

So (UKL) fails.

- $b_2 = 0$. For this case $\rho = \eta$, hence $\mu = 0$ and (using R'):

$$\Delta(\mathrm{i}\eta, \eta) = -b_1\mu + \mathrm{i}b_2(\rho + \eta) = 0.$$

So (UKL) fails.

Remark: At the points $(\pm\mathrm{i}\eta, \eta)$, the two families of eigenvalues μ meet. Such points are called glancing points. Glancing points are usually obstacles for continuous extension of $E_-(\tau, \eta)$.

- $b_1b_2 \neq 0$. In this case $|\rho| = \left|\frac{b_1^2+b_2^2}{b_1^2-b_2^2}\eta\right| > |\eta|$. So $\mu = -\mathrm{i}\sqrt{\rho^2 - \eta^2}$ for $\rho > 0$ and $\mu = \mathrm{i}\sqrt{\rho^2 - \eta^2}$ for $\rho < 0$.[11]

  ♦ $\rho > 0$, which implies $\mathrm{sgn}(\eta)\mathrm{sgn}(b_1^2 - b_2^2) = 1$. So $\mu = -\frac{2\mathrm{i}|b_1b_2|}{|b_1^2-b_2^2|}|\eta|$, and

$$\Delta(i\rho, \eta) = -\frac{2\mathrm{i}b_1b_2^2}{b_1^2 - b_2^2}\eta\left(\frac{\mathrm{sgn}(b_1)\mathrm{sgn}(b_2)\mathrm{sgn}(\eta)}{\mathrm{sgn}(b_1^2 - b_2^2)} + 1\right).$$

  So for $b_1b_2 < 0$, Δ vanishes at the points $(\mathrm{i}\rho, \eta)$, with $\rho = \frac{b_1^2+b_2^2}{b_1^2-b_2^2}\eta > 0$.

  ♦ $\rho < 0$, which implies $\mathrm{sgn}(\eta)\mathrm{sgn}(b_1^2 - b_2^2) = -1$. So $\mu = \frac{2\mathrm{i}|b_1b_2|}{|b_1^2-b_2^2|}|\eta|$, and

$$\Delta(\mathrm{i}\rho, \eta) = \frac{2\mathrm{i}b_1b_2^2}{b_1^2 - b_2^2}\eta\left(\frac{\mathrm{sgn}(b_2)\mathrm{sgn}(b_1)\mathrm{sgn}(\eta)}{\mathrm{sgn}(b_1^2 - b_2^2)} - 1\right).$$

  So for $b_1b_2 < 0$, Δ vanishes at the points $(\mathrm{i}\rho, \eta)$, with $\rho = \frac{b_1^2+b_2^2}{b_1^2-b_2^2}\eta < 0$.

  In conclusion, we see for $b_1b_2 < 0$, (UKL) fails at the points $(\mathrm{i}\frac{b_1^2+b_2^2}{b_1^2-b_2^2}\eta, \eta)$. The eigenvalues μ are purely imaginary at such points. So they are hyperbolic boundary frequency points. For $b_1b_2 > 0$, (UKL) holds.

We thus proved the following.

Proposition 5.2. *(UKL) holds if and only if $b_1b_2 > 0$.*

5.2. Example 2. We consider the wave equation

$$\partial_t^2 u = c^2\Delta u, \qquad (x_d > 0)$$

with a boundary condition of the form

$$\partial_t u + a\partial_d u + \vec{b}\cdot\nabla_y u = 0. \tag{11}$$

[11]For $\tau = \gamma + \mathrm{i}\rho$, $\gamma > 0$, since $\mu^2 = \gamma^2 - \rho^2 + \eta^2 + 2\gamma\rho\mathrm{i}$, so for $\rho > 0$, μ^2 is in the second quadrant. Hence the branch μ with negative real part should approach $-\mathrm{i}\sqrt{\rho^2 - \eta^2}$ as $\gamma \to 0+$. For $\rho < 0$, μ^2 is in the third quadrant, so the branch μ with negative real part approach $\mathrm{i}\sqrt{\rho^2 - \eta^2}$ as $\gamma \to 0+$.

5.2.1. For $u = \mathrm{e}^{\tau t + \mathrm{i}\eta\cdot y}U(x_d)$ ($\mathrm{Re}\tau \geq 0$) to be a solution of the wave equation, U shall satisfy

$$U'' = (|\eta|^2 + \frac{\tau^2}{c^2})U.$$

Let $\alpha = \sqrt{|\eta|^2 + \frac{\tau^2}{c^2}}$ **with** $\mathrm{Re}\alpha > 0$**,** and $V = U'$, we get

$$\begin{pmatrix} U \\ V \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix}.$$

The stable subspace is generated by $(1, -\alpha)^\top$**,** and note the boundary condition is $(i\vec{b}\cdot\eta + \tau, a)\begin{pmatrix} U \\ V \end{pmatrix} = 0$, then the Lopatinskii determinant is

$$\Delta(\tau, \eta) = (\mathrm{i}\vec{b}\cdot\eta + \tau, a)(1, -\alpha)^\top = -a\alpha + \mathrm{i}\vec{b}\cdot\eta + \tau.$$

Set $\alpha = e + d\mathrm{i}$, with ($\tau = \gamma + \mathrm{i}\rho$)

$$e^2 - d^2 = |\eta|^2 + (\gamma^2 - \rho^2)/c^2, \quad de = \gamma\rho/c^2. \tag{12}$$

Then

$$\Delta = (\gamma - ae) + (\rho + \vec{b}\cdot\eta - ad)\mathrm{i}. \tag{13}$$

Lemma 5.1. *The following statements hold:*

- *(KL) holds if and only if $a \leq 0$ or $a > c$.*
- *The (UKL) is satisfied if and only if*

$$a < 0, \qquad |\vec{b}| < c. \tag{14}$$

- *Boundary frequency point of elliptic part* [12] *is made of pairs $(\tau = \mathrm{i}\rho, \eta)$ such that $|\rho| < c|\eta|$.*
- *The boundary frequency glancing points* [13] *are the pairs $(\pm\mathrm{i}c|\eta|, \eta)$.*
- *The Lopatinskii determinant vanishes in the elliptic zone and nowhere else, if and only if*

$$a = 0, \quad |\vec{b}| < c. \tag{15}$$

- *Δ vanishes only at one glacing point if, and only if*

$$a \leq 0, \quad |\vec{b}| = c. \tag{16}$$

[12]By definition, at such a point $(\tau = \mathrm{i}\rho, \eta), \rho \in \mathbb{R}$, none of the eigenvalue is pure imaginary.

[13]By definition, at least two eigenvalues meet at such a point $(\tau = \mathrm{i}\rho, \eta), \rho \in \mathbb{R}$.

Proof. **1.** (KL) means $\Delta(\tau,\eta)\neq 0$ for $\mathrm{Re}\tau>0, \eta\in\mathbb{R}^{d-1}, |\tau|+|\eta|\neq 0$. We can see from (12) that $\gamma>0$ implies $e>0$, so for $a\le 0$, we always have $\gamma-ae\neq 0$ and so $\Delta\neq 0$, and (KL) holds.

Also, for $a=c$, one easily checks that $\Delta(\tau,0)=0$, and (KL) does not hold.

For $a>0$ but $a\neq c$, we may solve, if $\Delta=0$, that (recall $\gamma>0$)

$$\rho=\frac{c^2}{a^2-c^2}(\vec{b}\cdot\eta),\qquad \gamma^2=\frac{a^2c^2}{c^2-a^2}(|\eta|^2-\frac{(\vec{b}\cdot\eta)^2}{c^2-a^2}).$$

If $a>c$, then for the second equation to hold, we should have $|\eta|=\gamma=0$, contradictory to our assumption $\gamma>0$, this means $\Delta\neq 0$ for $a>c$. However, if $0<a<c$, no matter what $\vec{b}$ is, there always exists nonzero η (for example, those $\vec{b}\cdot\eta=0$) so that

$$\gamma=\frac{ac}{\sqrt{c^2-a^2}}\sqrt{|\eta|^2-\frac{(b\cdot\eta)^2}{c^2-a^2}}>0.$$

Hence (KL) does not hold at these $(\gamma+i\rho,\eta)$.

2. (UKL) means $\Delta(\tau,\eta)\neq 0$ on $\mathrm{Re}\tau\geq 0, \eta\in\mathbb{R}^{d-1}, |\tau|+|\eta|\neq 0$. Therefore, we restrict ourselves to the case $a\le 0$ or $a>c$. We also need only consider the case $\gamma=0$.

If $a=0$, then $\Delta(-\mathrm{i}\vec{b}\cdot\eta,\eta)=0$, so (UKL) does not hold.

If $a>c$ or $a<-c$, then we may solve $e=0$ and $d=(\rho+\vec{b}\cdot\eta)/a$, $d^2=\rho^2/c^2-|\eta|^2\geq 0$. Note that for $\rho>0$, we should choose the root $d>0$, and for $\rho<0$, choose the root $d<0$.[14] We have an equation of ρ :

$$f(\rho)=(1-a^2/c^2)\rho^2+2\rho(\vec{b}\cdot\eta)+(\vec{b}\cdot\eta)^2+a^2|\eta|^2=0.$$

This quadratic equation always has real roots, but remember we need $|\rho|\geq c|\eta|$. This requires $f(\pm c|\eta|)=(\pm c|\eta|+\vec{b}\cdot\eta)^2\geq 0$. and it always holds for $|a|>c$.

We may solve that

$$\rho_\pm=\frac{c^2}{a^2-c^2}\left(\vec{b}\cdot\eta\pm\frac{|a|}{c}\sqrt{(\vec{b}\cdot\eta)^2+|\eta|^2(a^2-c^2)}\right).$$

Hence

$$d_\pm=\frac{|a|}{a^2-c^2}(\vec{b}\cdot\eta)\pm\frac{c\mathrm{sgn}(a)}{a^2-c^2}\sqrt{(\vec{b}\cdot\eta)^2+|\eta|^2(a^2-c^2)}.$$

As $\pm\rho_\pm>0$, for $\Delta=0$, we also need $\pm d_\pm\geq 0$. For $a>c$, the requires $-\pm\vec{b}\cdot\eta\le c|\eta|$; for $a<-c$, it is $\pm\vec{b}\cdot\eta\geq c|\eta|$.

As no matter what $\vec{b}$ is, it is always possible to find η so that $-\pm\vec{b}\cdot\eta\le c|\eta|$. This means for $a>c$, Δ actually vanishes at some boundary frequency points and (UKL) does not hold.

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[14] As $e+di=\sqrt{|\eta|^2+(\gamma^2-\rho^2)/c^2+2\gamma\rho/c^2}$, if $|\eta|^2+(\gamma^2-\rho^2)/c^2<0$, then for $\rho>0$ ($\rho<0$), the root with $d>0$ ($d<0$) has positive real part for γ small.

If $|\vec{b}| \geq c$, then it is also possible to find η so that $\pm\vec{b}\cdot\eta \leq c|\eta|$. This means (UKL) does not hold for $a < -c$ and $|\vec{b}| \geq c$.

However, if $a < -c$ and $|\vec{b}| < c$, it is impossible for a nontrivial η so that $\pm\vec{b}\cdot\eta > c|\eta|$. Hence (UKL) holds.

Similarly, for $a = -c$ and $\vec{b}\cdot\eta \neq 0$, we may obtain $\rho = -\frac{1}{2}\vec{b}\cdot\eta - \frac{1}{2}\frac{c^2|\eta|^2}{\vec{b}\cdot\eta}$, thus $d = -\frac{1}{2c}[(\vec{b}\cdot\eta) - c^2|\eta|^2/(\vec{b}\cdot\eta)]$. If $\vec{b}\cdot\eta < 0$, thus $\rho > 0$, we need $d \geq 0$, that is $\vec{b}\cdot\eta \leq -c|\eta|$. If $\vec{b}\cdot\eta > 0$, thus $\rho < 0$, we need $d \leq 0$, that is $\vec{b}\cdot\eta \geq c|\eta|$. So if $|\vec{b}| \geq c$, (UKL) may not hold; if $0 \neq |\vec{b}| < c$, (UKL) holds.

For $a = -c$ and $\vec{b} = 0$ (or $\vec{b}\cdot\eta = 0$), for $\Delta(\mathrm{i}\rho, \eta) = 0$, we need $\eta = 0$, and then $d = -\rho/c$. So d and ρ are of opposite sign and $\Delta \neq 0$. So we conclude (UKL) holds also for $a = -c, \vec{b} = 0$.

If $-c < a < 0$, then $\Delta = 0$ implies as above, $e = 0$ and $f(\rho) = 0$. If f has no real root, $\Delta \neq 0$; if f has a real root, then it is necessary that at least one of $f(\pm c|\eta|) \leq 0$ holds. The latter requires (at least one of $\pm$) $-\pm\vec{b}\cdot\eta = c|\eta|$. So if $|\vec{b}| < c$, (UKL) will always hold (for $\eta \neq 0$, using Cauchy-Schwarz Inequality; for $\eta = 0$, using the argument of the above paragraph).

Finally we consider the case $-c < a < 0$ and $|\vec{b}| \geq c$. We show there are some points $(\mathrm{i}\rho, \eta)$ where Δ vanishes.

We first note there are η so that $|\vec{b}\cdot\eta| \geq (c^2 - a^2)^{1/2}|\eta|$, this means $f(\rho)$ has real roots $\rho_\pm$. Now there are the following two cases.

(a) $\vec{b}\cdot\eta = c|\eta|$. So one of the root of f is $\rho = -c|\eta|$. Hence $d = 0$, and we actually have $\Delta(-\mathrm{i}c|\eta|, \eta) = 0$.

(b) $-\vec{b}\cdot\eta = c|\eta|$. So one of the root of f is $\rho = c|\eta|$. Hence $d = 0$, and we actually have $\Delta(\mathrm{i}c|\eta|, \eta) = 0$.

In conclusion, we have a table in the following (some will be confirmed in the steps following).

3. For boundary frequency point $(\mathrm{i}\rho, \eta)$ of elliptic type, then α should not be purely imaginary. As now $\alpha = \sqrt{|\eta|^2 - \rho^2/c^2}$, this means $|\rho| < c|\eta|$.

a	$\vec{b}$	$\{\Delta(\tau,\eta)=0\}$
$a>c$		some boundary points
$0<a\le c$		some interior points
$a=0$	$\lvert\vec{b}\rvert<c$	$\Leftrightarrow$ on elliptic boundary points
$a=0$	$\lvert\vec{b}\rvert\ge c$	some boundary points
$-c<a<0$	$\lvert\vec{b}\rvert<c$	$\emptyset$
$-c<a<0$	$\lvert\vec{b}\rvert\ge c$	some boundary points
$a=-c$	$\vec{b}=0$	$\emptyset$
$a=-c$	$\lvert\vec{b}\rvert<c$	$\emptyset$
$a=-c$	$\lvert\vec{b}\rvert\ge c$	some boundary points
$a<-c$	$\lvert\vec{b}\rvert<c$	$\emptyset$
$a<-c$	$\lvert\vec{b}\rvert\ge c$	some boundary points

4. For boundary glancing point, two branches of eigenvalues meet at a purely imaginary one. If $\rho > c|\eta|$, then one branch of eigenvalue, with positive real part, will approach $\sqrt{\rho^2/c^2-|\eta|^2}\mathrm{i}$ and the other branch, with negative real part, will approach $-\sqrt{\rho^2/c^2-|\eta|^2}\mathrm{i}$. Similar phenomena occur for $\rho<-c|\eta|$. So $(\mathrm{i}\rho,\eta)$ with $|\rho|>c|\eta|$ are not glancing points.

While, $(\pm\mathrm{i}c|\eta|,\eta)$ are glancing points, as both branches will approach the eigenvalue $\alpha=0$ as $(\tau,\eta)\to(\pm\mathrm{i}c|\eta_0|,\eta_0)$ with $\mathrm{Re}\tau>0$.

5. $\Delta=0$ at $(i\rho,\eta)$ with $|\rho|<c|\eta|$ means $ae=0, \rho+\vec{b}\cdot\eta-ad=0$. If $a\neq 0$, then $e=0$ and $d^2=\rho^2/c^2-|\eta|^2<0$, contradiction. So we need $a=0$. Then $d=0$ and $e=\sqrt{|\eta|^2-\rho^2/c^2}$, and $\rho=-\vec{b}\cdot\eta$. This requires that $|\vec{b}\cdot\eta|^2<c^2|\eta|^2$. As we need find the range of $a,\vec{b}$ so that Δ only vanishes on elliptic points, this amounts $|\vec{b}|<c$. (If $|\vec{b}|\ge c$, we may find η so that $|\vec{b}\cdot\eta|^2\ge c^2|\eta|^2$, so $\Delta(-\mathrm{i}\vec{b}\cdot\eta,\eta)=0$ while $(-\mathrm{i}\vec{b}\cdot\eta,\eta)$ is not an elliptic point). Hence it is necessary that $a=0$ and $|\vec{b}|<c$, for $\Delta=0$ only on the elliptic point $(-\mathrm{i}\vec{b}\cdot\eta,\eta)$.

On the other hand, if $a=0$, as shown in Step 1, Δ can only vanish on $\mathrm{Re}\tau=0$. Hence $\Delta=0$ implies $\rho=-\vec{b}\cdot\eta$. As $|\vec{b}|<c$, so Δ vanishes actually only on elliptic points.

6. If $\Delta(\mathrm{i}\rho,\eta)=0$ at a glancing point, for $\rho=\pm c|\eta|$, then one solves $\rho=-\vec{b}\cdot\eta$ and thus require $|\vec{b}|\ge c$. However, if $|\vec{b}|>c$, we may find lots of η (not in a one-dimensional subspace) with unit length so that $\vec{b}\cdot\eta=c|\eta|$. Hence, to ensure the set where Δ vanishes contains only one glancing point (generated by η in a one-dimensional subspace of $\mathbb{R}^{d-1}$), it is necessary that $|\vec{b}|=c$.

As we require Δ must vanish on only one glancing point, (it also cannot vanish on other non-glancing points), then by the above necessary condition $|\vec{b}|=c$, we need check the following cases one by one.

1) $a > c, |\vec{b}| = c$;
2) $a = c, |\vec{b}| = c$;
3) $0 < a < c, |\vec{b}| = c$;
4) $a = 0, |\vec{b}| = c$;
5) $-c < a < 0, |\vec{b}| = c$;
6) $a = -c, |\vec{b}| = c$;
7) $a < -c, |\vec{b}| = c$.

As we always require (KL) holds, Cases 2) and 3) do not need be discussed.

Case 1). By Step 2, one requires $-\pm\vec{b}\cdot\eta \le c|\eta|$ for $\Delta = 0$. Even if $|\vec{b}| = c$, one may find η so this holds and then Δ vanishes at least at a non-glancing point.

Case 4). We know from Step 2 that $\Delta(-\mathrm{i}\vec{b}\cdot\eta, \eta) = 0$. As $|\vec{b}| = c$, we can find only one η so that $\vec{b}\cdot\eta = c|\eta|$, hence Δ vanishes only on one glancing point.

Case 5). As in Step 2, we see that, because $|\vec{b}| = c$, there is only one η (namely $\pm\vec{b}/|\vec{b}|$) so that $\Delta(-\pm \mathrm{i}c|\eta|, \eta) = 0$. Other points where $\Delta = 0$ are not glancing points.

Case 6). By Step 2, for Δ vanishes at a glancing point, it is necessary that $\vec{b}\cdot\eta \neq 0$, hence $\rho = -\frac{1}{2}\vec{b}\cdot\eta - \frac{1}{2}\frac{c^2|\eta|^2}{\vec{b}\cdot\eta}$, and for $\rho = \pm c|\eta|$, we get $\vec{b}\cdot\eta = -\pm c|\eta|$. By $|\vec{b}| = c$, such points η is unique, namely, $\pm\vec{b}/|\vec{b}|$.

Case 7). By Step 2, where we require $c|\eta| \le \pm\vec{b}\cdot\eta$. Using Cauchy-Schwarz Inequality, as $|\vec{b}| = c$, we can only take $\eta = \vec{b}/c$ ($\eta = -\vec{b}/c$) and then may check that $\rho_- = -c|\eta|$ ($\rho_+ = c|\eta|$). This shows Δ actually vanishes at only one glancing point.

In conclusion, we find Δ vanishes at one glancing point if and only if $a \le 0, |\vec{b}| = c$. □

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LECTURE NOTES 4: BOUNDARY CONDITIONS FOR EULER EQUATIONS

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This note is totaly based on Chapter 14 of [1]. Another good reference on the discussion of boundary conditions of Euler system is the paper [2].

1. Basic Facts on Euler Equations

The motion of a compressible, inviscid and non-heat-conducting fluid is governed by the Euler equations, consisting of the mass, momentum and energy conservation laws:

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0, \\ \partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = 0, \\ \partial_t(\rho(\frac{1}{2}|\mathbf{u}|^2 + e)) + \nabla\cdot((\rho(\frac{1}{2}|\mathbf{u}|^2+e)+p)\mathbf{u}) = 0. \end{cases} \tag{1.1}$$

This system of $(d+2)$ equations contains $(d+3)$ unknowns: the density $\rho\in\mathbb{R}^+$, the velocity $\mathbf{u}\in\mathbb{R}^d$, the internal energy $e\in\mathbb{R}^+$ and the pressure $p\in\mathbb{R}^+$. This system has to be closed by adding a suitable equation of state, or pressure law, $(\rho,e)\mapsto p(\rho,e)$. For polytropic gas, this is

$$p = (\gamma-1)\rho e, \qquad \gamma > 1.$$

1.1. **Constantly hyperbolicity.** A classical and elementary manipulation (cf. Chapter 2 of [3]) shows that, for smooth solutions, (1.1) is equivalent to

$$\begin{cases} \partial_t\rho + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0, \\ \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \rho^{-1}\nabla p = 0, \\ \partial_t e + \mathbf{u}\cdot\nabla e + \rho^{-1}p\nabla\cdot\mathbf{u} = 0. \end{cases} \tag{1.2}$$

Therefore, the hyperbolicity of (1.1) is equivalent to the uniform real diagonalizability of the matrix

$$A(U;\mathbf{n}) \doteq \begin{pmatrix} \mathbf{u}\cdot\mathbf{n} & \rho\mathbf{n}^\top & 0 \\ \rho^{-1}p'_\rho\mathbf{n} & (\mathbf{u}\cdot\mathbf{n})I_d & \rho^{-1}p'_e\mathbf{n} \\ 0 & \rho^{-1}p\mathbf{n}^\top & \mathbf{u}\cdot\mathbf{n} \end{pmatrix}$$

for all $U=(\rho,\mathbf{u},e)$ and $\mathbf{n}\in\mathbb{R}^d\setminus\{0\}$ (a column vector). Recall that I_d is the $d\times d$ identity matrix.

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Proposition 1.1. *The system* (1.1)*, for polytropic gas, is constantly hyperbolic, and strictly hyperbolic in dimension $d=1$. Its eigenvalues in the direction* $\mathbf{n}$ *are*

$$\lambda_1(U;\mathbf{n})=\mathbf{u}\cdot\mathbf{n}-c|\mathbf{n}|,\quad \lambda_2(U;\mathbf{n})=\mathbf{u}\cdot\mathbf{n},\quad \lambda_3(U;\mathbf{n})=\mathbf{u}\cdot\mathbf{n}+c|\mathbf{n}|, \tag{1.3}$$

where $c=\sqrt{\gamma p/\rho}$ denotes as usual the sound speed. The associated eigenvectors are respectively

$$r_1(U;\mathbf{n})=\begin{pmatrix}\rho\\ -c\frac{\mathbf{n}}{|\mathbf{n}|}\\ p/\rho\end{pmatrix},\quad r_2(U;\mathbf{n})=\begin{pmatrix}-\dot{\alpha}p'_e\\ \dot{\mathbf{u}}\\ \dot{\alpha}p'_\rho\end{pmatrix}\quad \text{with}\quad \dot{\mathbf{u}}\cdot\mathbf{n}=0,$$
$$r_3(U;\mathbf{n})=\begin{pmatrix}\rho\\ c\frac{\mathbf{n}}{|\mathbf{n}|}\\ p/\rho\end{pmatrix}. \tag{1.4}$$

Here $\dot{\alpha}$ is an arbitrary real number, and $\dot{\mathbf{u}}\in\mathbb{R}^d$. So r_2 actually spans a d-dimensional subspace in the state space $\mathbb{R}^{d+2}$, and λ_2 is of multiplicity d.

In addition, the characteristic field (λ_2,r_2) is linearly degenerate, that is, $\mathrm{d}\lambda_2\cdot r_2\equiv 0$ (where d stands for differentiation with respect to U), and the fields (λ_1,r_1) and (λ_3,r_3) (also called acoustic fields) are genuinely nonlinear.

1.2. **Symmetric hyperbolicity.** There are many different ways to write the Euler equations as a symmetric hyperbolic system. We use here the simplest one: Hadamard symmetrization in non-conservation variables. Using $(\rho,\mathbf{u},s)$ as independent variables, we may rewrite the quasilinear system (1.2) as [1]

$$\begin{cases}\partial_t p+\mathbf{u}\cdot\nabla p+\rho c^2\nabla\cdot\mathbf{u}=0,\\ \partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\rho^{-1}\nabla p=0,\\ \partial_t s+\mathbf{u}\cdot\nabla s=0.\end{cases} \tag{1.5}$$

The characteristic matrix of this system reads

$$A(p,\mathbf{u},s;\mathbf{n})\doteq\begin{pmatrix}\mathbf{u}\cdot\mathbf{n} & \rho c^2\mathbf{n}^\top & 0\\ \rho^{-1}\mathbf{n} & (\mathbf{u}\cdot\mathbf{n})I_d & 0\\ 0 & 0 & \mathbf{u}\cdot\mathbf{n}\end{pmatrix}.$$

The symmetrizer is

$$S(p,\mathbf{u},s)\doteq\mathrm{diag}\{(\rho c^2)^{-1},\rho,\cdots,\rho,1\},$$

[1] Notice that the strategy is to use appropriate coordinates in the state space. For details of the computation, see [3, Chapter].

and

$$S(p,\mathbf{u},s)A(p,\mathbf{u},s;\mathbf{n}) = \begin{pmatrix} \frac{\mathbf{u}\cdot\mathbf{n}}{\rho c^2} & \mathbf{n}^\top & 0 \\ \mathbf{n} & \rho(\mathbf{u}\cdot\mathbf{n})I_d & 0 \\ 0 & 0 & \mathbf{u}\cdot\mathbf{n} \end{pmatrix}. \tag{1.6}$$

2. Classification of fluid Initial-Boundary-Value Problems

We provide below a classification of Initial-Boundary-Value Problems (IBVPs) according to various physical situations, and discuss possible boundary conditions ensuring well-posedness.

As far as smooth domains are concerned, a crucial issue is the well-posedness of IBVPs in half-spaces (obtained using coordinate charts). To fix ideas, we consider IBVPs in the half-space $\{x_d \geq 0\}$ (without loss of generality, the Euler equations being invariant by rotations).

We recall that the characteristic speeds of the Euler equations in the direction $\mathbf{n} = (0,\cdots,0,1)^\top$ are

$$\lambda_1 = u - c, \quad \lambda_2 = u, \quad \lambda_3 = u + c,$$

where $u \doteq \mathbf{u}\cdot\mathbf{n}$ is the last component of $\mathbf{u}$ and c is the sound speed. When the boundary is a wall, u is clearly zero. Otherwise, if $u \neq 0$, we can distinguish between *incoming* flows, for which $u > 0$, and *outgoing* flows, for which $u < 0$. Another distinction to be made concerns the (normal) Mach number[2]

$$M \doteq |u|/c.$$

The flow is said to be *subsonic* if $M < 1$, and *supersonic* if $M > 1$. This yields the following classification, when c is non-zero (non-vacuum).

2.1. Non-characteristic problems.

For the following cases, we check that $\det A^d \neq 0$.

Out-Supersonic ($u < 0$ and $M > 1$, hence $\lambda_1, \lambda_2, \lambda_3 < 0$.)

There is no incoming characteristics. No boundary condition should be prescribed.

Out-Subsonic ($u < 0$ and $M < 1$, hence $\lambda_1, \lambda_2 < 0, \lambda_3 > 0$.)

There is one incoming characteristics. One and only one boundary condition should be prescribed.

[2]The Mach number of the flow is $|\mathbf{u}|/c$. Here only the flow along the normal direction x_d is of interests. Notice that a supersonic flow could be subsonic in the normal direction.

In-Subsonic ($u > 0$ and $M < 1$, hence $\lambda_1 < 0, \lambda_2, \lambda_3 > 0$.)

There are $(d+1)$ incoming characteristics, counting with multiplicity. (Recall that λ_2 is counted d times repeatedly.) This means that $(d+1)$ independent boundary conditions are needed.

In-Supersonic ($u > 0$ and $M > 1$, hence $\lambda_1, \lambda_2, \lambda_3 > 0$.)

All characteristics are incoming characteristics. This means that all components of the unknown $\mathbf{W} = (p, \mathbf{u}, s)$ should be prescribed on the boundary.

2.2. **Characteristic problems.** For the following cases, we check that $\det A^d = 0$.

Slip walls ($u = 0$, hence $\lambda_1 < 0$, $\lambda_2 = 0$, and $\lambda_3 > 0$.) The boundary $\{x_d = 0\}$ is characteristic (caused by λ_2) with constant rank 2 (rank $A^d = 2$ now).

One and only one boundary condition $b(\mathbf{W})$ should be prescribed. For the IBVP to be normal, the 2-eigenfield should be tangent to the level set of b.[3]

Out-Sonic ($u = -c$, hence $\lambda_1, \lambda_2 < 0$ and $\lambda_3 = 0$.) The boundary $\{x_d = 0\}$ is characteristic (caused by λ_3) with constant rank $d+1$ (rank $A^d = d+1$).

No boundary condition should be prescribed.

In-Sonic ($u = c$, hence $\lambda_1 = 0$, $\lambda_2, \lambda_3 > 0$.) The boundary $\{x_d = 0\}$ is characteristic (caused by λ_1) with constant rank $d+1$ (rank $A^d = d+1$).

A set of $(d+1)$ boundary conditions $b_1(\mathbf{W}), \cdots, b_{d+1}(\mathbf{W})$ (corresponding to eigenfields of λ_2, λ_3) should be prescribed. For the IBVP to be normal, the 1-eigenfield should be tangent to the level set of $b_1, \cdots, b_{d+1}$ (i.e., $r_1 \cdot \nabla b_j = 0, j = 1, \cdots, d+1$).

3. Dissipative initial boundary value problem

In this section, we look for dissipative boundary conditions. This notion depends on the symmetrization used. For concreteness, we use the simplest symmetrization, in $(p, \mathbf{u}, s)$ variables. We recall indeed that, away from vacuum, the Euler equations can be written as

$$S(p, \mathbf{u}, s)(\partial_t + A(p, \mathbf{u}, s; \nabla)) \begin{pmatrix} p \\ \mathbf{u} \\ s \end{pmatrix} = 0,$$

[3]For the IBVP to be normal, one needs that $\ker A^d \subset \ker B$, where B is the linearized boundary matrix, namely $B = \mathrm{d}b$. (To avoid confusion, we write $\mathrm{d}b$ to be the gradient of b with respect to the unknowns $\mathbf{W}$.) Now $\ker A^d$ consists of those eigenvectors of $\lambda_2 = 0$, i.e., r_2. So we shall have $B \cdot r_2 = 0$, which means exactly that r_2 is tangent to the level set of b.

where $S(p, \mathbf{u}, s)$ is symmetric positive-definite and

$$S(p, \mathbf{u}, s)A(p, \mathbf{u}, s; \mathbf{n}) = \begin{pmatrix} \frac{\mathbf{u}\cdot\mathbf{n}}{\rho c^2} & \mathbf{n}^\top & 0 \\ \mathbf{n} & \rho(\mathbf{u}\cdot\mathbf{n})I_d & 0 \\ 0 & 0 & \mathbf{u}\cdot\mathbf{n} \end{pmatrix}.$$

Definition 3.1. Assume that $L \doteq \partial_t + \sum_{j=1}^d A^j(x,t)\partial_j$ is a Friedrichs-symmetrizable operator, with Friedrichs symmetrizer S. The boundary matrix B is called *strictly dissipative* if there exist $\alpha > 0$ and $\beta > 0$ so that for all $(x,t) \in \partial\Omega \times \mathbb{R}$ and all $\mathbf{v} \in \mathbb{R}^n$:

$$\boxed{\mathbf{v}^\top S(x,t)A(x,t;\mathbf{n}(x))\mathbf{v} \geq \alpha|\mathbf{v}|^2 - \beta|B(y,t)\mathbf{v}|^2,} \tag{3.1}$$

where $\mathbf{n} = (n_1, \cdots, n_d)^\top$ denotes the unit outward normal to $\partial\Omega$, and $A(x,t;\mathbf{n}(x)) \doteq \sum_{j=1}^d A^j(x,t)n_j$.

Note: Strongly dissipativeness means $S(x,t)A(x,t;\mathbf{n})$ is positive-definite on $\ker B$; while dissipativeness means $SA(\mathbf{n})$ is nonnegative on $\ker B$. In the case $\Omega = \{x_d > 0\}$, the strict dissipativity of B means

$$v^* S(y,0,t)A^d(y,0,t)v \leq -\alpha|v|^2 + \beta|B(y,t)v|^2 \tag{3.2}$$

for all $(y,t) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and all $v \in \mathbb{C}^n$. Here and in what follows, we take $\mathbf{n} = (0, \cdots, 0, 1)^\top$.

According to Definition 3.1, dissipativeness of a set of boundary conditions encoded by a nonlinear mapping $b : (p, \mathbf{u}, s) \mapsto b(p, \mathbf{u}, s)$ requires that $-SA^d$ be non-negative on the tangent bundle of the manifold $\mathscr{B} = \{b(p, \mathbf{u}, s) = \underline{b}\}$, while strictly dissipativeness requires that $-SA^d$ be coercive on the same bundle.

A straightforward computation shows that for $\dot{U} = (\dot{p}, \dot{\check{\mathbf{u}}}, \dot{u}, \dot{s})^\top$, and $v \doteq 1/\rho$,

$$(SA^d\dot{U}, \dot{U}) = u\dot{s}^2 + \rho u(|\dot{\check{\mathbf{u}}}|^2 + \dot{u}^2) + 2\dot{p}\dot{u} + u\dot{p}^2/(\rho c^2). \tag{3.3}$$

Here $\dot{\check{\mathbf{u}}}$ is the (perturbed) tangential velocity, and $\dot{u}$ is the (perturbed) normal velocity. By organizing the terms, if $u \neq 0$, we have

$$\boxed{(-SA^d\dot{U}, \dot{U}) = -\rho u \left(\frac{1}{u^2}(u\dot{u} + v\dot{p})^2 - \frac{v^2}{u^2}(1 - M^2)\dot{p}^2 + |\dot{\check{\mathbf{u}}}|^2 + v\dot{s}^2 \right).} \tag{3.4}$$

We can now review the different cases.

Supersonic outflow ($u < 0$ and $M > 1$).

We see that $-SA^d$ is coercive on the whole space. So this case is harmless.

Subsonic outflow ($u<0$ and $M<1$).

The restriction of $-SA^d$ to the hyperplane $\{\dot{p}=0\}$ is obviously coercive. Thus a strictly dissipative condition is obtained by prescribing the pressure p at the boundary.

Another, possible, simple, choice is to prescribe the normal velocity u, since $-SA^d$ is also coercive when restricted on the hyperplane $\{\dot{u}=0\}$. [4]

Subsonic inflow ($u>0$ and $M<1$).

This is the most complicated case. Prescribing the pressure among the boundary conditions would obviously be a bad idea, for the same reason as it is a good one for subsonic outflows.

On the other hand, the easiest way to cancel some bad terms is to prescribe the tangential velocity $\check{\mathbf{u}}$ and the entropy s, which leaves only one boundary condition to be determined in such a way that $u\mathrm{d}u + v\mathrm{d}p = 0$ on the tangent bundle of $\mathscr{B}$. Recalling that, $\mathrm{d}e = -p\mathrm{d}v + T\mathrm{d}s$, the specific enthalpy $h = e + pv$ is such that

$$\mathrm{d}h = T\mathrm{d}s + v\mathrm{d}p,$$

so we see that the above requirement is achieved by giving $\frac{1}{2}u^2 + h$ since $\mathrm{d}(h + \frac{1}{2}u^2) = T\mathrm{d}s = 0$ as we have prescribed the entropy s. Hence a strictly dissipative set of boundary conditions is [5]

$$\left\{\frac{1}{2}u^2 + h, \check{\mathbf{u}}, s\right\}.$$

Other boundary conditions may be exhibited that are relevant from a physical point of view — for instance, using concepts of total pressure and total temperature.

Supersonic inflow ($u>0$ and $M>1$).

We see that SA^d (instead of $-SA^d$) is coercive. But since all components of the unknown should be prescribed on the boundary, the tangent spaces are reduced to $\{0\}$. [6] So this case is also harmless.

Slip walls ($u=0$).

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[4] Comparing to the case of steady compressible Euler flows.

[5] For polytropic gas, $h = \frac{c^2}{\gamma-1} = \frac{\gamma}{\gamma-1}\frac{p}{\rho}$.

[6] With more restrictions, the manifold becomes more smaller.

The kernel of $A^d = \begin{pmatrix} 0 & \rho c^2 \mathbf{n}^\top & 0 \\ \rho^{-1}\mathbf{n} & 0_d & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (here $\mathbf{n} = (0, \cdots, 0, 1)^\top$) is the d-dimensional subspace $\{(0, \dot{\check{\mathbf{u}}}, 0, \dot{s})\}$, which is part of the tangent subspace $\{\dot{u} = 0\}$ associated with the natural boundary condition on u (i.e., $u = 0$) — as required by the normality criteria ($\ker A^d \subset \ker B$). The matrix SA^d is null on $\{\dot{u} = 0\}$ (see (3.3), where $u = 0$ now), which means the boundary condition is dissipative but of course not strictly dissipative.

Out-sonic ($u = -c$).

Notice that now $M = 1$. The matrix $-SA^d$ is non-negative but has isotropic vectors (defined by $u\dot{u} + v\dot{p} = 0$ and $\dot{\check{\mathbf{u}}} = 0, \dot{s} = 0$, totally $1 + d - 1 + 1 = d + 1$ constraints). This is a one-dimensional isotropic space. Since no boundary condition here, the tangent space is the whole $\mathbb{R}^{d+2}$, and the boundary is dissipative but not strictly dissipative. [7]

In-sonic ($u = c$).

The only possible choice of dissipative boundary conditions is the one described for subsonic inflow, which cancels all terms in $(SA^d\dot{U}, \dot{U})$ (since $M = 1$ and $-(1 - M^2)\dot{p}^2$ is zero). [8]

The normality criteria is met by those boundary conditions because [9]

$$\ker A^d = \ker SA^d = \left\{(\dot{p}, \dot{\check{\mathbf{u}}}, \dot{u}, \dot{s}) : \; u\dot{u} + v\dot{p} = 0, \; \dot{\check{\mathbf{u}}} = 0, \; \dot{s} = 0\right\} = \ker B.$$

We now turn to a more systematic testing of boundary conditions, which is known (and will be shown) to be less restrictive.

4. Normal modes analysis

Our purpose is to discuss boundary conditions from the Kreiss-Lopatinskii point of view, for general fluids equipped with a complete equation of state. To get simpler

[7] We note although for the linear IBVP there is no boundary condition here, but for the nonlinear problem, we have one condition, that is $u = -c$ here.

[8] We note although for the linear IBVP there are $d+1$ boundary conditions here, but for the nonlinear problem, we have one more condition, that is $u = c$ here. So totally we shall have $d + 2$ conditions and hence $\mathbf{W}$ is given.

[9] Now looking at (1.6) and recall that $\mathbf{u} \cdot \mathbf{n} = u = c$, therefore $A^d = \begin{pmatrix} \frac{1}{\rho u} & \mathbf{n}^\top & 0 \\ \mathbf{n} & \rho u I_d & 0 \\ 0 & 0 & u \end{pmatrix}$ and its kernel could be calculated easily.

computations, we choose the specific volume $v = 1/\rho$ and the specific entropy s as the thermodynamical variables, and rewrite the Euler equations as

$$\begin{cases} \partial_t v + \mathbf{u} \cdot \nabla v - v \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + v p'_v \nabla v + v p'_s \nabla s = 0, \\ \partial_t s + \mathbf{u} \cdot \nabla s = 0, \end{cases} \tag{4.1}$$

with the short notations of partial derivatives of $p = p(v, s)$ [10]

$$p'_v = \left.\frac{\partial p}{\partial v}\right|_s, \qquad p'_s = \left.\frac{\partial p}{\partial s}\right|_v.$$

Later, we shall make the connection with the non-dimensional coefficients γ and Γ, in that

$$p'_v = -\gamma \frac{p}{v}, \quad p'_s = \Gamma \frac{T}{v},$$

and T is the temperature. Alternatively, we recall that $p'_v = -\frac{c^2}{v^2}$, where c is the sound speed. Our minimal assumption is that c is real (positive). Furthermore, we have seen in §2 that boundary conditions for supersonic flows are either trivial or absent. A normal modes analysis is irrelevant in those cases. Since sonic IBVP are so degenerate that a normal modes analysis is also useless, from now on we concentrate on the subsonic case, assuming that

$$0 < |u| < c. \tag{4.2}$$

4.1. **The stable subspace of interior equations.** Linearize (4.1) about a reference state $(v, \mathbf{u} = (\check{\mathbf{u}}, u), s)$, ignoring zeroth-order terms, we get

$$\begin{cases} (\partial_t + \check{\mathbf{u}} \cdot \check{\nabla} + u\partial_z)\dot{v} - v\nabla \cdot \dot{\mathbf{u}} = 0, \\ (\partial_t + \check{\mathbf{u}} \cdot \check{\nabla} + u\partial_z)\dot{\mathbf{u}} + v p'_v \nabla \dot{v} + v p'_s \nabla \dot{s} = 0, \\ (\partial_t + \check{\mathbf{u}} \cdot \check{\nabla} + u\partial_z)\dot{s} = 0, \end{cases} \tag{4.3}$$

where z stands for the co-ordinate x_d, normal to the boundary, and $\check{\nabla}$ is the gradient operator along the boundary. The tangential co-ordinates will be denoted by $\mathbf{y} \in \mathbb{R}^{d-1}$.

By definition, for $\operatorname{Re}\tau > 0$ and $\boldsymbol{\eta} \in \mathbb{R}^{d-1}$, the sought stable subspace $E_-(\tau, \boldsymbol{\eta})$ is the space spanned by vectors $(\dot{v}, \dot{\mathbf{u}}, \dot{s})$ such that there exists a mode ω of positive real part for

[10] For polytropic gas, $p = \exp(s/c_v)v^{-\gamma}$ with $c_v > 0$ and $\gamma > 1$. So $p'_v = -\gamma p\rho$ and $p'_s = p/c_v$ as shown below. Recall for polytropic gas, $e = c_v T$, $\Gamma = \gamma - 1$.

which $\boxed{\exp(\tau t)\exp(\mathrm{i}\boldsymbol{\eta}\cdot\mathbf{y})\exp(-\omega z)(\dot v,\dot{\mathbf{u}},\dot s)}$ solves (4.3). [11] We are thus led to the system

$$\begin{cases} (\tau+\mathrm{i}\boldsymbol{\eta}\cdot\breve{\mathbf{u}}-u\omega)\dot v - v(\mathrm{i}\boldsymbol{\eta}\cdot\dot{\breve{\mathbf{u}}})+v\omega\dot u=0,\\ (\tau+\mathrm{i}\boldsymbol{\eta}\cdot\breve{\mathbf{u}}-u\omega)\dot{\breve{\mathbf{u}}}+vp_v'\mathrm{i}\dot v\boldsymbol{\eta}+vp_s'\mathrm{i}\dot s\boldsymbol{\eta}=0,\\ (\tau+\mathrm{i}\boldsymbol{\eta}\cdot\breve{\mathbf{u}}-u\omega)\dot u - vp_v'\omega\dot v-vp_s'\omega\dot s=0,\\ (\tau+\mathrm{i}\boldsymbol{\eta}\cdot\breve{\mathbf{u}}-u\omega)\dot s=0, \end{cases} \tag{4.4}$$

where we have used the obvious notation $\dot{\mathbf{u}}=(\dot{\breve{\mathbf{u}}},\dot u)$. To simplify the writing we introduce

$$\tilde\tau \doteq \tau+\mathrm{i}\boldsymbol{\eta}\cdot\breve{\mathbf{u}}.$$

(Observe that $\operatorname{Re}\tau=\operatorname{Re}\tilde\tau$.) We need to solve the eigenvalue ω and associated eigenvector $(\dot v,\dot{\mathbf{u}},\dot s)$.

The only nontrivial modes are thus obtained.

- for (eigenvalue) $\omega=\tilde\tau/u$ and the eigenvector [12]

$$u(\mathrm{i}\boldsymbol{\eta}\cdot\dot{\breve{\mathbf{u}}})-\tilde\tau\dot u=0 \quad\text{and}\quad p_v'\dot v+p_s'\dot s=0, \tag{4.5}$$

- for ω solution of the dispersion relation [13]

$$(\tilde\tau-u\omega)^2+v^2p_v'(\omega^2-|\boldsymbol{\eta}|^2)=0,$$

[11] We recall the abstract theory. For $\partial_t u+\sum_{j=1}^d A^j\partial_j u=0$, suppose that $u=\mathrm{e}^{\tau t+\mathrm{i}\eta\cdot y}U(x_d)$, then we find $U'=\mathcal{A}(\tau,\eta)U$, with $\mathcal{A}(\tau,\eta)=-(A^d)^{-1}(\tau I_n+\mathrm{i}A(\eta))$ for the non-characteristic case. Since for $\operatorname{Re}\tau>0$, $\mathcal{A}$ although might not always be diagonalized, the number of eigenvalues with positive (resp. negative) real part is fixed, and there is no any purely imaginary eigenvalue – so $U(x_d)$ will either exponentially decay or grow as $x_d\to\infty$. Suppose $-\omega$ is an eigenvalue with $\operatorname{Re}\omega>0$, then by suitable change of independent variables, $U(x_d)=U_0\exp(-\omega x_d)$, and U_0 is a (generalized) eigenvector associated with $-\omega$. Collecting all such U_0, we get $E_-(\tau,\eta)$. The following does not follow this strategy strictly, by replacing $U(x_d)=\mathrm{e}^{-\omega x_d}U_0$. This works well if $\mathcal{A}(\tau,\eta)$ is diagonalizable. However, this is the genuine case for the Euler system. Once we get $E_-(\tau,\eta)$ this way, then by analytical continuation to the only exception, that is, glancing point where $\mathcal{A}(\tau,\eta)$ has a Jordan block (see below), we get $E_-(\tau,\eta)$ for all $\operatorname{Re}\tau>0$ and hence, by constant hyperbolicity of Euler system, to $\operatorname{Re}\tau\geq 0$.

[12] The first comes from the first equation in (4.4), replacing ω by $\tilde\tau/u$. The second comes from the second, or the third equation in (4.4). Since here are two constraints, the corresponding space is of dimension $d+2-2=d$.

[13] This is obtained by substituting (4.6) into the first equation in (4.4).

and the eigenvector given by [14]

$$\begin{cases} (\tilde{\tau} - u\omega)\dot{\check{\mathbf{u}}} + vp'_v \mathrm{i}\dot{v}\boldsymbol{\eta} = 0, \\ (\tilde{\tau} - u\omega)\dot{u} - vp'_v \omega\dot{v} = 0, \\ \dot{s} = 0. \end{cases} \tag{4.6}$$

We see that the dispersion equation for the unknown ω also reads

$$\boxed{(\tilde{\tau} - u\omega)^2 = c^2(\omega^2 - |\boldsymbol{\eta}|^2),} \tag{4.7}$$

which has no purely imaginary root when $\mathrm{Re}\,\tau > 0$. [15] By looking at the easier case $\boldsymbol{\eta} = 0$ and using our usual continuity argument, we find that because of the subsonic condition (4.2), (4.7) has exactly one root of positive real part ($\tilde{\tau}/(u+c)$ when $\boldsymbol{\eta} = 0$), which we denote by ω_+, and one root of negative real part ($\tilde{\tau}/(u-c)$ when $\boldsymbol{\eta} = 0$), ω_-. By definition, the stable subspace $E_-(\tau, \boldsymbol{\eta})$ involved in the K-L condition is made of normal modes with $\mathrm{Re}\,\omega > 0$ for $\mathrm{Re}\,\tau > 0$. (Recall that — with our notation — decaying modes at $z = +\infty$ are obtained for $\mathrm{Re}\,\omega > 0$.) So the root ω_- does not contribute to $E_-(\tau, \boldsymbol{\eta})$, and we only need to consider ω_+, and furthermore, **if** $u > 0$, $\omega_0 \doteq \tilde{\tau}/u$ (recall that for the case considered in (4.5), where if $u > 0$, then $\mathrm{Re}\,\tau > 0 \Leftrightarrow \mathrm{Re}\,\omega > 0$ as required). To simplify again the writing, we simply denote ω_+ by ω when no confusion is possible.

4.1.1. *Outflow subsonic case.* If $u < 0$ (outflow case), $E_-(\tau, \boldsymbol{\eta})$ is a line, spanned by the solution $\mathbf{e}(\tau, \boldsymbol{\eta}) = (\dot{v}, \dot{\check{\mathbf{u}}}, \dot{u}, \dot{s})^\top$ of (4.6), defined by

$$\mathbf{e}(\tau, \boldsymbol{\eta}) \doteq \begin{pmatrix} v(\tilde{\tau} - u\omega) \\ \mathrm{i}c^2\boldsymbol{\eta} \\ -c^2\omega \\ 0 \end{pmatrix}. \tag{4.8}$$

4.1.2. *Inflow subsonic case.* If $u > 0$ (inflow case), $E_-(\tau, \boldsymbol{\eta})$ is a hyperplane ($\dim E_-(\tau, \boldsymbol{\eta}) = d + 1$). For convenience, we introduce the additional notation

$$a \doteq u\tilde{\tau} + \omega(c^2 - u^2). \tag{4.9}$$

An elementary manipulation of (4.7) then shows that [16]

$$a(\tilde{\tau} - u\omega) = c^2(\tilde{\tau}\omega - u|\boldsymbol{\eta}|^2).$$

[14] These comes from the last three equations in (4.4). Since here are $d + 1$ constraints, the dimension of solution space is then 1.

[15] If not, then the right-hand side is negative, so $\tilde{\tau} - u\omega$ should be purely imaginary. Since ω is purely imaginary as assumed, $\tilde{\tau}$ must be purely imaginary, contradicts to the assumption $\mathrm{Re}\,\tau > 0$.

[16] Note that $a = u(\tilde{\tau} - \omega u) + \omega c^2$.

Using this relation and combining (4.5) and (4.6) together, we get the very simple description [17]

$$E_-(\tau,\boldsymbol{\eta}) = \ell(\tau,\boldsymbol{\eta})^\perp,$$
$$\ell(\tau,\boldsymbol{\eta}) \doteq (a,\ -\mathrm{i}vu\boldsymbol{\eta}^\top,\ v\tilde{\tau},\ ap'_s/p'_v).$$

Observe that ℓ is homogeneous degree 1 in $(\tau,\boldsymbol{\eta})$ like a; while for a, $\tilde{\tau}$ and ω are homogeneous degree 1 for the variables $(\tau,\boldsymbol{\eta})$, and ω is determined by $\tilde{\tau},\boldsymbol{\eta}$. This description has the advantage of unifying the treatment of regular points and Jordan points $\tilde{\tau} = u|\boldsymbol{\eta}|$ of the matrix $\mathcal{A}(\tau,\boldsymbol{\eta})$ [18] — where ω coincides with ω_0, see (4.7).

Remark 4.1. In the particular "one-dimensional" case, i.e. with $\boldsymbol{\eta} = \mathbf{0}$ (no tangential variables), one easily checks that

$$\omega = \frac{\tau}{u+c}, \quad a = \tau c, \quad \ell(\tau,\mathbf{0}) = \tau(c,\ v,\ cp'_s/p'_v).$$

4.2. **Derivation of the Lopatinskii determinant.** Once we have description of $E_-(\tau,\boldsymbol{\eta})$, we easily arrive at the Lopatinskii condition. We consider the two cases separately.

4.2.1. *Outflow subsonic case.* If $u < 0$, one boundary condition $b(v,\mathbf{u},s)$ is required. The existence of nontrivial modes in the line $E_-(\tau,\boldsymbol{\eta})$ is thus equivalent to

$$\Delta(\tau,\boldsymbol{\eta}) \doteq \mathrm{d}b\cdot\mathbf{e}(\tau,\boldsymbol{\eta}) = 0.$$

Recall that $\mathrm{d}b$ here is the gradient of b with respect to the variables (v,u,s). We see in particular that this condition does not depend on $\partial b/\partial s$. By definition of $\mathbf{e}(\tau,\boldsymbol{\eta})$,

$$\Delta(\tau,\boldsymbol{\eta}) = v(\tilde{\tau}-u\omega)\frac{\partial b}{\partial v} + \mathrm{i}c^2\mathrm{d}_{\check{\mathbf{u}}}b\cdot\boldsymbol{\eta} - c^2\omega\frac{\partial b}{\partial u}.$$

We thus recover (as pointed out in §3) that prescribing the pressure ensures the uniform Lopatinskii condition, since for $b(v,\mathbf{u},s) = p(v,s)$, we have [19]

$$\Delta(\tau,\boldsymbol{\eta}) = v(\tilde{\tau}-u\omega)p'_v \neq 0 \quad \text{for} \quad \operatorname{Re}\tau \geq 0, (\tau,\boldsymbol{\eta}) \neq (0,\mathbf{0}).$$

[17] First consider (4.5). Since we have only two equations, so E_- corresponding to ω_0 is of dimension $d+2-2 = d$. We easily check that it is perpendicular to ℓ, thanks to (4.8). Then consider (4.6). Here we have $d-1+2 = d+1$ equations, hence E_- corresponds to ω_+ is of dimension $d+2-d-1 = 1$. Using (4.6), or (4.8) we see that it is perpendicular to ℓ, that is, $a\dot{v} - \mathrm{i}vu\boldsymbol{\eta}\cdot\dot{\check{\mathbf{u}}} + v\tilde{\tau}\dot{u} + ap'_s/p'_v\dot{s} = 0$.

[18] We did not take the other possibility $\tilde{\tau} = -u|\boldsymbol{\eta}|$ when $\omega_+ = \omega_0$, because it is required that $\operatorname{Re}\tilde{\tau} > 0$. At the point $\tilde{\tau} = u|\boldsymbol{\eta}|$, the solutions are $\omega_- = -\frac{c^2+u^2}{c^2-u^2}|\boldsymbol{\eta}|$ and $\omega = \omega_0 = |\boldsymbol{\eta}|$. However, this is not what happens by extending to $\operatorname{Re}\tau = 0$. The dimension of E_- is still $d+1$ in this case; It follows from Hersh Theorem, by analyticity of eigenvalues of ω_+ on $(\tau,\boldsymbol{\eta})$ for $\operatorname{Re}\tau > 0$. This can not be checked directly by (4.5), as $\mathbf{e}(\tau|_{\tilde{\tau}=u|\boldsymbol{\eta}|},\boldsymbol{\eta}) = (0,\mathrm{i}c^2\boldsymbol{\eta},-c^2|\boldsymbol{\eta}|,0)^\top$, which does satisfy (4.5).

[19] Note, if $\tilde{\tau}-u\omega = 0$, then (4.7) implies $\omega = \pm|\boldsymbol{\eta}|$. Since $u < 0$, for $\operatorname{Re}\tau \geq 0$, we should take $\omega = -|\boldsymbol{\eta}|$. However, for a nontrivial mode, we require $\omega > 0$.

This is less obvious with the alternative boundary condition $b(v,\mathbf{u},s)=u$, because in this case

$$\Delta(\tau,\boldsymbol{\eta})=-c^2\omega,$$

and it demands a little effort to check that ω does not vanish. For clarity, we state this point in the following.

Proposition 4.1. *For $0>u>-c$, the root ω_+ of (4.7) that is of positive real part for $\operatorname{Re}\tilde{\tau}>0$ has a continuous extension to $\operatorname{Re}\tilde{\tau}=0$ that does not vanish for $(\tau,\boldsymbol{\eta})\neq(0,0)$.*

Proof. **1.** We can solve from (4.7) that the root ω to be

$$\omega=\frac{-u\tilde{\tau}\pm c\sqrt{\tilde{\tau}^2+(c^2-u^2)|\boldsymbol{\eta}|^2}}{c^2-u^2}.$$

The point is to determine which one is ω_+. We see by (4.7), the only points where it could happen that ω_+ vanishes are such that $\tilde{\tau}^2=-c^2|\boldsymbol{\eta}|^2$. In particular, $\omega_+=0$ implies $\tilde{\tau}\in\mathrm{i}\mathbb{R}$, and also $-\tilde{\tau}^2\geq(c^2-u^2)|\boldsymbol{\eta}|^2$.

2. Set $\tilde{\tau}=\mu+\mathrm{i}\rho$. We have $\tilde{\tau}^2+(c^2-u^2)|\boldsymbol{\eta}|^2=\mu^2+(-\rho^2+(c^2-u^2)|\boldsymbol{\eta}|^2)+2\mathrm{i}\mu\rho$. Since as $\mu=0$, $(-\rho^2+(c^2-u^2)|\boldsymbol{\eta}|^2)<0$, so for $\mu>0$ small, we see $\operatorname{Re}(\tilde{\tau}^2+(c^2-u^2)|\boldsymbol{\eta}|^2)<0$, and $\operatorname{Im}(\tilde{\tau}^2+(c^2-u^2)|\boldsymbol{\eta}|^2)$ has the same sign as $\rho=\operatorname{Im}\tilde{\tau}$. So we should take [20]

$$\omega_\pm=\frac{-u\tilde{\tau}\pm\mathrm{i}c\ \mathrm{sign}(\operatorname{Im}\tilde{\tau})\ \sqrt{-\tilde{\tau}^2-(c^2-u^2)|\boldsymbol{\eta}|^2}}{c^2-u^2}. \tag{4.10}$$

3. So for $\tilde{\tau}^2=-c^2|\boldsymbol{\eta}|^2$, this gives (using the fact that u is negative)

$$\omega_-=0,\quad \omega_+=\frac{-2u\tilde{\tau}}{c^2-u^2},$$

the latter being non-zero unless $(\tau,\boldsymbol{\eta})=(0,0)$. ∎

More generally, we can find alternative boundary conditions that satisfy the uniform Lopatinskii condition without being dissipative. For instance, take $\alpha\in(0,1)$ and

$$b(v,\mathbf{u},s)=\frac{\alpha}{2}u^2+h(p(v,s),s),$$

[20] Taking ω_+ as a complex-valued function of $\tilde{\tau}\in\mathbb{C}$. By the Cauchy-Riemann equations, we have

$$\frac{\partial\operatorname{Im}\omega_+}{\partial\operatorname{Im}\tilde{\tau}}=\frac{\partial\operatorname{Re}\omega_+}{\partial\operatorname{Re}\tilde{\tau}}\geq 0.$$

The second is nonnegative, since as $\operatorname{Re}\tilde{\tau}>0$, we should have $\operatorname{Re}\omega_+>0$ (it is zero at $\operatorname{Re}\tilde{\tau}=0$). This may help us get directly the expression.

where $h = e + pv$ is the specific enthalpy. We find that, [21]

$$\Delta(\tau,\boldsymbol{\eta}) = -c^2(\tilde{\tau} - (1-\alpha)u\omega) \neq 0 \quad \text{for} \quad \operatorname{Re}\tau \geq 0, \quad (\tau,\boldsymbol{\eta}) \neq (0,0).$$

This means the uniform Lopatinskii condition is satisfied. Nevertheless, the quadratic form defined in (3.4) may be non-definite on the tangent hyperplane $\{\alpha u\dot{u} + v\dot{p} + T\dot{s} = 0\}$. More specifically, this happens for $\alpha \in (\frac{1}{1+\sqrt{1-M^2}}, 1)$.

4.2.2. *Inflow case.* If $u > 0$, $(d+1)$ boundary conditions $b_1(v,\mathbf{u},s), \cdots, b_{d+1}(v,\mathbf{u},s)$ are needed. The existence of nontrivial normal modes in the hyperplane $E_-(\tau,\boldsymbol{\eta})$ is thus equivalent to [22]

$$\Delta(\tau,\boldsymbol{\eta}) \doteq \det \begin{pmatrix} \mathrm{d}b_1 \\ \vdots \\ \mathrm{d}b_{d+1} \\ \ell(\tau,\boldsymbol{\eta}) \end{pmatrix} = 0.$$

We may consider, for example, as the first d conditions

$$b_1 = \check{\mathbf{u}}_1, \ \cdots, \ b_{d-1} = \check{\mathbf{u}}_{d-1}, \ b_d = s.$$

Then, up to a minus sign, [23]

$$\Delta(\tau,\boldsymbol{\eta}) = -v\tilde{\tau}\frac{\partial b_{d+1}}{\partial v} + a\frac{\partial b_{d+1}}{\partial u}.$$

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[21]Recall that $u < 0$ and $0 < \alpha < 1$, this is clear for $\operatorname{Re}\tau > 0$ (hence $\operatorname{Re}\omega > 0$). For $\operatorname{Re}\tau = 0$, we have $\tilde{\tau} = \mathrm{i}\rho$, then by (4.10), $\Delta(\tau,\boldsymbol{\eta}) = -\frac{\mathrm{i}c^2}{c^2-u^2}[\rho(c^2-\alpha u^2) - u(1-\alpha)c\operatorname{sign}(\rho)\sqrt{\rho^2-(c^2-u^2)|\boldsymbol{\eta}|^2}]$, which has nontrivial real parts if $\rho^2 < (c^2-u^2)|\boldsymbol{\eta}|^2$. For $\rho \geq \sqrt{c^2-u^2}|\boldsymbol{\eta}|$, the term in the brackets $[\cdot]$ is also positive. For $\rho \leq -\sqrt{c^2-u^2}|\boldsymbol{\eta}|$, the term in the brackets $[\cdot]$ is negative.

[22]The boundary matrix is $B = \begin{pmatrix} \mathrm{d}b_1 \\ \mathrm{d}b_2 \\ \vdots \\ \mathrm{d}b_{d+1} \end{pmatrix}$. Existence of nontrivial modes means that there is a nonzero vector $\mathbf{e}$ in $\ker B \cap E_-(\tau,\boldsymbol{\eta})$. So $B\mathbf{e} = 0$ and $\ell \cdot \mathbf{e} = 0$. Hence $\Delta = 0$. On the contrary, $\Delta = 0$ implies a nonzero $\mathbf{e}$ with the above property by linear algebra.

[23]In this case, $\Delta(\tau,\boldsymbol{\eta}) = \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ & \cdots & & \cdots & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ \frac{\partial b_{d+1}}{\partial v} & * & * & \cdots & * & \frac{\partial b_{d+1}}{\partial u} & * \\ a & * & * & \cdots & * & v\tilde{\tau} & * \end{vmatrix}$. Then applying the Laplace expansion formula for determinant to the first d rows.

In particular, for

$$b_{d+1}(v,\mathbf{u},s)=\frac{\alpha}{2}u^2+h(p(v,s),s),$$

we have [24]

$$\Delta(\tau,\boldsymbol{\eta})\doteq(c^2+\alpha u^2)\tilde{\tau}+\alpha u\omega(c^2-u^2)\neq 0\quad\text{for}\quad\operatorname{Re}\tau\geq 0,(\tau,\boldsymbol{\eta})\neq(0,0),$$

provided that $\alpha>0$. For α large enough, this gives again an example of boundary conditions satisfying the uniform Lopatinskii condition without being dissipative.

We note that if $b_{d+1}=p$, then as $p=A(s)v^{-\gamma}$ for polytropic gas, we find

$$\Delta(\tau,\boldsymbol{\eta})=\gamma p\bar{\tau}.$$

So this boundary condition, giving pressure at the subsonic inlet, violates uniform Lopatinskii condition at $\bar{\tau}=0$, that is, at $\tau=-\mathrm{i}\eta\check{\mathbf{u}}$ (since $\bar{\tau}=\tau+\mathrm{i}\boldsymbol{\eta}\cdot\check{\mathbf{u}}$). However, the Lopatinskii condition still holds.

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[24]As $h=c^2/(\gamma-1)=\gamma Av^{1-\gamma}/(\gamma-1)$, one has $\partial h/\partial v=-\gamma p$. Since $c^2-u^2>0$, $\operatorname{Re}\tau=\operatorname{Re}\tilde{\tau}>0$, $\operatorname{Re}\omega>0$ (as proved in Proposition 4.1), so the sum cannot be zero for $\operatorname{Re}\tau>0$. For the case that $\tilde{\tau}=\omega u$, $\Delta(\tau,\boldsymbol{\eta})=(1+\alpha)c^2\tilde{\tau}\neq 0$. For the case ω given by (4.10), with $\tilde{\tau}=\mathrm{i}\rho$, we have $\Delta(\tau,\boldsymbol{\eta})=\mathrm{i}[\rho c^2+\alpha uc\operatorname{sign}(\rho)\sqrt{\rho^2-(c^2-u^2)|\boldsymbol{\eta}|}]$, which is nonzero for $(\rho,\boldsymbol{\eta})\neq(0,0)$.

LECTURE NOTES 5: PSEUDO-DIFFERENTIAL CALCULUS AND ESTIMATES OF LINEAR HYPERBOLIC OPERATORS WITH SMOOTH COEFFICIENTS

HAIRONG YUAN

To obtain L^2 estimates for hyperbolic systems, the key point is that such system should be "symmetric" — therefore using integration by parts, a derivative can be thrown to the coefficients. It is not always possible to find a matrix-valued function that symmetrizes a generally given hyperbolic system. As a generalization, taking the system as an operator, we find if there are operators (called functional symmetrizers) that symmetrize the hyperbolic operator, we can still obtain L^2 estimates. Such symmetrizers can be realized as pseudo-differential operators corresponding to certain symbolic symmetrizers. Symbolic symmetrizers are matrices depending on all time, space, and frequency variables. We show for constantly hyperbolic operators, such symbolic symmetrizers always exist.

Once we obtained L^2 estimates, it is rather easier to get H^s estimates, by using the pseudo-differential operator Λ^s with symbol $\lambda^s(\xi) = (1 + |\xi|^2)^{s/2}$. Here we usually need some estimates of commutators, which is often pseudo-differential operators of lower order.

The disadvantage of pseudo-differential operators is that their symbols must be smooth functions, and the related operator norms may depend on lots of orders of the derivatives of the symbols. [1] So it can be hardly used to deal with systems with less-regular coefficients. [2] However, by linearization of nonlinear problems, the obtained system is often with non-smooth coefficients. Therefore, as a generalization, people introduced para-differential calculus, which will be introduced later. However, the basic idea of how to obtain estimate is the same.

In this note we first review basic notions and results on pseudo-differential calculus, and then applying this theory to derive H^s estimates for the Cauchy problem of linear hyperbolic systems under the assumption that the coefficients are smooth. The note is totally based on Appendix C.3 and Section 2.1 of [1]. Another excellent reference is [2].

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[1] A careful check of the proof of boundedness of pseudo-differential operators on Sobolev spaces shows that, if $a(x,\xi) \in \mathbf{S}^{-d-1}$, $\|\mathrm{Op}(a)\|_{\mathscr{B}(L^2)}$ depends only on decay of $\partial_\xi^\beta a(x,\xi)$ for $|\beta| \le d$ (d is the space dimension), and for $a \in \mathbf{S}^{-1}$, as $\mathrm{Op}(a)^k \in \mathrm{OPS}^{-d-1}$ for large k, we get $\|\mathrm{Op}(a)\|_{\mathscr{B}(L^2)}$. But as the symbol of $\mathrm{Op}(a)^k$, given by asymptotic expansion through a, actually depends on infinite number of derivatives $\partial_x^\alpha \partial_\xi^\beta a(x,\xi)$. So smoothness and decay in the definition of symbols are quite essential to this theory.

[2] Such partial differential operators can not be regarded as pseudo-differential operators.

1. Review: Pseudo-differential calculus

1.1. Symbols and approximate symbols.

Definition 1.1. For any real number m, we define the set $\mathbf{S}^m$ of functions $a \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{N\times N})$ such that for all d-uples α and β, there exists $C_{\alpha,\beta} > 0$ so that

$$\boxed{\left\|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right\| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}.} \tag{1.1}$$

Functions [3] belonging to $\mathbf{S}^m$ are called symbols of order m. The set of symbols of all orders is

$$\mathbf{S}^{-\infty} \doteq \bigcap_m \mathbf{S}^m.$$

Basic examples

Example 1.1 (Differential symbols)**.** Functions of the form

$$a(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)(\mathrm{i}\xi)^\alpha,$$

where all the coefficients a_α are $\mathscr{C}^\infty$ and bounded, as well as all their derivatives, belong to $\mathbf{S}^m$.

Example 1.2 ("Homogeneous" functions)**.** A function $a \in \mathscr{C}^\infty(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}))$ that is bounded as well as all its derivatives in x and homogeneous degree m in ξ is "almost" a symbol of order m. This means it becomes a symbol provided that we remove the singularity at $\xi = 0$. As a matter of fact, considering a $\mathscr{C}^\infty$ function χ vanishing in a neighborhood of 0 and such that $\chi(\xi) = 1$ for $|\xi| \geq 1$, we have the result that

$$\tilde{a}(x,\xi) = \chi(\xi)a(x,\xi) \in \mathbf{S}^m.$$

Any other symbol constructed in this way differs from $\tilde{a}$ by a symbol in $\mathbf{S}^{-\infty}$. For convenience we shall denote $\dot{\mathbf{S}}^m$ the set of such functions a.

Example 1.3 (Sobolev symbols)**.** Some special symbols are extensively used in the theory, which we refer to as Sobolev symbols since they are naturally involved in Sobolev norms. Denoting

$$\lambda^s(\xi) = (1+|\xi|^2)^{s/2},$$

it is easily seen that λ^s is a symbol of order s. The important point is that the Sobolev space H^s can be equipped with the norm

$$\|u\|_{H^s} = \|\lambda^s \hat{u}\|_{L^2}.$$

[3]Be careful these functions may be considered as matrix-valued.

The following lemma is used to prove Gårding inequality later.

Lemma 1.1. *For all $a \in \mathbf{S}^0$ (respectively $a \in \dot{\mathbf{S}}^0$), such that $a(x,\xi)$ is Hermitian and uniformly positive-definite for $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$ (resp. $(x,\xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$), there exists $b \in \mathbf{S}^0$ (resp. $b \in \dot{\mathbf{S}}^0$), such that $b = b^*$ and $b(x,\xi)^* b(x,\xi) = a(x,\xi)$.*

Proof. The proof proceeds in the same way for both cases ($a \in \mathbf{S}^0$ or $a \in \dot{\mathbf{S}}^0$).

1. By assumption of (1.1), $a(x,\xi)$ lies in a bounded subset of the cone of Hermitian positive-definite matrices and thus the set of eigenvalues of $a(x,\xi)$ is included in some real interval $[\alpha,\beta] \subset (0,\infty)$. In particular, there exists a positively orientated contour Γ lying in $\mathbb{C} \setminus (-\infty, 0]$ that is symmetric with respect to the real axis and contains $[\alpha,\beta]$ in its interior. Therefore, considering the holomorphic complex square root $\sqrt{\cdot}$ in $\mathbb{C} \setminus (-\infty, 0]$, the Dunford-Taylor Integral

$$b(x,\xi) \doteq \frac{1}{2\mathrm{i}\pi} \int_\Gamma \sqrt{z}\big(zI_N - a(x,\xi)\big)^{-1} \,\mathrm{d}z$$

answers the equation.

2. We first show, by the symmetry of Γ, $b(x,\xi)$ is Hermitian. As a matter of fact, as $\overline{\sqrt{z}} = \sqrt{\bar{z}}$ in our case, we have

$$\begin{aligned}
b(x,\xi)^* &= -\frac{1}{2\mathrm{i}\pi} \int_\Gamma \overline{\sqrt{z}}(\bar{z}I_N - a(x,\xi)^*)^{-1} \,\overline{\mathrm{d}z} \\
&= -\frac{1}{2\mathrm{i}\pi} \int_{-\Gamma} \sqrt{z'}(z'I_N - a(x,\xi))^{-1} \,\mathrm{d}z' \qquad (z' = \bar{z}) \\
&= \frac{1}{2\mathrm{i}\pi} \int_\Gamma \sqrt{z'}(z'I_N - a(x,\xi))^{-1} \,\mathrm{d}z' = b(x,\xi).
\end{aligned}$$

3. For another contour Γ' enjoying the same properties as Γ and containing it in its interior, we have

$$b^*b = \frac{-1}{4\pi^2} \int_{\Gamma'} \int_\Gamma \sqrt{z}\sqrt{z'}(zI_N - a)^{-1}(z'I_N - a)^{-1} \,\mathrm{d}z\mathrm{d}z'.$$

By the well-known resolvent equation

$$(zI_N - a)^{-1} - (z'I_N - a)^{-1} = (z' - z)(zI_N - a)^{-1}(z'I_N - a)^{-1}$$

(checked easily by writing $(z - z')I_N = (zI_N - a) - (z'I_N - a)$), we thus have

$$b^*b = \frac{1}{(2\mathrm{i}\pi)^2} \int_{\Gamma'} \int_\Gamma \sqrt{z}\sqrt{z'} \frac{(zI_N - a)^{-1} - (z'I_N - a)^{-1}}{z' - z} \,\mathrm{d}z\mathrm{d}z'.$$

On the other hand, for $z' \in \Gamma'$, the function $z \mapsto \sqrt{z}/(z'-z)$ is holomorphic in the interior of Γ, thus

$$\int_\Gamma \frac{\sqrt{z}}{z'-z}\,\mathrm{d}z = 0.$$

Hence we obtain

$$\begin{aligned}
b^*b &= \frac{1}{(2\mathrm{i}\pi)^2}\int_{\Gamma'}\int_\Gamma \sqrt{z}\sqrt{z'}\frac{(zI_N-a)^{-1}}{z'-z}\,\mathrm{d}z\mathrm{d}z' \\
&= \frac{1}{(2\mathrm{i}\pi)}\int_\Gamma \sqrt{z}(zI_N-a)^{-1}\left(\frac{1}{(2i\pi)}\int_{\Gamma'}\frac{\sqrt{z'}}{z'-z}\,\mathrm{d}z'\right)\mathrm{d}z \\
&= \frac{1}{(2\mathrm{i}\pi)}\int_\Gamma z(zI_N-a)^{-1}\,\mathrm{d}z \\
&= a.
\end{aligned}$$

For the third and last equality, we used Cauchy Integration Formula for holomorphic function.

4. In view of smoothness of the mapping $(z,a)\mapsto(z-a)^{-1}$, it is clear by Chain Rule and Lebesgue's Convergence Theorem, b is as smooth as a. By an induction on derivatives of composite functions $\partial_\xi^n F(u(\xi))$, we may also prove b satisfies the estimate (1.1) with $m=0$, if $a\in\mathbf{S}^0$.

5. If $a\in\dot{\mathbf{S}}^0$, it is obvious that b is also homogeneous of degree 0 in ξ. □

1.2. **Definition of pseudo-differential operators.** The introduction of pseudo-differential operators is based on the following observation. If $a\in\mathbf{S}^m$ is polynomial in ξ, like in Example 1.1, it is naturally associated with the differential operator [4]

$$\mathrm{Op}(a) = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha$$

in the sense that

$$(\mathrm{Op}(a)u)(x) \doteq \mathscr{F}^{-1}(a(x,\cdot)\hat{u}(\cdot))$$

for all $u\in\mathcal{S}=\mathcal{S}(\mathbb{R}^n)$ (the class of fast decreasing Schwarz functions on $\mathbb{R}^n$) and $x\in\mathbb{R}^d$. But this formula can be used to define operators associated with more general symbols. This is the purpose of the following.

Proposition 1.1. *Let a be a symbol of order m. Then there exists a continuous linear operator on $\mathcal{S}$, denoted by* $\mathrm{Op}(a)$, *such that*

$$(\mathrm{Op}(a)u)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}x\cdot\xi}a(x,\xi)\hat{u}(\xi)\,\mathrm{d}\xi \tag{1.2}$$

for all $u\in\mathcal{S}$. Furthermore, the mapping $a\mapsto\mathrm{Op}(a)$ is one-to-one.

[4]Note for $\alpha=(\alpha_1,\cdots,\alpha_d)$, we write $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}}$.

The proof can be found in [4, p.22, Proposition 3.1].

Observe that for "constant-coefficient operators", that is, symbols independent of x, (2.2) reduces to the same formula as for the differential operators (Fourier multiplier)

$$\mathrm{Op}(a)u = \mathscr{F}^{-1}(a\hat{u}).$$

In short, for symbols a depending only on ξ, we have by definition

$$\mathrm{Op}(a) = \mathscr{F}^{-1}a\mathscr{F}.$$

Definition 1.2. The set of pseudo-differential operators of order m is [5]

$$\boxed{\mathbf{OPS}^m \doteq \{\mathrm{Op}(a) : \ a \in \mathbf{S}^m\} \subset \mathscr{B}(\mathcal{S}).}$$

For $a \in \mathbf{S}^m$, the operator $\mathrm{Op}(a)$ is called a pseudo-differential operator of order m and symbol a.

It is more subtle to show that pseudo-differential operators extend to operators on $\mathcal{S}'$ (the space of tempered distribution). By a standard duality argument, this amounts to showing the following.

Theorem 1.1. *The adjoint of a pseudo-differential operator of order m is a pseudo-differential operator of order m. Furthermore, the symbol of the adjoint operator $\mathrm{Op}(a)^*$ differs from a^* (where $a^*(x,\xi) = a(x,\xi)^*$ merely in the sense of matrices) by a symbol of order $m-1$, which means that*

$$\boxed{(\mathrm{Op}(a))^* - \mathrm{Op}(a^*) \in \mathbf{OPS}^{m-1}} \tag{1.3}$$

for all $a \in \mathbf{S}^m$.

For a proof, see Proposition 6.4, Corollary 6.5 in [4, pp.33-35].

1.3. **Basic properties of pseudo-differential operators.** The first important property of pseudo-differential operators is the following, considering boundedness acting on chains of Sobolev spaces.

Theorem 1.2. *Let P be a pseudo-differential operator of order m, extended to $\mathcal{S}'$ by the formula*

$$\langle Pu, \phi\rangle = \langle u, P^*\phi\rangle$$

for all $u \in \mathcal{S}'$ and $\phi \in \mathcal{S}$. Then, for all $s \in \mathbb{R}$, P belongs to $\mathscr{B}(H^s; H^{s-m})$.

For a proof, see Proposition 9.12 in [4, p.43].

[5] $\mathscr{B}(\mathcal{S})$ is the set of bounded linear operators on $\mathcal{S}$.

Example 1.4. For all $s \in \mathbb{R}$, let Λ^s denote pseudo-differential operator of symbol $\lambda^s(\xi) = (1+|\xi|^2)^{s/2}$. Then for all real numbers s we have

$$\|u\|_{H^s} = \|\Lambda^s u\|_{L^2}\,.$$

Therefore, we have for all $m \in \mathbb{R}$

$$\|\Lambda^m u\|_{H^{s-m}} = \left\|\Lambda^{s-m}\Lambda^m u\right\|_{L^2} = \|\Lambda^s u\|_{L^2} = \|u\|_{H^s}\,.$$

Theorem 1.3. *If P and Q are pseudo-differential operators of order m and n respectively, then*

i) *the composed operator PQ is a pseudo-differential operator of order $m+n$, and its symbol differs from the product of symbols by a lower-order term, which means that*

$$\mathrm{Op}(a)\mathrm{Op}(b) - \mathrm{Op}(ab) \in \mathbf{OPS}^{m+n-1} \tag{1.4}$$

for all $a \in \mathbf{S}^m$ and $b \in \mathbf{S}^n$.

ii) **if one of the operators is scalar-valued,** *the commutator*

$$[P,Q] = PQ - QP$$

is of order $m+n-1$ (and its symbol differs from the Poisson bracket of symbols

$$\{a,b\} = \sum_j \frac{\partial a}{\partial \xi_j}\frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial b}{\partial \xi_j}$$

by a lower-order term).

For a proof of i), see Proposition 5.4 in [4, p.31]. Claim ii) follows from i), and the asymptotic expansion given in this cited proposition. It will be used to derive a priori estimates in Sobolev spaces H^s, with $Q = \Lambda^s$.

1.4. **Gårding Inequality.** Finally, other important results are the Gårding Inequality, which relates the positivity of an operator (up to a lower-order error) to the positivity of its symbol, and the sharp form of Gårding Inequality, which applies to non-negative symbols. We begin with the standard form of Gårding inequality (for matrix-valued symbols) and its elementary proof.

Theorem 1.4 (Gårding Inequality)**.** *If A is a pseudo-differential operator of symbol $a \in \mathbf{S}^m$, or A is associated with $\dot{\mathbf{S}}^m$ by a low-frequency cut-off, such that for some positive number α,*

$$a(x,\xi) + a(x,\xi)^* \geq \alpha\lambda^m(\xi) I_N, \quad \lambda^m(\xi) = (1+|\xi|^2)^{m/2}$$

(in the sense of Hermitian matrices) for all $x \in \mathbb{R}^d$ and $|\xi|$ large, then there exists C so that

$$\mathrm{Re}\,(Au, u) \geq \frac{\alpha}{4} \|u\|^2_{H^{m/2}} - C \|u\|^2_{H^{\frac{m}{2}-1}} \tag{1.5}$$

for all $u \in H^{m/2}$.

Proof. We may always define $\tilde{a}$ so that $\tilde{a} = a$ for $|\xi|$ large and $\tilde{a}(x,\xi)+\tilde{a}(x,\xi)^* \geq \alpha\lambda^m(\xi)I_N$ holds true for all ξ. Since $a - \tilde{a}$ has compact support in ξ, it belongs to $\mathbf{S}^{-\infty}$. Such a difference will not take difference to the result, as can be seen from the proof below. So we can assume $a(x,\xi) + a(x,\xi)^* \geq \alpha\lambda^m(\xi)I_N$ holds for all ξ in the following proof.

1. Case $m = 0$**.** We have $\mathrm{Re}\,(Au, u) = \mathrm{Re}\,(\frac{1}{2}(A + A^*)u, u)$, and by (2.3), we know that

$$A + A^* - \mathrm{Op}(a + a^*) = \mathrm{Op}(e') \in \mathbf{OPS}^{-1}.$$

Therefore, there exists $c > 0$ so that, for $u \in L^2$,

$$\begin{aligned} ((A + A^*)u, u) &= (\mathrm{Op}(a + a^*)u, u) + (\mathrm{Op}(e')u, u) \\ &\geq (\mathrm{Op}(a + a^*)u, u) - \|\mathrm{Op}(e')u\|_{L^2} \|u\|_{L^2} \\ &\geq (\mathrm{Op}(a + a^*)u, u) - c \|u\|_{H^{-1}} \|u\|_{L^2} \\ &\geq (\mathrm{Op}(a + a^*)u, u) - \frac{\alpha}{16} \|u\|^2_{L^2} - \frac{4c^2}{\alpha} \|u\|^2_{H^{-1}} . \end{aligned}$$

Then the result will be proved if we show that

$$(\mathrm{Op}(a + a^*)u, u) \geq \frac{9\alpha}{16} \|u\|^2_{L^2} - C \|u\|^2_{H^{-1}} .$$

In some sense this reduces the problem to Hermitian symbols.

2. By assumption, the Hermitian symbol $\tilde{a} = a + a^* - \alpha' I_N$, with $\alpha' = 3\alpha/4$, is positive-definite. By Lemma 1.1, there exists $b \in \mathbf{S}^0$ such that $b^*b = \tilde{a}$. Denoting $B = \mathrm{Op}(b)$ and $\tilde{A} = \mathrm{Op}(a + a^* - \alpha' I_N)$, we know from Theorem 1.1 and Theorem 2.2 i) that [6]

$$B^*B - \tilde{A} \in \mathbf{OPS}^{-1}.$$

Consequently, there exists $\tilde{c} > 0$ so that

$$\begin{aligned} (\tilde{A}u, u) &\geq (B^*Bu, u) - \tilde{c} \|u\|_{H^{-1}} \|u\|_{L^2} \geq \underbrace{\|Bu\|^2_{L^2}}_{\geq 0} - \frac{\alpha'}{4} \|u\|^2_{L^2} - \frac{\tilde{c}^2}{\alpha'} \|u\|^2_{H^{-1}} \\ &\geq -\frac{\alpha'}{4} \|u\|^2_{L^2} - \frac{\tilde{c}^2}{\alpha'} \|u\|^2_{H^{-1}} . \end{aligned}$$

[6] $BB^* - \tilde{A} = \mathrm{Op}(b)\mathrm{Op}(b)^* - \mathrm{Op}(\tilde{a}) = \mathrm{Op}(b)(\mathrm{Op}(b^*) + \mathbf{OPS}^{-1}) - \mathrm{Op}(\tilde{a}) = \mathrm{Op}(b)\mathrm{Op}(b^*) - \mathrm{Op}(\tilde{a}) + \mathbf{OPS}^{-1} = \mathrm{Op}(bb^*) - \mathrm{Op}(\tilde{a}) + \mathbf{OPS}^{-1} = \mathbf{OPS}^{-1}$. Notice that $bb^* = \tilde{a}$.

This implies

$$(\mathrm{Op}(a+a^*)u,u) \geq \frac{3\alpha'}{4}\|u\|_{L^2}^2 - \frac{\tilde{c}^2}{\alpha'}\|u\|_{H^{-1}}^2.$$

3. We then have the inequality (1.5), for $m=0$, with $C = 4(3c^2+\tilde{c}^2)/(3\alpha)$.

4. General case. We consider $B = \Lambda^{-m/2}A\Lambda^{-m/2}$, which is of order 0, and its symbol $s(B)$ satisfies

$$s(B) - \lambda^{-m/2}a\lambda^{-m/2} = e \in \mathbf{S}^{-1}.$$

So

$$s(B)+s(B)^* = \lambda^{-m/2}(a+a^*)\lambda^{-m/2} + e + e^* \geq \alpha I_N + (e+e^*).$$

Since $e, e^* \in \mathbf{S}^{-1}$, it decays as $|\xi| \to \infty$, so we still has $s(B)+s(B)^* \geq \alpha I_N$ for large $|\xi|$, and we may use the proved result, for $u \in H^{m/2}$:

$$\begin{aligned}
\mathrm{Re}\,(Au,u) &= \mathrm{Re}\,(\Lambda^{m/2}B\Lambda^{m/2}u,u)\\
&= (B\Lambda^{m/2}u,\Lambda^{m/2}u)\\
&\geq \frac{\alpha}{4}\left\|\Lambda^{m/2}u\right\|_{L^2}^2 - C\left\|\Lambda^{m/2}u\right\|_{H^{-1}}^2\\
&= \frac{\alpha}{4}\|u\|_{H^{m/2}}^2 - C\|u\|_{H^{\frac{m}{2}-1}}^2
\end{aligned}$$

□

We complete this section by stating the Sharp Gårding Inequality, which amounts to allowing $\alpha = 0$ in the standard one. In other words, it shows that non-negative symbols imply a gain of derivatives: an operator of order m with non-negative symbol satisfies a lower bound as though it were of order $m-1$.

Theorem 1.5 (Sharp Gårding Inequality)**.** *If A is a pseudo-differential operator of symbol $a \in \mathbf{S}^m$, or A is associated with $a \in \dot{\mathbf{S}}^m$ by a low-frequency cut-off, such that*

$$a(x,\xi) + a(x,\xi)^* \geq 0$$

(in the sense of Hermitian matrices) for all $x \in \mathbb{R}^d$ and $|\xi|$ large, then there exists C so that

$$\mathrm{Re}\,(Au,u) \geq -C\|u\|_{H^{(m-1)/2}} \tag{1.6}$$

for all $u \in H^{m/2}$.

2. Symmetrizers and energy estimates

The purpose is to deal with linear variable-coefficients systems of the form

$$\frac{\partial u}{\partial t} + \sum_{\alpha=1}^{d} A^\alpha(x,t)\frac{\partial u}{\partial x_\alpha} = B(x,t)u + f(x,t), \tag{2.1}$$

where the $n \times n$ matrices A^α and B depend "smoothly" on (x,t). In the following, unless otherwise stated, it will be implicitly assumed that B and A^α are $\mathscr{C}^\infty$ (real) functions that are bounded as well as all their derivatives.

To simplify notations, we may alternatively write (1.1) as

$$\partial_t u = P(t)u + f,$$

where $P(t)$ is the spatial differential operator

$$P(t) : u \mapsto P(t)u = -\sum_{\alpha=1}^{d} A^\alpha(\cdot,t)\partial_\alpha u + B(\cdot,t)u. \tag{2.2}$$

Or, in short, (2.1) equivalently reads

$$Lu = f,$$

where L denotes the evolution operator

$$L : u \mapsto Lu \doteq \partial_t u - P(t)u. \tag{2.3}$$

2.1. **L^2 estimate for Friedrichs symmetrizable systems.** There is a special class of systems for which energy estimates are almost as natural as for scalar equations, or wave equations. This is the class of Friedrichs-symmetrizable systems, which fulfill the following definition.

Definition 2.1. The system (2.1) is Friedrichs-symmetrizable if there exists a $\mathscr{C}^\infty$ mapping $S_0 : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{M}_n(\mathbb{R})$, bounded as well as its derivatives, such that $S_0(x,t)$ is symmetric and uniformly positive-definite, and the matrices $S_0(x,t)A^\alpha(x,t)$ are symmetric for all (x,t).

Like scalar equations, **Friedrichs-symmetrizable systems enjoy a priori estimates that keep track of coefficients.** We give the L^2 estimates below, which is proved elementarily and will be extensively used in the nonlinear analysis on quasi-linear Cauchy problems.

Proposition 2.1. *Assume that* (2.1) *is Friedrichs-symmetrizable, with a symmetrizer* S_0 *satisfying*

$$\beta I_n \le S_0 \le \beta^{-1} I_n, \quad \beta > 0$$

in the sense of quadratic forms. We also assume that S_0, A^α, and their first derivatives are bounded, as well as B.

Then, for all $T>0$ and $u \in \mathscr{C}([0,T];\ H^1) \cap \mathscr{C}^1([0,T];\ L^2)$, we have

$$\beta^2 \|u(t)\|_{L^2}^2 \le \mathrm{e}^{\gamma t} \|u(0)\|_{L^2}^2 + \int_0^t \mathrm{e}^{\gamma(t-\tau)} \|Lu(\tau)\|_{L^2}^2 \, \mathrm{d}\tau, \qquad \forall t \in [0,T], \tag{2.4}$$

where L is defined as in (2.3) and γ is chosen to be large enough, so that

$$\beta(\gamma-1) \ge \left\| \partial_t S_0 + \sum_{\alpha=1}^d \partial_\alpha(S_0 A^\alpha) + S_0 B + B^\top S_0 \right\|_{L^\infty(\mathbb{R}^d \times [0,T])}. \tag{2.5}$$

We note, here we only need $S_0,\ A^\alpha, B$ and their first derivatives be bounded. This track of regularity of coefficients is important for applications to nonlinear problems.

Proof. **1.** Using integration by parts, and symmetry of $S_0,\ S_0A^\alpha$, we have

$$0 = \int_{\mathbb{R}^d} \partial_\alpha(S_0 A^\alpha u \cdot \bar{u}) \, \mathrm{d}x = (\partial_\alpha(S_0A^\alpha)u, u) + 2(S_0 A^\alpha \partial_\alpha u, u).$$

So we get the crucial identity [7]

$$\boxed{(S_0 A^\alpha \partial_\alpha u,\ u) = -\frac{1}{2}(\partial_\alpha(S_0A^\alpha)u,\ u),} \tag{2.6}$$

hence

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(S_0 u, u) &= (\partial_t S_0 u, u) + 2(S_0 \partial_t u, u) \\ &= 2(S_0 L u + S_0 B u - \sum_\alpha S_0 A^\alpha \partial_\alpha u, u) + (\partial_t S_0 u, u) \\ &= 2(S_0 u, Lu) - 2\sum_\alpha (S_0 A^\alpha \partial_\alpha u, u) + ((\partial_t S_0 + S_0 B + B^\top S_0)u, u) \\ &= 2(S_0 u, Lu) + (Ru, u), \end{aligned}$$

where

$$R \doteq \partial_t S_0 + S_0 B + B^\top S_0 + \sum_\alpha \partial_\alpha(S_0 A^\alpha).$$

[7] We see here symmetry helps us to throw a derivative to the coefficients and then such terms can be controlled by L^2 norm of the solution. Otherwise an L^2 estimate is impossible.

2. Integrating in time and using the Cauchy-Schwarz Inequality (for the inner product $(S_0\cdot,\cdot)$) and $2ab < a^2+b^2$, we arrive at

$$\begin{aligned}
(S_0(\cdot,t)u(t),u(t)) &\le (S_0(\cdot,0)u(0),u(0)) + 2\int_0^t (S_0(\cdot,\tau)u(\tau),Lu(\tau))\,\mathrm{d}\tau \\
&\quad + \|R\|_{L^\infty}\int_0^t \|u(\tau)\|_{L^2}\,\mathrm{d}\tau \\
&\le (S_0(\cdot,0)u(0),u(0)) + \int_0^t (S_0(\cdot,\tau)Lu(\tau),Lu(\tau))\,\mathrm{d}\tau \\
&\quad + \int_0^t (1+\beta^{-1}\|R\|_{L^\infty})(S_0(\cdot,\tau)u(\tau),u(\tau))\,\mathrm{d}\tau.
\end{aligned}$$

3. Then using Gronwall's Inequality, we get

$$(S_0u(t),u(t)) \le \mathrm{e}^{\gamma t}(S_0u(0),u(0)) + \int_0^t \mathrm{e}^{\gamma(t-\tau)}(S_0Lu(\tau),Lu(\tau))\,\mathrm{d}\tau,$$

with $\gamma \ge 1+\beta^{-1}\|R\|_{L^\infty}$. This yields the final estimate after multiplication of β. □

2.2. H^s **estimates for systems admit functional symmetrizers.** More generally, a priori estimates hold true for systems admitting a *functional symmetrizer*, defined as follows.

Definition 2.2. Given a family of first-order (pseudo-)differential operators $\{P(t)\}_{t\ge 0}$ acting on functions defined on $\mathbb{R}^d$, a *functional symmetrizer* is a $\mathscr{C}^1$ mapping [8]

$$\Sigma : \mathbb{R}^+ \to \mathscr{B}(L^2(\mathbb{R}^d;\mathbb{C}^n))$$

such that, for $0\le t\le T$,

$$\Sigma(t) = \Sigma^*(t) \ge \alpha I_n \tag{2.7}$$

for some positive α depending only on T, and [9]

$$\mathrm{Re}\,(\Sigma P(t)) \doteq \frac{1}{2}(\Sigma P(t) + P(t)^*\Sigma) \in \mathscr{B}(L^2(\mathbb{R}^d)) \tag{2.8}$$

with a uniform bound of the operator norms on $[0,T]$.

Example 2.1. For a Friedrichs-symmetrizable system with symmetrizer S_0, the simple multiplication operator $\Sigma(t) : u \mapsto \Sigma(t)u \doteq S_0(\cdot,t)u$ is a functional symmetrizer (so

[8]This means both $\|\Sigma(t)\|_{L^2\to L^2}$ and $\left\|\frac{\mathrm{d}}{\mathrm{d}t}\Sigma(t)\right\|_{L^2\to L^2}$ are uniformly bounded for $t\in[0,T]$, with a bound depending only on T.

[9]Note formally $\mathrm{Re}\,(\Sigma P(t))$ is of first order, however, we require it to be zero order here. So symmetry help us for cancelation of one order, cf. (2.6). Recall here that A^* is the adjoint operator of A.

$\Sigma(t) = S_0(\cdot, t)$). As a matter of fact, (2.7) follows from the analogous property of the matrices $S_0(x,t)$. And, because the matrices $S_0(x,t)A^\alpha(x,t)$ are symmetric, $2\mathrm{Re}\,(\Sigma(t)P(t))$ reduces to the multiplication operator associated with [10]

$$\Big(\sum_\alpha \partial_\alpha(S_0A^\alpha) + S_0B + B^*S_0\Big)(\cdot,t).$$

Theorem 2.1. *If a family of operators $\{P(t)\}$ admits a functional symmetrizer, then, for all $s \in \mathbb{R}$ and $T > 0$, there exists $C > 0$ so that for $u \in \mathscr{C}^1([0,T]; H^s) \cap \mathscr{C}([0,T]; H^{s+1})$, we have*

$$\|u(t)\|_{H^s}^2 \le C\left(\|u(0)\|_{H^s}^2 + \int_0^t \|Lu(\tau)\|_{H^s}^2 \,\mathrm{d}\tau\right), \tag{2.9}$$

where L is defined by (2.3).

Proof. **Case** $s = 0$**.** From (2.7) we know that

$$(\Sigma(t)u(t), u(t)) \ge \alpha\,\|u(t)\|_{L^2}^2\,. \tag{2.10}$$

To bound the left-hand side we write (recall that $u_t = Lu + P(t)u$)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Sigma u, u) = 2\mathrm{Re}\,(\Sigma Lu, u) + 2\mathrm{Re}\,(\Sigma Pu, u) + (\frac{\mathrm{d}\Sigma}{\mathrm{d}t}u, u).$$

Each term here above can be estimated by using the Cauchy-Schwarz Inequality. For the first and last ones, we use uniform bounds in t of $\|\Sigma(t)\|_{\mathscr{B}(L^2)}$ and $\|\mathrm{d}\Sigma(t)/\mathrm{d}t\|_{\mathscr{B}(L^2)}$. For the middle term we use (2.8) and a uniform bound in t of $\mathrm{Re}\,(\Sigma P)$. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Sigma u, u) \le C_1(\|u\|_{L^2}^2 + \|Lu\|_{L^2}^2).$$

Hence, by integration and applying (2.10), we have

$$\alpha\,\|u(t)\|_{L^2}^2 \le C_0\,\|u(0)\|_{L^2}^2 + C_1\int_0^t (\|u(\tau)\|_{L^2}^2 + \|Lu(\tau)\|_{L^2}^2)\,\mathrm{d}\tau,$$

where $C_0 = \|\Sigma(0)\|_{\mathscr{B}(L^2)}$. We conclude by Gronwall's Inequality that (2.9) holds for $s = 0$ with $C = C'\exp(C'T)$, with $C' \doteq \max\{C_1,\ C_0\}/\alpha$.

General case. Let s be an arbitrary real number. For $u \in \mathscr{C}^1([0,T]; H^s) \cap \mathscr{C}([0,T]; H^{s+1})$ the inequality previously derived for $s = 0$ applies to $\Lambda^s u$ and yields

$$\|\Lambda^s u(t)\|_{L^2}^2 \le C\left(\|\Lambda^s u(0)\|_{L^2}^2 + \int_0^t \|L\Lambda^s u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau\right).$$

Writing

$$L\Lambda^s u = \Lambda^s Lu + [\Lambda^s, P]u,$$

[10]Note that now $P(t)^*\Sigma u = \sum_\alpha \partial_\alpha\big((A^\alpha)^*S_0u\big) + B^*S_0u$, and $\Sigma P(t)u = -\sum_\alpha S_0A^\alpha\partial_\alpha u + S_0Bu$.

and observing that both Λ^s and $P(t)$ are pseudo-differential operators, then the commutator $[\Lambda^s, P]$ is of order $s+1-1=s$, and hence $\|[\Lambda^s, P]u\|_{L^2} \le C' \|u\|_{H^s}$. The above inequality is rewritten as

$$\|u(t)\|_{H^s}^2 \le C \left(\|u(0)\|_{H^s}^2 + \int_0^t (\|Lu(\tau)\|_{H^s}^2 + C' \|u(\tau)\|_{H^s}^2) \, \mathrm{d}\tau \right).$$

An application of Gronwall'e inequality gives (2.9). □

Remark 2.1. By reversing time, that is, changing t to $T-t$ and $P(t)$ to $-P(T-t)$ in Theorem 2.1, we also obtain the estimate

$$\|u(t)\|_{H^s}^2 \le C \left(\|u(T)\|_{H^s}^2 + \int_t^T \|Lu(\tau)\|_{H^s}^2 \, \mathrm{d}\tau \right), \quad \forall t \in [0,T], \tag{2.11}$$

for $u \in \mathscr{C}^1([0,T]; H^s) \cap \mathscr{C}([0,T]; H^{s+1})$. This estimate of the adjoint problem, namely the equation $L^* v = g$ with terminal value $v(T) = 0$, is used to show existence of a L^2 weak solution of the original Cauchy problem $Lu = f$ with initial value $u(0) = 0$, via a duality argument.

2.3. **Construction of functional symmetrizer by symbolic symmetrizer.** The problem is now to construct functional symmetrizers. Except for Friedrichs-symmetrizable systems, this is not an easy task. We shall conveniently use symbolic calculus. We denote

$$A(x,t,\xi) = \sum_{\alpha=1}^{d} \xi_\alpha A^\alpha(x,t), \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \quad \xi \in \mathbb{R}^d,$$

which can be viewed up to a $-\mathrm{i}$ factor as the symbol of the principle part of the operator $P(t)$ defined in (2.2).

Definition 2.3. A symbolic symmetrizer associated with $A(x,t,\xi)$ is a $\mathscr{C}^\infty$ mapping

$$S: \quad \mathbb{R}^d \times \mathbb{R}^+ \times (\mathbb{R}^d \setminus \{0\}) \to \mathbb{M}_n(\mathbb{C}),$$

homogeneous degree zero in its last variables ξ, bounded as well as all its derivatives with respect to (x,t,ξ) on $|\xi| = 1$, such that, for all (x,t,ξ),

$$S(x,t,\xi) = S(x,t,\xi)^* \ge \beta I, \tag{2.12}$$

for some positive number β, uniformly on sets of the form $\mathbb{R}^d \times [0,T] \times (\mathbb{R}^d \setminus \{0\})$ (where $T \ge 0$), and

$$S(x,t,\xi)A(x,t,\xi) = A(x,t,\xi)^* S(x,t,\xi). \tag{2.13}$$

Of course, a Friedrichs-symmetrizable system admits an obvious "symbolic" symmetrizer independent of ξ:

$$S(x,t,\xi) = S_0(x,t).$$

Note that, in general, a symbolic symmetrizer is not exactly a symbol of a pseudo-differential operator, due to the singularity allowed at $\xi = 0$. However, truncating about 0 does yield a pseudo-differential symbol in $\mathbf{S}^0$, which is unique modulo $\mathbf{S}^{-\infty}$. This enables us to associate S with a family of pseudo-differential operators $\tilde{\Sigma}(t)$ of order zero modulo infinitely smoothing operators. This in turn will enable us to construct a functional symmetrizer $\Sigma(t)$.

Remark 2.2. In the constant-coefficient case, neither $A(x,t,\xi)$ nor $S(x,t,\xi)$ depend on x, t, and it is elementary to construct a functional symmetrizer based on S. This symmetrizer is of course independent of t and is just given by

$$\Sigma \doteq \mathscr{F}^{-1} S \mathscr{F}$$

(where $\mathscr{F}$ denotes the usual Fourier Transform). Then (2.7) holds with $\alpha = \beta$, because, by Plancherel's Theorem,

$$(\Sigma u, v) = (\mathscr{F}^{-1}(S\hat{u}), v) = (S\hat{u}, \hat{v}).$$

This shows Σ is also Hermitian and positive-definite. And (2.8) follows from (2.13) because of the relations

$$\begin{aligned}(\Sigma P u, v) + (u, \Sigma P v) &= (S\widehat{Pu}, \hat{v}) + (\hat{u}, S\widehat{Pv})\\ &= (S(-\mathrm{i}A + B)\hat{u}, v) + (\hat{u}, S(-\mathrm{i}A + B)\hat{v}) = ((SB + B^* S)\hat{u}, \hat{v}),\end{aligned}$$

and the fact that S, B are uniformly bounded.

Theorem 2.2. *Assuming that $A(x,t,\xi)$ admits a symbolic symmetrizer $S(x,t,\xi)$ (according to Definition 2.3), then the family $\{P(t)\}$ defined in (2.2) admits a functional symmetrizer $\Sigma(t)$ (as in Definition 2.2).*

Proof. The proof consists of a pseudo-differential extension of Remark 2.2.

1. As mentioned above, $S(\cdot, t, \cdot)$ can be associated with a pseudo-differential operator of order 0, $\tilde{\Sigma}(t)$. We recall that the operator $\tilde{\Sigma}(t)$ is not necessarily self-adjoint, even though its symbol, the matrix $\tilde{S}(x,t,\xi)$, is Hermitian. But $\tilde{\Sigma}(t)^*$ differs from $\tilde{\Sigma}(t)$ by an operator of order -1 (since they are both of order 0).

2. Let us define

$$\Sigma(t) = \frac{1}{2}(\tilde{\Sigma}(t) + \tilde{\Sigma}(t)^*).$$

By Gårding's Inequality (recall $S(x,t,\xi)$ is positive-definite), there exists $C_T > 0$ so that

$$(\Sigma(t)u, u) \geq \frac{\beta}{2} \|u\|_{L^2}^2 - C_T \|u\|_{H^{-1}}^2$$

for all $t \in [0,T]$ and $u \in L^2(\mathbb{R}^d)$. Now, noting that

$$\|u\|_{H^{-1}}^2 = (\Lambda^{-2}u, u),$$

we can change $\Sigma(t)$ into $\Sigma(t) + C_T\Lambda^{-2}$ in order to have

$$(\Sigma(t)u, u) \geq \frac{\beta}{2} \|u\|_{L^2}^2 .$$

This modification does not alter the self-adjointness of $\Sigma(t)$ and gives (2.7) with $\alpha = \beta/2$.

3. Furthermore, $\Sigma(t)P(t) + P(t)^*\Sigma(t)$ coincides with the operator of symbol (using (2.13))

$$\tilde{S}(-\mathrm{i}A + B) + (-\mathrm{i}A + B)^*\tilde{S} = \tilde{S}B + B^*\tilde{S}$$

up to a remainder of order $0 + 1 - 1 = 0$. (Here we used the product formula of pseudo-differential operators. To simplify notations, we have omitted the dependence on the parameter t of the symbols.) Since $(\tilde{S}B + B^*\tilde{S})(\cdot, t, \cdot)$ belongs to $\mathbf{S}^0$, so $\Sigma(t)P(t) + P(t)^*\Sigma$ is of order 0 and hence is a bounded operator on L^2. $\square$

As a consequence of Theorems 2.1 and 2.2, we have the following.

Corollary 2.1. *If $A(x,t,\xi) = \sum_\alpha \xi^\alpha A^\alpha(x,t)$ admits a symbolic symmetrizer, then for all $s \in \mathbb{R}$ and $T > 0$, there exists $C > 0$ so that for $u \in \mathscr{C}^1([0,T]; H^s) \cap \mathscr{C}([0,T]; H^{s+1})$, we have*

$$\|u(t)\|_{H^s}^2 \leq C \left(\|u(0)\|_{L^2}^2 + \int_0^t \|Lu(\tau)\|_{H^s}^2 \,\mathrm{d}\tau \right),$$

where $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha - B$.

2.4. **Construction of symbolic symmetrizer for constantly hyperbolic operators.** Except Friedrichs-symmetrizable systems, another important class of hyperbolic systems that do admit a symbolic symmetrizer is the one of constantly hyperbolic systems.

Theorem 2.3. *We assume that the system* (1.1) *is constantly hyperbolic, that is, the matrices $A(x,t,\xi)$ are diagonalizable with distinct real eigenvalues $\lambda_1, \cdots, \lambda_p$ of constant multiplicities on $\mathbb{R}^d \times \mathbb{R}^+ \times (\mathbb{R}^d \setminus \{0\})$. We also assume that these matrices are independent of x for $|x| \geq R$. Then they admit a symbolic symmetrizer.*

Together with Theorem 2.2, this shows that constantly hyperbolic systems are symmetrizable and thus enjoy H^s estimate.

Proof. **1.** The proof is based on spectral projections associated with $A(x,t,\xi)$, which are well-defined due to spectral separation. As a matter of fact, the assumptions imply that the spectral gap $|\lambda_j(x,t,\xi)-\lambda_k(x,t,\xi)|$ is bounded by below for $1\le j\ne k\le p, (x,t)\in\mathbb{R}^d\times[0,T]$ and $|\xi|=1$. Let us define [11]

$$\rho \doteq \frac{1}{2}\min\{|\lambda_j(x,t,\xi)-\lambda_k(x,t,\xi)|:\ 1\le j\ne k\le p,\ (x,t)\in\mathbb{R}^d\times[0,T],\ |\xi|=1\}$$

and the projectors

$$\boxed{Q_j(x,t,\xi)=\frac{1}{2\mathrm{i}\pi}\int_{\lambda\in\mathbb{C}:\ |\lambda-\lambda_j(x,t,\xi)|=\rho|\xi|}(\lambda I_n-A(x,t,\xi))^{-1}\,\mathrm{d}\lambda}$$

for $1\le j\le p$. Since A and its eigenvalues λ_j are homogeneous degree 1 in ξ, we easily see by changing of variables that Q_j is homogeneous degree 0 on ξ. Furthermore, Q_j is independent of x for $|x|\ge R$.

2. Then we introduce

$$S(x,t,\xi)=\sum_{j=1}^{p}Q_j(x,t,\xi)^*Q_j(x,t,\xi).$$

By construction, the matrix S is Hermitian. Moreover, we have for any vector $v\in\mathbb{C}^n$,

$$v^*Sv=\sum_{j=1}^{p}|Q_jv|^2\ge\beta|v|^2,$$

where

$$\beta=\min\left\{\sum_{j=1}^{p}|Q_j(x,t,\xi)v|^2:\ |v|=1,\ |x|\le R,\ 0\le t\le T,\ |\xi|=1\right\}>0$$

since $\sum_j Q_j=I_n$. (See [3, p.7].) This proves (2.12).

3. Finally, since $Q_jA=AQ_j=\lambda_jQ_j$ for $\lambda_j\in\mathbb{R}$ (cf. [3, p.8]), we have

$$\begin{aligned}(SAv,w)&=\sum_{j=1}^{p}(Q_jAv,Q_jw)=\sum_{j=1}^{p}\lambda_j(Q_jv,Q_jw)\\&=\sum_{j=1}^{p}\lambda_j\overline{(Q_jw,Q_jv)}=\overline{(SAw,v)}=(v,SAw).\end{aligned}$$

for all $v,w\in\mathbb{C}^n$, and thus SA is Hermitian. $\square$

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LECTURE NOTES 6: PARA-PRODUCT, PARA-LINEARIZATION AND PARA-DIFFERENTIAL CALCULUS

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In this note, following the presentations in appendices in [1], mostly word by word, we introduce para-product and para-differential calculus based on the Littlewood-Paley decomposition. The note is only used for purpose of teaching.

1. Littlewood-Paley decomposition and Sobolev spaces

1.1. **Introduction.** Let $\psi(\xi) \in \mathscr{D}(\mathbb{R}^d) = C_0^\infty(\mathbb{R}^d)$ be a monotonically decreasing function along rays that satisfies

$$\begin{aligned} \psi(\xi) = 1, \quad & \text{if} \quad |\xi| \le \frac{1}{2}, \\ 0 \le \psi(\xi) \le 1, \quad & \text{if} \quad \frac{1}{2} \le |\xi| < 1, \\ \psi(\xi) = 0, \quad & \text{if} \quad |\xi| \ge 1. \end{aligned}$$

Then we define

$$\phi(\xi) \doteq \psi(\xi/2) - \psi(\xi) \quad \text{and} \quad \phi_p(\xi) \doteq \phi(2^{-p}\xi) \quad \text{for} \quad p \in \mathbb{Z}.$$

One readily checks that

$$\operatorname{supp}\phi \subset \left\{\frac{1}{2} \le |\xi| \le 2\right\}, \quad \operatorname{supp}\phi_p \subset \left\{2^{p-1} \le |\xi| \le 2^{p+1}\right\}, \tag{1.1}$$
$$\operatorname{supp}\phi_p \cap \operatorname{supp}\phi_q = \emptyset, \quad \text{if } |p-q| \ge 2,$$

as well as the point-wise identity

$$1 = \psi(\xi) + \sum_{p=0}^{\infty} \phi_p(\xi).$$

These facts lead to [1]

$$\frac{1}{2} \le \psi^2 + \sum_{q \ge 0} \phi_q^2 \le 1.$$

For convenience, we also denote $\phi_{-1} = \psi$.

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[1]Let $I_0 = \sum_{p=\text{even}}^{\infty} \phi_p(\xi)$, $I_1 = \psi(\xi) + \sum_{p=\text{odd}}^{\infty} \phi_p(\xi)$. Then $1 = (I_0 + I_1)^2 \le 2(I_0^2 + I_1^2)$. But $I_0^2 = \sum_{p=\text{even}}^{\infty} \phi_p(\xi)^2$, $I_1^2 = \psi(\xi)^2 + \sum_{p=\text{odd}}^{\infty} \phi_p(\xi)^2$.

Now for $u \in \mathscr{S}'(\mathbb{R}^d)$ (Schwartz tempered distribution), there holds

$$\hat{u} = \psi(\xi)\hat{u} + \sum_{p=0}^{\infty} \phi_p(\xi)\hat{u},$$

where $\hat{u} = \mathscr{F}(u)$ is the Fourier transform of u. We define the operators

$$\triangle_{-1}u = \mathscr{F}^{-1}(\psi(\xi)\hat{u}), \quad \triangle_p u = \mathscr{F}^{-1}(\phi_p(\xi)\hat{u}), \quad p \in \mathbb{N} \cup \{0\}.$$

Then we have the *nonhomogeneous Littlewood-Paley (L-P) decomposition* of a distribution u:

$$u = \sum_{p=-1}^{\infty} \triangle_p u.$$

Note each term $\triangle_p u$ is a smooth function. The infinite sum converges in the sense of tempered distribution. For $q \geq 0$, we define the partial sum as

$$S_q u = \sum_{p=-1}^{q-1} \triangle_p u = \mathscr{F}^{-1}(\psi(2^{-q}\xi)\hat{u}).$$

One notes that $\operatorname{supp} \mathscr{F}(S_q u) \subset \{|\xi| \leq 2^q\}$. We also set $\triangle_p = 0$ for $p < -1$ and $S_q = 0$ for $q \leq -1$.

By definition, we have, for all $u \in \mathscr{S}'$,

$$\mathscr{F}(\triangle_q u) = \phi_q \hat{u}, \quad \text{and} \quad \mathscr{F}(S_q u) = \psi_q \hat{u},$$

with the rescaled functions $\psi_q = \psi(2^{-q}\xi)$ for $q \geq 0$.

A first interesting property of the operators Δ_q is that the L^∞ norms of $\triangle_q u, S_q u$ and their derivatives are all controlled by the L^∞ norm of u. The cost of one derivative is found to be adding a factor 2^q in the constant.

Proposition 1.1 (Bernstein)**.** *For all $m \in \mathbb{N}$, there exists $C_m > 0$ so that for all $u \in L^\infty$, for all d-uple α, $|\alpha| \leq m$, for all $q \geq -1$,*

$$\|\partial^\alpha(\triangle_q u)\|_{L^\infty} \leq C_m 2^{q|\alpha|} \|u\|_{L^\infty} \quad \textit{and} \quad \|\partial^\alpha(S_q u)\|_{L^\infty} \leq C_m 2^{q|\alpha|} \|u\|_{L^\infty} \tag{1.2}$$

Proof. We have

$$\triangle_q u = (\mathscr{F}^{-1}\phi_p) * u, \quad \text{and} \quad S_q u = (\mathscr{F}^{-1}\psi_p) * u.$$

All functions $(\mathscr{F}^{-1}\phi_q)(\xi) = 2^{qd}(\mathscr{F}^{-1}\phi)(2^q\xi)$ are integrable because of the regularity of ϕ (with compact support and smooth, then belongs to $\mathscr{S}$), with the additional invariance property

$$\left\|\mathscr{F}^{-1}\phi_q\right\|_{L^1} = \left\|\mathscr{F}^{-1}\phi\right\|_{L^1}$$

for all $q \geq -1$. We also easily compute that

$$\left\|\partial^\alpha(\mathscr{F}^{-1}\phi_q)\right\|_{L^1} = 2^{q|\alpha|}\left\|\partial^\alpha(\mathscr{F}^{-1}\phi)\right\|_{L^1}.$$

The same is also true for ψ_q. Then Young's Inequality on Convolution yields the conclusion with $C = \max_{|\alpha|\leq m}\left\{\|\partial^\alpha(\mathscr{F}^{-1}\psi)\|_{L^1}, \|\partial^\alpha(\mathscr{F}^{-1}\phi)\|_{L^1}\right\}$. □

1.2. **Basic estimates concerning Sobolev spaces.** All results displayed in this section but the very last one are concerned with the most classical Sobolev spaces H^s on the whole space $\mathbb{R}^d$.

1.2.1. *An equivalent definition of Sobolev spaces H^s.* First, we note that if $u \in H^s$, the equality $u = \sum_q \triangle_q u$ holds true not only in $\mathscr{S}'$ but also in H^s. As a matter of fact, we have $\mathscr{F}(S_q u - u) \to 0$ point-wise as $q \to \infty$, and $|\lambda^s(\xi)\mathscr{F}(S_q u - u)(\xi)|^2 \leq (1 + \|\psi\|_{L^\infty})^2|\lambda^s(\xi)\hat{u}(\xi)|^2$ for $\lambda^s(\xi) = (1 + |\xi|^2)^{s/2}$, so Lebesgue Dominant Convergence Theorem yields

$$\|S_q u - u\|_{H^s}^2 = \|\lambda^s(\xi)(\mathscr{F}(S_q u - u))(\xi)\|_{L^2}^2 \to 0$$

as $q \to \infty$. Furthermore, the operators $\triangle_q$ appear to give rise to equivalent norms on the Sobolev spaces.

Proposition 1.2. *For all $s \in \mathbb{R}$, there exist $C_s > 1$ such that for all $u \in H^s$,*

$$\frac{1}{C_s}\sum_{q\geq -1} 2^{2qs}\|\triangle_q u\|_{L^2}^2 \leq \|u\|_{H^s}^2 \leq C_s \sum_{q\geq -1} 2^{2qs}\|\triangle_q u\|_{L^2}^2. \tag{1.3}$$

Proof. **1.** We begin with the case $s = 0$. We claim that the estimate in (1.3) works with $C_0 = 2$. One may remark that the equality, that is, (1.3) with $C_0 = 1$, could be true if the $\triangle_q u$ were pairwise orthogonal. But we only have

$$(\triangle_p u, \triangle_q u) = 0 \ \text{ provided that } \ |p - q| \geq 2. \tag{1.4}$$

The inequalities in (1.3) can be viewed as measuring the default of orthogonality. The proof is almost straightforward. As a matter of fact, the inequalities $\frac{1}{2} \leq \psi^2 + \sum_{q\geq 0}\phi_q^2 \leq 1$ imply that

$$\sum_{q\geq -1}|\phi_q(\xi)\hat{u}(\xi)|^2 \leq |\hat{u}(\xi)|^2 \leq 2\sum_{q\geq -1}|\phi_q(\xi)\hat{u}(\xi)|^2 \tag{1.5}$$

for all $u \in L^2$ and almost all $\xi \in \mathbb{R}^d$. Integrating in ξ we get, in view of the definitions of $\triangle_q u$,

$$\sum_{q\geq -1}\left\|\widehat{\triangle_q u}\right\|_{L^2}^2 \leq \|\hat{u}\|_{L^2}^2 \leq 2\sum_{q\geq -1}\left\|\widehat{\triangle_q u}\right\|_{L^2}^2$$

and we just concluded by Plancherel's Theorem.

2. We then consider the general case. From (1.5) we have

$$\sum_{q\geq -1} |\lambda^s(\xi)\phi_q(\xi)\hat{u}(\xi)|^2 \leq |\lambda^s(\xi)\hat{u}(\xi)|^2 \leq 2\sum_{q\geq -1} |\lambda^s(\xi)\phi_q(\xi)\hat{u}(\xi)|^2. \tag{1.6}$$

Assume, for instance, s is positive. Then for $q \geq 0$ and for

$$\xi \in \operatorname{supp}\phi_q \subset \{2^{q-1} \leq |\xi| \leq 2^{q+1}\},$$

we have

$$2^{-2s}2^{2qs} \leq \lambda^{2s}(\xi) = (1+|\xi|^2)^s \leq 2^{3s}2^{2qs}, \tag{1.7}$$

while for

$$\xi \in \operatorname{supp}\phi_{-1} \subset \{|\xi| \leq 1\}$$

we have

$$2^{2s}2^{-2s} = 1 \leq \lambda^{2s}(\xi) \leq 2^s = 2^{3s}2^{-2s}.$$

Therefore, we get by (1.6)

$$2^{-2s}\sum_{q\geq -1} 2^{2qs}|\phi_q(\xi)\hat{u}(\xi)|^2 \leq |\lambda^s(\xi)\hat{u}(\xi)|^2 \leq 2^{3s+1}\sum_{q\geq -1} 2^{2qs}|\phi_q(\xi)\hat{u}(\xi)|^2$$

and by integrating on $\xi \in \mathbb{R}^d$, we get (1.3) with $C_s = 2^{3s+1}$.

3. For $s < 0$, the estimates on λ^{2s} are reversed and (1.3) holds true with $C_s = 2^{-3s+1}$, cf. (1.7). □

In particular, this proposition shows that for all $u \in H^s$ and all $q \geq -1$,

$$\|\triangle_q u\|_{L^2} \leq \sqrt{C_s}2^{-qs}\|u\|_{H^s}. \tag{1.8}$$

Of course C_s becomes 1 if we replace the usual H^s norm by the equivalent norm

$$\|u\|_{H^s} = \left(\sum_{q\geq -1} 2^{2qs}\|\triangle_q u\|_{L^2}^2\right)^{\frac{1}{2}}. \tag{1.9}$$

Proposition 1.3. *For all $m \in \mathbb{N}$, there exists $C_m > 0$ so that for all $u \in L^2$, for all d-uple α, $|\alpha| \leq m$, for all $q \geq -1$,*

$$\|\partial^\alpha(\triangle_q u)\|_{L^2} \leq C_m 2^{q|\alpha|}\|u\|_{L^2} \quad \textit{and} \quad \|\partial^\alpha(S_q u)\|_{L^2} \leq C_m 2^{q|\alpha|}\|u\|_{L^2} \tag{1.10}$$

This implies, for all positive integers s, there exists $C > 0$ so that for all $q \geq -1$ and $u \in L^2$,

$$\|\triangle_q u\|_{H^s} \leq C_s 2^{qs}\|u\|_{L^2} \quad \text{and} \quad \|S_q u\|_{H^s} \leq C_s 2^{qs}\|u\|_{L^2}. \tag{1.11}$$

Note the constant C_s depends on s and is increasing with s.

The proof of Proposition 1.3 is exactly the same as that of Proposition 1.1, replacing the $L^1 - L^\infty$ convolution estimate by $L^1 - L^2$ estimate.

1.2.2. *Some embedding theorems.* Another noteworthy remark is that the L^∞ norms of $\triangle_q u$ and $S_q u$ can be controlled even for unbounded u, provided that u belongs to some H^s (which is not embedded in L^∞ for $s \le d/2$), as shown in the following.

Proposition 1.4. *For all $s \in \mathbb{R}$, there exists $C > 0$ so that for all $u \in H^s(\mathbb{R}^d)$ and all $q \ge -1$,*

$$\|\triangle_q u\|_{L^\infty} \le C 2^{-q(s-d/2)} \|u\|_{H^s} \quad \textit{and} \quad \|S_q u\|_{L^\infty} \le C 2^{-q(s-d/2)} \|u\|_{H^s}. \tag{1.12}$$

Proof. Since both $\widehat{\triangle_q u}$ and $\widehat{S_q u}$ are supported by the ball $\{|\xi| \le 2^{q+1}\}$, on which $\psi_{q+2} = 1$, we have, for instance,

$$\widehat{\triangle_q u} = \psi_{q+2} \widehat{\triangle_q u},$$

and similarly for $\widehat{S_q u}$. Therefore,

$$\triangle_q u = (\mathscr{F}^{-1}\psi_{q+2}) * \triangle_q u.$$

Now, to get the correct estimate we just have to pay attention to the fact that L^2 norm is not invariant by the scaling. We have indeed

$$\|\psi_q\|_{L^2} = 2^{qd/2} \|\psi\|_{L^2}$$

and thus Plancherel's Theorem and a basic convolution inequality yield

$$\|\triangle_q u\|_{L^\infty} \le 2^{(q+2)d/2} \|\psi\|_{L^2} \|\triangle_q u\|_{L^2}.$$

This together with (1.8) gives (1.12) with $C = 2^d \|\psi\|_{L^2} \sqrt{C_s}$. The same computation shows the inequality for $S_q u$. □

A straightforward consequence of this proposition is, of course, the well-known Sobolev Embedding $H^s(\mathbb{R}^d) \hookrightarrow L^\infty$ for $s > d/2$. For, the inequality in (1.12) shows the series $\sum \triangle_q u$ is normally convergent in L^∞ if u belongs to $H^s(\mathbb{R}^d)$ for $s > d/2$, and its sum must be u (by uniqueness of limits in the space of distribution).

Remark 1.1. By a similar calculation as in the above proof, we have L^2 estimates of $\triangle_q u$ for $u \in L^1$. Namely, there exists $C > 0$ so that

$$\|\triangle_q u\|_{L^2} \le C 2^{qd/2} \|u\|_{L^1}$$

for all $u \in L^1(\mathbb{R}^d)$ and $q \ge -1$.

Indeed, by definition of $\triangle_q u$, Plancherel's Theorem shows that

$$\|\triangle_q u\|_{L^2} = \|\phi_q \hat{u}\|_{L^2} \le \|\phi_q\|_{L^2} \|\hat{u}\|_{L^\infty}$$

for $q \ge 0$ (for $q = -1$, just replace ϕ_q by ψ) and

$$\|\phi_q\|_{L^2} = 2^{qd/2} \|\phi\|_{L^2},$$

while, of course, $\|\hat{u}\|_{L^\infty} \le \|u\|_{L^1}$. So the constant $C = \max\left\{ \|\phi\|_{L^2}, 2^{d/2}\|\psi\|_{L^2} \right\}$.

As a consequence of these estimates and Proposition 1.2, we find the embedding

$$L^1(\mathbb{R}^d) \hookrightarrow H^{-s}(\mathbb{R}) \quad \text{for} \quad s > d/2.$$

To see this, just note that [2]

$$\|u\|_{H^{-s}}^2 \le C_s \sum_{q\ge -1} 2^{-2qs} \|\triangle_q u\|_{L^2}^2 \le C'_{s,d} \sum_{q\ge -1} 2^{qd-2qs} \|u\|_{L^1}^2 \le C' \|u\|_{L^1}^2 .$$

1.2.3. *More on control of* $\|\triangle_q u\|_{L^\infty}$. To complete this section, we prove an additional result in the same sprit as Proposition 1.4, which gives an estimate of $\|\triangle_q u\|_{L^\infty}$ in terms of $\|\triangle_q u\|_{W^{m,\infty}}$ (instead of $\|\triangle_q u\|_{L^2}$ in the proof of Proposition 1.4).

Proposition 1.5. *For all $m \in \mathbb{N}$, there exists $C_m > 0$ so that for all $u \in L^\infty$ and all $q \ge 0$,*

$$\|\triangle_q u\|_{L^\infty} \le C_m 2^{-qm} \sum_{|\alpha|=m} \|\partial^\alpha(\triangle_q u)\|_{L^\infty} . \tag{1.13}$$

Proof. There is nothing to prove for $m = 0$. Let us assume $m \ge 1$. We consider some function $\chi \in \mathscr{D}(\mathbb{R}^d)$ vanishing near 0 and being equal to 1 on the support of ϕ (for instance take $\chi(\xi) = \psi(\xi/4) - \psi(2\xi)$), so that $\phi = \chi\phi$. With obvious notations we also have

$$\phi_q = \chi_q \phi_q$$

for all $q \ge 0$. Since χ vanishes near 0 we can define for all d-uples α of length m a function $\chi^\alpha \in \mathscr{D}(\mathbb{R}^d)$ by

$$\chi^\alpha(\xi) = \frac{(\mathrm{i}\xi)^\alpha}{\sum_{|\beta|=m} (\mathrm{i}\xi)^{2\beta}} \chi(\xi).$$

By construction we have

$$\chi(\xi) = \sum_{|\alpha|=m} (\mathrm{i}\xi)^\alpha \chi^\alpha(\xi),$$

and

$$\chi_q(\xi) = 2^{-qm} \sum_{|\alpha|=m} (\mathrm{i}\xi)^\alpha \chi_q^\alpha(\xi),$$

with still the obvious notation $\chi_q^\alpha(\xi) = \chi^\alpha(2^{-q}\xi)$. Then we have

$$\widehat{\triangle_q u} = \chi_q(\xi)\widehat{\triangle_q u} = 2^{-qm} \sum_{|\alpha|=m} (\mathrm{i}\xi)^\alpha \chi_q^\alpha(\xi) \widehat{\triangle_q u}.$$

[2]We also note, this result is not quite surprising. Since L^1 is dense in H^{-s}, so if $i : L^1 \to H^{-s}$ is compact, its adjoint operator $i^* = i : H^s \to L^\infty$ is also compact by standard property of adjoint operator from functional analysis.

This easily implies that

$$\triangle_q u = 2^{-qm} \sum_{|\alpha|=m} (\mathscr{F}^{-1}\chi_q^\alpha) * \partial^\alpha(\triangle_q u).$$

The result follows again from a convolution inequality and the identities

$$\left\|\mathscr{F}^{-1}\chi_q^\alpha\right\|_{L^1} = \left\|\mathscr{F}^{-1}(\chi^\alpha)\right\|_{L^1}.$$

We find then $\|\triangle_q u\|_{L^\infty} \le 2^{-qm}\left(\max_{|\alpha|=m}\|\mathscr{F}^{-1}\chi^\alpha\|_{L^1}\right)\sum_{|\alpha|=m}\|\partial^\alpha(\triangle_q u)\|_{L^\infty}$. □

The proof here above obviously fails for $q=-1$, because $\triangle_{-1}u$ does involve small frequencies. However, by Proposition 1.1,

$$\|\triangle_{-1}u\|_{L^\infty} \le C_0\|u\|_{L^\infty} \le C_0 2^k \|u\|_{L^\infty}$$

for all $k \in \mathbb{N}$. Therefore, using the commutation property $\partial^\alpha\triangle_q = \triangle_q\partial^\alpha$, a consequence of Proposition 1.5 is the following.

Corollary 1.1. *For all $k \in \mathbb{N}$, there exists $C_k > 0$ so that for all $u \in W^{k,\infty}$, with $W^{k,\infty} = \{u : \|\partial_x^\alpha u\|_{L^\infty} < \infty, \quad \forall\alpha \in \mathbb{N}^d, \quad |\alpha| \le k\}$, there holds*

$$\forall q \ge -1, \quad \|\triangle_q u\|_{L^\infty} \le C_k 2^{-qk}\|u\|_{W^{k,\infty}}. \tag{1.14}$$

Proof. We need only consider the case $q \ge 0$. Then $\|\triangle_q u\|_{L^\infty} \le C_k 2^{-qk}\sum_{|\alpha|=k}\|\partial^\alpha\triangle_q u\|_{L^\infty} = C_k 2^{-qk}\sum_{|\alpha|=k}\|\triangle_q(\partial^\alpha u)\|_{L^\infty} \le C_k 2^{-qk}\sum_{|\alpha|=k}\|\partial^\alpha u\|_{L^\infty} \le C_k 2^{-qk}\|u\|_{W^{k,\infty}}$. □

2. The Para-products

2.1. **Para-products.** It is well-known that for two distributions $u, v \in \mathscr{S}'(\mathbb{R}^d)$, generally their production uv can not be well-defined. However, one can utilize L-P decomposition to define, in a certain sense, the product of two distributions.

Let $u=\sum_{p=-1}^{\infty}\triangle_p u$, $v=\sum_{q=-1}^{\infty}\triangle_q v$ be the L-P decomposition of u,v respectively. Then formally we may write [3]

$$\begin{aligned} uv &= \sum_{p,q\geq -1}\triangle_p u\triangle_q v \\ &= \sum_{p=2}^{\infty}\sum_{q=-1}^{p-3}\triangle_p u\triangle_q v+\sum_{q=2}^{\infty}\sum_{p=-1}^{q-3}\triangle_p u\triangle_q v+\sum_{|p-q|\leq 2}\triangle_p u\triangle_q v \\ &= \sum_{p=2}^{\infty}\triangle_p u S_{p-2}v+\sum_{q=2}^{\infty}\triangle_q v S_{q-2}u+\sum_{|p-q|\leq 2}\triangle_p u\triangle_q v. \end{aligned}$$

We introduce the para-product of v by u as

$$\boxed{T_u v \doteq \sum_{q=-1}^{\infty}\triangle_q v S_{q-2}u=\sum_{q=2}^{\infty}\triangle_q v S_{q-2}u.} \tag{2.1}$$

Then we formally have the symmetric decomposition

$$uv = T_v u + T_u v + R(u,v) \tag{2.2}$$

with the remainder term

$$R(u,v)\doteq \sum_{|p-q|\leq 2}\triangle_p u\triangle_q v. \tag{2.3}$$

Remark 2.1. From the definition it is easy to see that $\partial_j(T_u v)=T_u(\partial_j v)+T_{\partial_j u}v$.

Example. Let us consider the special case when u is a constant c to understand better the para-product $T_u v$. Indeed we have now $\hat{u}=c\delta$, with δ the Dirac measure supported at $\{0\}$, hence $\mathscr{F}(\triangle_p u)=0$ whenever $p\geq 0$, and $\mathscr{F}(S_q u)=\mathscr{F}(S_0 u)=c\delta$ for all $q\geq 0$, thus $S_q c=c$. Therefore $T_u v=c\sum_{q=2}^{\infty}\triangle_q v$. We may further compute $uv-T_u v=u\sum_{q=-1}^{1}\triangle_q v$, that has compact spectrum (i.e., its Fourier transform has compact support set), hence is smooth.

2.2. **Reasonability of the definition of para-product.** Since (2.1) involves infinite sums, we'd better explain whether the sum is meaningful. To this end, we look the spectrum of each term, and find, since $\operatorname{supp}\mathscr{F}(\triangle_q v)\subset\{2^{q-1}\leq|\xi|\leq 2^{q+1}\}$, $\operatorname{supp}\mathscr{F}(S_{q-2}u)\subset\{|\xi|\leq 2^{q-2}\}$, there holds, by $\operatorname{supp} f*g\subset\operatorname{supp} f+\operatorname{supp} g$,

$$\operatorname{supp}\mathscr{F}(\triangle_q v S_{q-2}u)\subset\left\{\frac{1}{4}2^q\leq|\xi|\leq\frac{9}{4}2^q\right\}. \tag{2.4}$$

[3]Although the first two terms in the right-hand side make sense even if $u,v\in\mathscr{S}'$, but there are two reasons that uv in general makes no sense. The first one is the last term is only sum of terms with bounded support (not in annulus); the second one is this identity involves change order of limits, which can not be true in general.

Therefore (2.1) is meaningful at least for $u, v \in \mathscr{S}'(\mathbb{R}^d)$, in the sense of tempered distributions.

2.3. **Basic estimates on para-product.**

Proposition 2.1. *For all $s \in \mathbb{R}$, there exists $C > 0$ so that for all $u \in L^\infty$ and all $v \in H^s$, there holds*

$$\boxed{\|T_u v\|_{H^s} \leq C \|u\|_{L^\infty} \|v\|_{H^s}.} \tag{2.5}$$

Proof. Using the equivalent norm of H^s characterized by the L-P decomposition, we need to prove

$$\sum_{p \geq -1} 2^{2ps} \|\triangle_p(T_u v)\|_{L^2}^2 \leq C_s \|u\|_{L^\infty}^2 \sum_{p \geq -1} 2^{2ps} \|\triangle_p v\|_{L^2}^2. \tag{2.6}$$

1. We first estimate the term $\triangle_p(T_u v)$. Taking Fourier transform, we have

$$\mathscr{F}(\triangle_p(T_u v)) = \phi_p(\xi) \sum_{q \geq 2} \mathscr{F}(\triangle_q v S_{q-2} u).$$

While for each term, by (1.1) and (2.4), one easily checks

$$\phi_p(\xi)\mathscr{F}(\triangle_q v S_{q-2} u) = 0 \quad \text{if} \quad |p - q| \geq 4.$$

Hence

$$\|\triangle_p(T_u v)\|_{L^2}^2 \leq C \sum_{|q-p| \leq 3} \|\triangle_q v S_{q-2} u\|_{L^2}^2 \leq C \sum_{|q-p| \leq 3} \|\triangle_q v\|_{L^2}^2 \|S_{q-2} u\|_{L^\infty}^2.$$

We note, by $\mathscr{F}^{-1}(f(\varepsilon\xi)) = (\mathscr{F}^{-1}(f))_\varepsilon$,

$$\|S_q u\|_{L^\infty} = \left\|\mathscr{F}^{-1}(\psi(2^{-q}\xi)) * u\right\|_{L^\infty} \leq \left\|\mathscr{F}^{-1}(\psi(2^{-q}\xi))\right\|_{L^1} \|u\|_{L^\infty} \leq C \|u\|_{L^\infty}, \tag{2.7}$$

therefore

$$\|\triangle_p(T_u v)\|_{L^2}^2 \leq C \|u\|_{L^\infty}^2 \sum_{|q-p| \leq 3} \|\triangle_q v\|_{L^2}^2.$$

2. Then we get

$$2^{2ps} \|\triangle_p(T_u v)\|_{L^2}^2 \leq C \|u\|_{L^\infty}^2 2^{2ps} \sum_{|q-p| \leq 3} \|\triangle_q v\|_{L^2}^2 \leq C_s \|u\|_{L^\infty}^2 \sum_{|q-p| \leq 3} 2^{2qs} \|\triangle_q v\|_{L^2}^2.$$

Hence

$$\begin{aligned} \sum_{p=-1}^{\infty} 2^{2ps} \|\triangle_p(T_u v)\|_{L^2}^2 &\leq C_s \|u\|_{L^\infty}^2 \sum_{p=-1}^{\infty} \sum_{|q-p| \leq 3} 2^{2qs} \|\triangle_q v\|_{L^2}^2 \\ &\leq C_s' \|u\|_{L^\infty}^2 \sum_{p=-1}^{\infty} 2^{2ps} \|\triangle_p v\|_{L^2}^2 \end{aligned}$$

as desired. □

Using similar ideas, we can prove

Proposition 2.2. *For all $r \in \mathbb{Z}$ and $s > 0$ there exists $C > 0$ so that for all $u \in L^\infty$ and $v \in H^s$,*

$$\left\| \sum_{q\geq -1} S_{q-r} u \triangle_q v \right\|_{H^s} \leq C \left\| u \right\|_{L^\infty} \left\| v \right\|_{H^s}. \tag{2.8}$$

Proof. We need prove

$$\sum_{p\geq -1} 2^{2ps} \left\| \triangle_p (\sum_{q\geq -1} S_{q-r} u \triangle_q v) \right\|_{L^2}^2 \leq C \left\| u \right\|_{L^\infty}^2 \sum_{p\geq -1} 2^{2ps} \left\| \triangle_p v \right\|_{L^2}^2.$$

1. We still first estimate the distribution of spectrum of the terms $\triangle_p(S_{q-r} u \triangle_q v)$. We have, by a property of convolution,

$$\operatorname{supp} \mathscr{F}(S_{q-r} u \triangle_q v) \subset \{|\xi| \leq 2^{q-r}\} + \{2^{q-1} \leq |\xi| \leq 2^{q+1}\}.$$

Then for $r \geq 2$, $\operatorname{supp} \mathscr{F}(S_{q-r} u \triangle_q v)$ is contained in the spherical shell $\{\frac{1}{4} 2^q \leq |\xi| \leq 2^{q+2}\}$. So, by (1.1), there is a $k \in \mathbb{N}$ such that $\triangle_p(S_{q-r} u \triangle_q v) = 0$ whenever $|p-q| \geq k+1$.

For $r \leq 1$, the spectrum of $\operatorname{supp} \mathscr{F}(S_{q-r} u \triangle_q v)$ might not be bounded away from 0: $\operatorname{supp} \mathscr{F}(S_{q-r} u \triangle_q v) \subset \{|\xi| \leq 2^{q+3-r}\}$. Thus we can only find $k \in \mathbb{N}$ depending on r such that

$$\triangle_p(S_{q-r} u \triangle_q v) = 0 \quad \text{if} \quad p - q \geq k+1. \tag{2.9}$$

2. Now by (2.9), we have

$$\triangle_p (\sum_{q\geq -1} S_{q-r} u \triangle_q v) = \sum_{q \geq p-k} \triangle_p (S_{q-r} u \triangle_q v).$$

Applying Cauchy–Schwartz inequality of l^2 and $s > 0$, and taking L^2 norms, there comes

$$\begin{aligned} \left\| \triangle_p (\sum_{q\geq -1} S_{q-r} u \triangle_q v) \right\|_{L^2}^2 &\leq \sum_{q\geq p-k} 2^{-qs} \sum_{q \geq p-k} 2^{qs} \left\| \triangle_p (S_{q-r} u \triangle_q v) \right\|_{L^2}^2 \\ \text{((1.8) with } s=0) \quad &\leq \frac{2^{-(p-k)s}}{1-2^{-s}} C \sum_{q\geq p-k} 2^{qs} \left\| S_{q-r} u \triangle_q v \right\|_{L^2}^2 \\ \text{((2.7))} \quad &\leq C_{s,k} 2^{-ps} \sum_{q \geq p-k} 2^{qs} \left\| u \right\|_{L^\infty}^2 \left\| \triangle_q v \right\|_{L^2}^2. \end{aligned}$$

So, using $\sum_{p=-1}^{q+k} 2^{(p-q)s} \leq C2^{-qs}2^{(q+k)s} = C2^{ks}$ when $s > 0$, we find

$$\begin{aligned}
\sum_{p\geq -1} 2^{2ps} \left\| \triangle_p(\sum_{q\geq -1} S_{q-r}u\triangle_q v)\right\|_{L^2}^2 &\leq C_{s.k} \|u\|_{L^\infty}^2 \sum_{p\geq -1}\sum_{q\geq p-k} 2^{(p+q)s} \|\triangle_q v\|_{L^2}^2 \\
&\leq C_{s,k} \|u\|_{L^\infty}^2 \sum_{q\geq -1} (2^{2qs}\|\triangle_q v\|_{L^2}^2) \left(\sum_{p=-1}^{q+k} 2^{(p-q)s}\right) \\
&\leq C'_{s,k} 2^{ks} \|u\|_{L^\infty}^2 \sum_{q\geq -1} (2^{2qs}\|\triangle_q v\|_{L^2}^2).
\end{aligned}$$

This finishes the proof. □

2.4. **Estimate on errors.**

Proposition 2.3. *For all $s > 0$, there exists $C > 0$ so that for all $u, v \in L^\infty \cap H^s$,*

$$\|uv - T_v u\|_{H^s} \leq C \|u\|_{L^\infty} \|v\|_{H^s}. \tag{2.10}$$

Proof. The assumption $s > 0$ ensures both $u = \sum \triangle_p u$ and $v = \sum \triangle_p v$ converge in the L^2 norm. This justifies the formula

$$uv = \sum_{p,\,q\geq -1} \triangle_p u \triangle_q v.$$

Then by definition of $T_v u = \sum_{p\geq 2} S_{p-2}v\triangle_p u = \sum_{p\geq 2}\sum_{q\leq p-3} \triangle_p u\triangle_q v$, we get

$$\begin{aligned}
uv - T_v u &= \sum_{q\geq -1}\sum_{p\geq -1} \triangle_q v\triangle_p u - \sum_{p\geq 2}\sum_{q\leq p-3} \triangle_q v \triangle_p u \\
&= \sum_{q\geq -1} \triangle_q v(\sum_{p\geq -1}\triangle_p u) - \sum_{q\geq -1}\triangle_q v(\sum_{p\geq q+3}\triangle_p u) \\
&= \sum_{q\geq -1} \triangle_q v \sum_{p=-1}^{q+2} \triangle_p u = \sum_{q\geq -1} \triangle_q v S_{q+3} u.
\end{aligned}$$

So by (2.8), with $r = -3$, we proved (2.10). □

2.5. **Estimates on products.** We easily obtain the following estimate of product thanks to (2.5) and (2.10).

Theorem 2.1. *For all $s > 0$ there exists $C > 0$ so that for all $u, v \in L^\infty \cap H^s$,*

$$\boxed{\|uv\|_{H_s} \leq C(\|u\|_{L^\infty}\|v\|_{H^s} + \|v\|_{L^\infty}\|u\|_{H^s}).} \tag{2.11}$$

Theorem 2.2 (Gagliardo-Nirenberg inequality)**.** *For any $s \in \mathbb{N}$, there exists $C > 0$ so that for all $u \in L^\infty \cap H^s$ and multi-index $|\alpha| \leq s$,*

$$\|\partial^\alpha u\|_{L^{2s/|\alpha|}} \leq C \|u\|_{L^\infty}^{1-|\alpha|/s} \|u\|_{H^s}^{|\alpha|/s}.$$

This inequality can be proved by utilizing scaling-related properties. We omit the details. We remark that the estimates listed in this and following subsections are essential and widely used to study various nonlinear problems in the framework of Sobolev spaces.

The following estimate strengthens (2.11).

Theorem 2.3. *For any $s \in \mathbb{N}$, there exists $C > 0$ such that for all $u, v \in L^\infty \cap H^s$ and all d-uples α, β with $|\alpha| + |\beta| = s$, we have*

$$\boxed{\left\|(\partial^\alpha u)(\partial^\beta v)\right\|_{L^2} \leq C(\|u\|_{L^\infty} \|v\|_{H^s} + \|v\|_{L^\infty} \|u\|_{H^s}).}$$

Proof. By Hölder inequality and G-N inequality above, since $\frac{1}{2} = \frac{|\alpha|}{2s} + \frac{|\beta|}{2s}$, we have

$$\begin{aligned}
\left\|(\partial^\alpha u)(\partial^\beta v)\right\|_{L^2} &\leq \|\partial^\alpha u\|_{L^{2s/|\alpha|}} \left\|\partial^\beta v\right\|_{L^{2s/|\beta|}} \\
&\leq C(\|u\|_{L^\infty}^{1-\frac{|\alpha|}{s}} \|u\|_{H^s}^{\frac{|\alpha|}{s}})(\|v\|_{L^\infty}^{1-\frac{|\beta|}{s}} \|v\|_{H^s}^{\frac{|\beta|}{s}}) \\
&\leq C(\|u\|_{L^\infty} \|v\|_{H^s})^{\frac{|\beta|}{s}} (\|v\|_{L^\infty} \|u\|_{H^s})^{\frac{|\alpha|}{s}} \\
&\leq C'(\|u\|_{L^\infty} \|v\|_{H^s} + \|v\|_{L^\infty} \|u\|_{H^s}).
\end{aligned}$$

In the last equality we also used Young's inequality $a^p b^q \leq pa + bq$ if $p + q = 1$. □

2.6. Estimates on commutators.

Theorem 2.4. *If $s > 1$ and α is a d-uple with $|\alpha| \leq s$, then there exists $C > 0$ such that for all $u, a \in H^s$ with $\nabla u, \nabla a \in L^\infty$,*

$$\boxed{\|[\partial^\alpha, a\nabla]u\|_{L^2} \leq C(\|\nabla u\|_{L^\infty} \|a\|_{H^s} + \|\nabla a\|_{L^\infty} \|u\|_{H^s}).} \tag{2.12}$$

Proof. We recall

$$[\partial^\alpha, a\nabla]u = \partial^\alpha(a\nabla u) - a\nabla(\partial^\alpha u).$$

For $j = 1, \cdots, d$, we only need to establish (2.12) with ∇ replaced by ∂_j.

Step 1. By Leibniz rule there are coefficients C_α^β with $C_\alpha^0 = 1$ such that

$$\partial^\alpha(a\partial_j u) = \sum_{|\beta| \leq |\alpha|} C_\alpha^\beta \partial^\beta a \partial^{\alpha-\beta} \partial_j u.$$

Hence

$$[\partial^\alpha, a\nabla]u = \sum_{1 \leq |\beta| \leq |\alpha|} C_\alpha^\beta \partial^\beta a \partial^{\alpha-\beta} \partial_j u.$$

and therefore

$$\|[\partial^\alpha, a\nabla]u\|_{L^2} \leq C \sum_{1 \leq |\beta| \leq |\alpha|} \left\|\partial^\beta a \partial^{\alpha-\beta} \partial_j u\right\|_{L^2}.$$

Step 2. For each term $\partial^\beta a \partial^{\alpha-\beta}\partial_j u = (\partial^{\beta_k}\partial_k a)(\partial^{\alpha-\beta}\partial_j u)$, where $|\beta_k| = |\beta| - 1$ and k can be found by virtue of $|\beta| \geq 1$. Observing $|\alpha - \beta + \beta_k| = |\alpha| - 1$, we use Theorem 2.3 with $s = |\alpha| - 1$ to obtain

$$\begin{aligned} \left\|(\partial^{\beta_k}\partial_k a)(\partial^{\alpha-\beta}\partial_j u)\right\|_{L^2} &\leq C(\|\partial_k a\|_{L^\infty}\|\partial_j u\|_{H^{|\alpha|-1}} + \|\partial_j u\|_{L^\infty}\|\partial_k a\|_{H^{|\alpha|-1}}) \\ &\leq C(\|\nabla a\|_{L^\infty}\|u\|_{H^{|\alpha|}} + \|\nabla u\|_{L^\infty}\|a\|_{H^{|\alpha|}}). \end{aligned}$$

Then (2.12) easily follows. □

2.7. **Estimate on remaining term.** To illustrate the power of para-products, let us just show the following result on the remainder R, where we see that the regularity of $R(u, v)$ is "almost" the one of u plus the one of v.

Theorem 2.5. *For all s, t with $s + t > 0$, there exists $C > 0$ such that for all $u \in H^s, v \in H^t$, $R(u, v)$ is well-defined by* (2.3), *and meets the estimate*

$$\|R(u,v)\|_{H^{s+t-d/2}} \leq C\|u\|_{H^s}\|v\|_{H^t}. \tag{2.13}$$

Proof. **1.** At first, we check that the assumption $s + t > 0$ ensures that $R(u, v)$ is well-defined and

$$R(u,v) = \sum_{q\geq -1} R_q(u,v), \quad \text{with} \quad R_q(u,v) \doteq \sum_{r=q-2}^{q+2} \triangle_r u \triangle_q v.$$

As a matter of fact, we have

$$\begin{aligned} \|R_q(u,v)\|_{L^1} &\leq \sum_{r=q-2}^{q+2}\|\triangle_r u\|_{L^2}\|\triangle_q v\|_{L^2} \leq C\sum_{r=q-2}^{q+2} 2^{-rs}\|u\|_{H^s}2^{-qt}\|v\|_{H^t} \\ &\leq 5\times 2^{2|s|}C\|u\|_{H^s}\|v\|_{H^t}2^{-q(s+t)}, \end{aligned}$$

which shows the series $R_q(u, v)$ is normally convergent in L^1.

2. To prove the estimate in (2.13), we must evaluate the L^2 norm of $\triangle_p R(u, v)$. Similarly as in the proof of Proposition 2.2, we note as $\mathscr{F}(R_q(u, v))$ has compact support (not necessarily in a shell), so $\triangle_p R(u, v)$ only involves some finite number of terms $\triangle_p R_q(u, v)$. This is due to the fact that there is an integral k such that

$$\triangle_p(\triangle_r u \triangle_q v) \equiv 0 \quad \text{for} \quad p - q \geq k+1 \quad \text{and} \quad |r-q| \leq 2. \tag{2.14}$$

Therefore, we have

$$\triangle_p R(u,v) = \sum_{q\geq p-k}\triangle_p R_q(u,v).$$

3. For all $p \geq -1$ we have

$$\triangle_p R_q(u,v) = \sum_{r=q-2}^{q+2} (\mathscr{F}^{-1}\phi_p) * (\triangle_r u \triangle_q v)$$

and thus a standard convolution inequality and Plancherel's Theorem shows that

$$\|\triangle_p R_q(u,v)\|_{L^2} \leq \|\phi_p\|_{L^2} \sum_{r=q-2}^{q+2} \|\triangle_r u \triangle_q v\|_{L^1}.$$

Since for $p \geq 0$ we have

$$\|\phi_p\|_{L^2} = 2^{pd/2} \|\phi\|_{L^2},$$

the latter inequality implies that for all $p \geq -1$,

$$\|\triangle_p R_q(u,v)\|_{L^2} \leq c 2^{pd/2} \sum_{r=q-2}^{q+2} \|\triangle_r u\|_{L^2} \|\triangle_q v\|_{L^2},$$

with $c \doteq \max\left\{ \|\phi\|_{L^2}, 2^{d/2}\|\psi\|_{L^2} \right\}$.

4. We now apply Cauchy-Schwarz Inequality of ℓ^2 to obtain

$$\begin{aligned} |\triangle_p R(u,v)|^2 &= \left(\sum_{q\geq p-k} 2^{-q(t+s)/2} \times 2^{q(t+s)/2} \triangle_p R_q(u,v)\right)^2 \\ &\leq \sum_{q\geq p-k} 2^{-q(t+s)} \sum_{q\geq p-k} 2^{q(t+s)} |\triangle_p R_q(u,v)|^2 \\ &\leq C_{t+s,k} 2^{-p(t+s)} \sum_{q\geq p-k} 2^{q(t+s)} |\triangle_p R_q(u,v)|^2, \end{aligned}$$

with $C_{t+s,k} = \sum_{l=-k}^{\infty} 2^{-l(s+t)}$. There then comes

$$\|\triangle_p R(u,v)\|_{L^2}^2 \leq C_{t+s,k} 2^{-p(t+s)} \sum_{q\geq p-k} 2^{q(t+s)} \|\triangle_p R_q(u,v)\|_{L^2}^2.$$

Consequently, we get

$$\|\triangle_p R(u,v)\|_{L^2}^2 \leq C' 2^{-p(s+t-d)} \sum_{q\geq p-k} 2^{q(t+s)} \|\triangle_q v\|_{L^2}^2 \sum_{r=q-2}^{q+2} \|\triangle_r u\|_{L^2}^2,$$

with $C' = 5 \times c^2 C_{s+t,k}$, and thus

$$2^{2p(s+t-d/2)} \|\triangle_p R(u,v)\|_{L^2}^2 \leq 5 \times 2^{2|s|} C' \sum_{q\geq p-k} 2^{(p-q)(t+s)} 2^{2qt} \|\triangle_q v\|_{L^2}^2 \sum_{r=q-2}^{q+2} 2^{2rs} \|\triangle_r u\|_{L^2}^2.$$

Finally, we obtain

$$\begin{aligned}
&\sum_{p\geq -1} 2^{2p(s+t-d/2)} \left\|\triangle_p R(u,v)\right\|_{L^2}^2 \\
\leq\ & 5\times 2^{2|s|}C' \sum_{q\geq -1} 2^{2qt} \left\|\triangle_q v\right\|_{L^2}^2 \sum_{-1\leq p\leq q+k} 2^{(p-q)(t+s)} \sum_{r=q-2}^{q+2} 2^{2rs} \left\|\triangle_r u\right\|_{L^2}^2 \\
\leq\ & 5\times 2^{2|s|}C' \sum_{q\geq -1} 2^{2qt} \left\|\triangle_q v\right\|_{L^2}^2 \sum_{l\leq k} 2^{l(t+s)} \sum_{r\geq -1} 2^{2rs} \left\|\triangle_r u\right\|_{L^2}^2 \\
\leq\ & C'' \left\|u\right\|_{H^s}^2 \left\|v\right\|_{H^t}^2,
\end{aligned}$$

with $C'' = 5\times 2^{2|s|}C'C_{s+t,k}C_sC_t$. This gives (2.13) with $C=\sqrt{C''C_{s+t-d/2}}$. □

2.8. **Further estimates on products.** This result on the remainder $R(u,v)$ gives a slightly bigger index than in the classical result recalled below for the full product uv. This is reasonable since $R(u,v)$ should be much smoother.

Theorem 2.6. *For all s and t with $s+t>0$, if $u\in H^s$ and $v\in H^t$, then the product $uv\in H^r$ for $r\leq \min\{s,\ t\}$ such that $r<s+t-d/2$. Furthermore, there exists C (depending only on r,s,t and d) such that*

$$\boxed{\left\|uv\right\|_{H^r} \leq C\left\|u\right\|_{H^s}\left\|v\right\|_{H^t}.}$$

In the case $r=s=t$ (hence they are larger than $d/2$ and $H^s(\mathbb{R}^d)\hookrightarrow L^\infty(\mathbb{R}^d)$ holds), this theorem is a consequence of Theorem 2.1. The proof of the general case follows easily from Theorem 2.5 and the following additional result on para-product.

Proposition 2.4. *For all s and t, if $u\in H^s$ and $v\in H^t$, then the para-product T_uv is well-defined and belongs to H^r for all $r\leq s+t-d/2$. Furthermore, there exists $C>0$ independent of u and v so that*

$$\boxed{\left\|T_uv\right\|_{H^r} \leq C\left\|u\right\|_{H^s}\left\|v\right\|_{H^t}.} \tag{2.15}$$

Proof. We almost repeat the proof of Proposition 2.1.

1. We first estimate the term $\triangle_p(T_uv)$. Taking Fourier transform, we have

$$\mathscr{F}(\triangle_p(T_uv)) = \phi_p(\xi)\sum_{q\geq 2}\mathscr{F}(\triangle_q vS_{q-2}u).$$

While for each term, by (1.1) and (2.4), one easily checks

$$\phi_p(\xi)\mathscr{F}(\triangle_q vS_{q-2}u) = 0 \quad \text{if } \ |p-q|\geq 4.$$

Hence

$$\left\|\triangle_p(T_uv)\right\|_{L^2}^2 \leq C\sum_{|q-p|\leq 3}\left\|\triangle_q vS_{q-2}u\right\|_{L^2}^2 \leq C\sum_{|q-p|\leq 3}\left\|\triangle_q v\right\|_{L^2}^2\left\|S_{q-2}u\right\|_{L^\infty}^2.$$

Using (1.12),

$$\|S_q u\|_{L^\infty} \le C 2^{-q(s-d/2)} \|u\|_{H^s},$$

therefore

$$\|\triangle_p(T_u v)\|_{L^2}^2 \le C 2^{-2p(s-d/2)} \|u\|_{H^s}^2 \sum_{|q-p|\le 3} \|\triangle_q v\|_{L^2}^2 .$$

2. Now we get

$$\begin{aligned} 2^{2pr} \|\triangle_p(T_u v)\|_{L^2}^2 &\le 2^{2pr} C 2^{-2p(s-d/2)} \|u\|_{H^s}^2 \sum_{|q-p|\le 3} \|\triangle_q v\|_{L^2}^2 \\ &\le C 2^{2p(r-s-t+d/2)} \|u\|_{H^s}^2 \sum_{|q-p|\le 3} 2^{2qt} \|\triangle_q v\|_{L^2}^2 . \end{aligned}$$

Hence

$$\begin{aligned} \sum_{p=-1}^{\infty} 2^{2pr} \|\triangle_p(T_u v)\|_{L^2}^2 &\le C \|u\|_{H^s}^2 \sum_{p=-1}^{\infty} 2^{2p(r-s-t+d/2)} \sum_{|q-p|\le 3} 2^{2qt} \|\triangle_q v\|_{L^2}^2 \\ &\le C' \|u\|_{H^s}^2 \|v\|_{H^t}^2 \sum_{p=-1}^{\infty} 2^{2p(r-s-t+d/2)} \le C'' \|u\|_{H^s}^2 \|v\|_{H^t}^2 , \end{aligned}$$

as the series converges because of $r - s - t + d/2 < 0$. $\square$

An easy consequence of Theorem 2.6 is the following commutator estimate.

Corollary 2.1. *If m is an integral greater than $d/2+1$ and α is a d-uple of length $|\alpha| \in [1, m]$, then there exists $C > 0$ such that for all $a \in H^m$ and all $u \in H^{|\alpha|-1}$,*

$$\boxed{\|[\partial^\alpha, a]u\|_{L^2} \le C \|a\|_{H^m} \|u\|_{H^{|\alpha|-1}} .}$$

Proof. Since we have

$$[\partial^\alpha, a]u = \sum_{1\le|\beta|\le|\alpha|} c_\alpha^\beta \partial^\beta a \partial^{\alpha-\beta} u,$$

so we only need estimate L^2 norm for each term $\left\|\partial^\beta a \partial^{\alpha-\beta} u\right\|_{L^2}$. Let $v = \partial^\beta a \in H^{m-|\beta|}$ and $w = \partial^{\alpha-\beta} u \in H^{|\alpha|-1-|\alpha|+|\beta|}$, so $s = m - |\beta| \ge 0$ and $t = |\beta| - 1 \ge 0$. Let $r = 0$, then the requirements in Theorem 2.6 is fulfilled and we have

$$\|vw\|_{L^2} \le C \|v\|_{H^s} \|w\|_{H^t} \le C \|a\|_{H^m} \|u\|_{H^{|\alpha|-1}}$$

as desired. $\square$

2.9. **Smoothing effect of para-product.** A useful result that was not pointed out yet is the smoothing effect of the operator $a - T_a$ when a is at least Lipschitz.

Theorem 2.7. *For all $k \in \mathbb{N}$, there exists $C > 0$ such that for all $a \in W^{k,\infty}$ and all $u \in L^2$,*

$$\|au - T_a u\|_{H^k} \leq C \|a\|_{W^{k,\infty}} \|u\|_{L^2}.$$

The case $k = 0$ (with no smoothing effect) is a trivial consequence of Proposition 2.1 and the Triangle Inequality. For the difficult case $k \geq 1$, a detailed proof can be found in [2, p.83, Theorem 5.2.8].

A straightforward consequence of Theorem 2.7 is the following, which enables us to replace a term $A\partial_j u$ in a quasi-linear hyperbolic system by a para-product, which is a special para-differential operator.

Corollary 2.2. *There exists $C > 0$ such that for all $a \in W^{1,\infty}$ and $u \in L^2$, for $j = 1, \cdots, d$,*

$$\boxed{\|a\partial_j u - T_a \partial_j u\|_{L^2} \leq C \|a\|_{W^{1,\infty}} \|u\|_{L^2}.}$$

Proof. First suppose $u \in H^1$, and we observe that

$$a\partial_j u - T_a \partial_j u = \partial_j (au - T_a u) - ((\partial_j a)u - T_{\partial_j a} u).$$

The L^2 norm of the first term is bounded by $C \|a\|_{W^{1,\infty}} \|u\|_{L^2}$ (by the case $k = 1$ in Theorem 2.7). The L^2 norm of the second term is bounded by $C \|\partial_j a\|_{L^\infty} \|u\|_{L^2}$ (by the case $k = 0$ in Theorem 2.7). So the inequality holds for $u \in H^1$.

The case for $u \in L^2$ is then proved by approximation, □

3. Para-linearization

Proposition 2.1 and Theorem 2.5 show in particular that for all $s > 0$, if $u \in H^s \cap L^\infty$, then

$$u^2 = 2T_u u + R(u, u) = T_{2u} u + R(u, u),$$

with the uniform estimates

$$\|T_{2u} u\|_{H^s} \leq C \|u\|_{L^\infty} \|u\|_{H^s}, \qquad \|R(u, u)\|_{H^{2s-d/2}} \leq C \|u\|_{H^s}^2.$$

A very strong result from para-differential calculus is the following, says that this decomposition of $F(u) = u^2$ can be generalized to any $\mathscr{C}^\infty$ function F vanishing at 0, under the only assumption that $s > d/2$.

Theorem 3.1 (Bony-Meyer)**.** *If $F \in \mathscr{C}^\infty(\mathbb{R}), F(0) = 0$, and $s > d/2$, then for all $u \in H^s(\mathbb{R}^d)$, we have*

$$F(u) = T_{F'(u)}u + R(u), \tag{3.1}$$

with $R(u) \in H^{2s-d/2}$.

Note the assumption $s > d/2$ automatically implies that $u \in L^\infty \cap H^s(\mathbb{R}^d)$.

Equation (3.1) is often referred to as the para-linearization formula of Bony. In particular, it shows that $F(u) \in H^s$ (since $F'(u) \in L^\infty$). We will not give the proof of Theorem 3.1. One can show that $F(u)$ enjoys the same estimate as its para-linearization counterpart $T_{F'(u)}u$, i.e., the following theorem.

Theorem 3.2. *If $F \in \mathscr{C}^\infty(\mathbb{R}), F(0) = 0$, and $s > d/2$, then there exists a continuous function $C : [0,\infty) \to [0,\infty)$ such that for all $u \in H^s(\mathbb{R}^d)$,*

$$\boxed{\|F(u)\|_{H^s} \leq C(\|u\|_{L^\infty}) \|u\|_{H^s}.}$$

Proof. The assumption $s > d/2$ implies that each $u \in H^s(\mathbb{R}^d)$ necessarily belongs to $L^\infty \cap L^2$.

1. We begin by showing the estimate for $F(S_0 u)$ instead of $F(u)$. To do so, it is sufficient to bound $\|\partial^\alpha F(S_0 u)\|_{L^2}$ for all d-uples α of length $|\alpha| \leq m$ with $m - 1 \leq s < m$.

For $|\alpha| = 0$, this is almost trivial. By Propositions 1.1 and 1.3,

$$\|S_0 u\|_{L^\infty} \leq C\|u\|_{L^\infty}, \qquad \|S_0 u\|_{L^2} \leq C\|u\|_{L^2}$$

and thus the Mean Value Theorem applied to F in the ball of radius $R = C\|u\|_{L^\infty}$ (that is, $F(S_0 u) = F'_u(\theta S_0 u) S_0 u$ with some $0 < \theta < 1$) implies that

$$\|F(S_0 u)\|_{L^2} \leq \max_{|v| \leq R} |F'_u(v)| \, \|S_0 u\|_{L^2} \leq C_R \|u\|_{L^2}$$

for some $C_R = C\|F'_u\|_{\mathscr{C}(\overline{B_R(0)})}$ depending continuously on R.

For $|\alpha| \geq 1$, the Chain Rule shows that there exist coefficients c_α^b, with $b = \{\beta^1, \cdots, \beta^m\}$ being a family of d-uples of positive length and of sum $\beta^1 + \cdots + \beta^m = \alpha$, so that

$$\partial^\alpha(F(S_0 u)) = \sum_{1 \leq n \leq |\alpha|, \quad \beta^1 + \cdots + \beta^n = \alpha, \quad |\beta^i| \geq 1} c_\alpha^b F^{(n)}(S_0 u) \partial^{\beta^1}(S_0 u) \cdots \partial^{\beta^n}(S_0 u).$$

By Proposition 1.1 and Proposition 1.3, we have

$$\left\|\partial^{\beta^i}(S_0 u)\right\|_{L^\infty} \leq C\|u\|_{L^\infty}, \qquad \left\|\partial^{\beta^i}(S_0 u)\right\|_{L^2} \leq C\|u\|_{L^2}.$$

Therefore, using L^∞ bounds for the successive derivatives $F^{(n)}$, $n \leq m$, on the ball of radius R we obtain a uniform estimate

$$\|\partial^\alpha(F(S_0 u))\|_{L^2} \leq C_R^{(m)} \|u\|_{L^2}$$

for all α with $|\alpha| \leq m$. In particular, up to modifying $C_R^{(m)}$, we have

$$\|F(S_0u)\|_{H^s} \leq C_R^{(m)} \|u\|_{L^2}$$

for all $s \leq m$.

2. The other main part of the proof consists in bounding the “error” $F(u) - F(S_0u)$. Since S_pu is known to tend to u in H^s, we *formally* have

$$F(u) - F(S_0u) = \sum_{p=0}^{\infty}(F(S_{p+1}u) - F(S_pu)).$$

To justify this decomposition we must show that the series involved is convergent in H^s. At first, we note that

$$F(S_{p+1}u) - F(S_pu) = G(S_pu, \triangle_pu)\triangle_pu,$$

where

$$G(v, w) = \int_0^1 F'(v + tw)\,\mathrm{d}t$$

is a $\mathscr{C}^\infty$ function of both its arguments. By Proposition 1.1 and a piece of calculations we can bound $G(S_pu, \triangle_pu)$ in L^∞ in exactly the same way we bounded $F(S_0u)$ in L^2. Thus we find another constant depending continuously on R, still denoted by $C_R^{(m)}$, so that

$$\|\partial^\alpha G(S_pu, \triangle_pu)\|_{L^\infty} \leq C_R^{(m)} 2^{p|\alpha|}$$

for all α with $|\alpha| \leq m$. Then a fine result, postponed to Lemma 3.1 below, enables us to conclude. As a matter of fact, Lemma 3.1 applies to $M_p = G(S_pu, \triangle_pu)$ and shows that

$$\left\|\sum_{p=0}^{\infty}(F(S_{p+1}u) - F(S_pu))\right\|_{H^s} = \left\|\sum_{p\geq 0} G(S_pu, \triangle_pu)\triangle_pu\right\|_{H^s} \leq cC_R^{(m)} \|u\|_{H^s}.$$

This justifies the convergence of the series.

3. We have

$$F(u) = F(S_0u) + \sum_{p=0}^{\infty}(F(S_{p+1}u) - F(S_pu)).$$

Collecting and summing the estimates of $F(S_0u)$ and the series $\sum(F(S_{p+1}u) - F(S_pu))$ we find that

$$\|F(u)\|_{H^s} \leq C(\|u\|_{L^\infty}) \|u\|_{H^s},$$

with $C(\|u\|_{L^\infty}) = (1 + c)C_R^{(m)}$. □

Lemma 3.1 (Meyer). *Let $\{M_p\}_{p\geq 0}$ be a sequence of $\mathscr{C}^\infty$ functions enjoying the uniform estimates*

$$\|\partial^\alpha M_p\|_{L^\infty} \leq c_m 2^{p|\alpha|} \tag{3.2}$$

for all α *with* $|\alpha| \le m$*. Then for all* $0 < s < m$*, there exists* c *so that for all* $u \in H^s$ *the series* $\sum M_p \triangle_p u$ *is convergent in* H^s *and*

$$\left\| \sum_{p\ge 0} M_p \triangle_p u \right\|_{H^s} \le c c_m \|u\|_{H^s}. \tag{3.3}$$

Proof. The proof resembles the one of Proposition 2.2, in that the sequence $\{M_q\}$ satisfies by assumption the same estimates as $S_q u$ (derived in Proposition 1.1). However, there is an additional difficulty due to the fact that, unlike $S_q u$, $\widehat{M_q}$ is not supposed to be compactly supported. This is why we first perform a frequency decomposition of M_p.

1. For this we use the dilated functions $\phi_q(2^{-p-3}\,\cdot\,)$ and define

$$M_{p,q} \doteq \mathscr{F}^{-1}(\phi_q(2^{-p-3}\,\cdot\,)\widehat{M_p})$$

for all $q \ge -1$. Observe that, for $q \ge 0$, we merely have

$$M_{p,q} \doteq \triangle_{q+p+3} M_p,$$

of which the spectrum is included in

$$\{\xi : 2^{p+q+2} \le |\xi| \le 2^{p+q+4}\},$$

and that the first term of the expansion,

$$M_{p,-1} = \mathscr{F}^{-1}(\psi(2^{-p-3}\,\cdot\,)\widehat{M_p}) \tag{3.4}$$

has a spectrum included in

$$\{\xi : |\xi| \le 2^{p+3}\}.$$

Because of the partition of unity $\psi + \sum_{q\ge 0}\phi_q \equiv 1$ (evaluated at $2^{-p-3}\xi$), we have

$$M_p = \sum_{q\ge -1} M_{p,q}$$

in the sense of $\mathscr{S}'$. In fact, this series is normally convergent in L^∞, since by Proposition 1.5,

$$\begin{aligned} &\|M_{p,q}\|_{L^\infty} = \|\triangle_{p+q+3} M_p\|_{L^\infty} \\ \le\ & C_m \sum_{|\alpha|=m} \|\partial^\alpha \triangle_{p+q+3} M_p\|_{L^\infty} 2^{-(p+q+3)m} = C_m \sum_{|\alpha|=m} \|\partial^\alpha M_{p,q}\|_{L^\infty} 2^{-(p+q+3)m} \end{aligned}$$

for $q \ge 0$, and by Proposition 1.1 applied to $\partial^\alpha M_p$, as $\partial^\alpha M_p = \sum_{q\ge -1}\partial^\alpha M_{p,q}$,

$$\|\partial^\alpha M_{p,q}\|_{L^\infty} \le C_m \|\partial^\alpha M_p\|_{L^\infty},$$

(we have used here the fact that $[\partial^\alpha, \triangle_q] = 0$ for all q) and so the assumption (3.2) implies that

$$\|M_{p,q}\|_{L^\infty} \le \tilde{C}_m 2^{-qm}.$$

2. Let us now look at the two-parameter family $\{M_{p,q}\triangle_p u\}_{p\geq 0,\ q\geq -1}$. By (1.8) and the previous inequality, we have

$$\sum_{p\geq 0}\sum_{q\geq -1}\|M_{p,q}\triangle_p u\|_{L^2}\leq \sum_{p\geq 0}\sum_{q\geq -1}\|M_{p,q}\|_{L^\infty}\|\triangle_p u\|_{L^2}\leq \tilde{C}_m\sum_{p\geq 0}\sum_{q\geq -1}2^{-qm}2^{-ps}\|u\|_{H^s}<\infty,$$

which justifies the interchanging formula

$$\sum_{p\geq 0}\sum_{q\geq -1}M_{p,q}\triangle_p u=\sum_{q\geq -1}\sum_{p\geq 0}M_{p,q}\triangle_p u.$$

This equivalently reads

$$\sum_{p\geq 0}M_p\triangle_p u=\sum_{q\geq -1}\Sigma_q,$$

with

$$\Sigma_q\doteq\sum_{p\geq 0}M_{p,q}\triangle_p u. \tag{3.5}$$

3. We can now estimate Σ_q in H^s.

We begin with the special case $q=-1$. We have

$$\operatorname{supp}\mathscr{F}(M_{p,-1}\triangle_p u)\subset\{|\xi|\leq 2^{p+4}\}$$

and thus

$$\triangle_r(M_{p,-1}\triangle_p u)\equiv 0\quad\text{for}\quad r\geq p+5.$$

Therefore,

$$\triangle_r\Sigma_{-1}=\sum_{p=r-4}^{\infty}\triangle_r(M_{p,-1}\triangle_p u)$$

for all $r\geq -1$.

Now we use the procedure as in the proof of Proposition 2.2. We have

$$\begin{aligned}
\|\triangle_r\Sigma_{-1}\|_{L^2}^2 &\leq \left(\sum_{p=r-4}^{+\infty}2^{-ps}\right)\left(\sum_{p=r-4}^{+\infty}2^{ps}\|\triangle_r(M_{p,-1}\triangle_p u)\|_{L^2}^2\right)\\
\leq&\ \left(\sum_{l\geq 0}^{+\infty}2^{-(l+r-4)s}\right)\left(\sum_{p=r-4}^{+\infty}2^{ps}C^2\|(M_{p,-1}\triangle_p u)\|_{L^2}^2\right)\ (l=p-r+4\text{ and Proposition 1.3})\\
\leq&\ C^2C_s2^{-rs}\left(\sum_{p=r-4}^{+\infty}2^{ps}\|(M_{p,-1}\|_{L^\infty}^2\|\triangle_p u)\|_{L^2}^2\right)\qquad (C_s=\sum_{l\geq 0}2^{-(l-4)s}<\infty)\\
\leq&\ C^2C^2C_sC^2C_0^22^{-rs}\sum_{p\geq r-4}2^{ps}\|\triangle_p u\|_{L^2}^2.
\end{aligned}$$

For the last inequality, we used the uniform estimate

$$\|M_{p,-1}\|_{L^\infty}\leq C\|M_p\|_{L^\infty}\leq CC_0$$

obtained from (3.4) and (3.2). So we get

$$\begin{aligned}\sum_{r\geq -1} 2^{2rs} \left\|\triangle_r \Sigma_{-1}\right\|_{L^2}^2 &\leq C' C_s \sum_{r\geq -1} 2^{(r-p)s} \sum_{p\geq r-4} 2^{2ps} \left\|\triangle_p u\right\|_{L^2}^2 \\ &\leq C' C_s \sum_{p\geq -1} \left(\sum_{r-p\leq 4} 2^{(r-p)s}\right) 2^{2ps} \left\|\triangle_p u\right\|_{L^2}^2 \\ &\leq C'' C_s \sum_{p\geq -1} 2^{2ps} \left\|\triangle_p u\right\|_{L^2}^2\end{aligned}$$

since for any given p, the sum $\sum_{r\leq p+4} 2^{(r-p)s} \leq C$ for a constant C independent of p. Using the equivalent norm of H^s, we have

$$\left\|\Sigma_{-1}\right\|_{H^s} \leq C_s \left\|u\right\|_{H^s}.$$

4. The general case $q \geq 0$ is no more difficult. We have

$$\operatorname{supp} \mathscr{F}(M_{p,q}\triangle_p u) \subset \left\{2^{p+q+1} \leq |\xi| \leq 2^{p+q+5}\right\}$$

and thus

$$\triangle_r(M_{p,q}\triangle_p u) \equiv 0 \qquad \text{for } \ r \geq p+q+6 \ \text{ or } \ r \leq p+q-1.$$

Consequently,

$$\triangle_r \Sigma_p = \sum_{p\geq r-q-5}^{r-q} \triangle_r(M_{p,q}\triangle_p u)$$

and, by (3.5),

$$\left\|\triangle_r(M_{p,q}\triangle_p u)\right\|_{L^2} \leq C \left\|(M_{p,q}\triangle_p u)\right\|_{L^2} \leq C \left\|M_{p,q}\right\|_{L^\infty} \left\|\triangle_p u\right\|_{L^2} \leq \tilde{C}_m 2^{-qm} \left\|\triangle_p u\right\|_{L^2}.$$

By the Cauchy-Schwarz Inequality, we obtain

$$\begin{aligned}\sum_{r\geq -1} 2^{2rs} \left\|\triangle_r \Sigma_q\right\|_{L^2}^2 &= \sum_{r\geq -1} 2^{2rs} \left\|\sum_{p=r-q-5}^{r-q} \triangle_r(M_{p,q}\triangle_p u)\right\|_{L^2}^2 \\ &\leq 6^2 \sum_{r\geq -1} 2^{2rs} \sum_{p=r-q-5}^{r-q} \left\|\triangle_r(M_{p,q}\triangle_p u)\right\|_{L^2}^2 \\ &\leq 6^2 \tilde{C}_m^2 \sum_{r\geq -1} \sum_{p=r-q-5}^{r-q} 2^{2rs-2qm-2ps} 2^{2ps} \left\|\triangle_p u\right\|_{L^2}^2 \\ &\leq 6^2 \tilde{C}_m^2 \sum_{p\geq -1} 2^{2ps} \left\|\triangle_p u\right\|_{L^2}^2 \left(\sum_{r=p+q}^{p+q+5} 2^{2(r-p-q)s}\right) 2^{2q(s-m)} \\ &= 6^2 \tilde{C}_m^2 C_s \left\|u\right\|_{H^s}^2 2^{2q(s-m)},\end{aligned}$$

with $C_s = \sum_{l=0}^{5} 2^{2ls}$. So we proved

$$\|\Sigma_q\|_{H^s} \le C_s \tilde{C}_m \|u\|_{H^s} 2^{q(s-m)}.$$

5. Then, as $s < m$, we conclude

$$\left\|\sum_{p\ge 0} M_p \triangle_p u\right\|_{H^s} \le \sum_{q\ge -1} \|\Sigma_q\|_{H^s} \le C_s \|u\|_{H^s} + C_s\tilde{C}_m \sum_{q\ge 0} 2^{q(s-m)} \|u\|_{H^s} \le C'_{s,m} \|u\|_{H^s}$$

as desired. ☐

The following is an easy consequence of Theorem 3.2 and Theorem 2.1.

Corollary 3.1. *If $F \in \mathscr{C}^\infty(\mathbb{R})$ and $s > d/2$, then there exists a continuous function $C : (0,\infty) \to (0,\infty)$ such that for all u and v in H^s,*

$$\boxed{\|F(u) - F(v)\|_{H^s} \le C(\max(\|u\|_{H^s}, \|v\|_{H^s})) \|u - v\|_{H^s}.}$$

Proof. Without loss of generality, we may assume $F'(0) = 0$, otherwise use

$$F(u) - F(v) = (F(u) - F'(0)u) - (F(v) - F'(0)v) + F'(0)(u - v).$$

By Taylor's Formula and Theorem 2.1, we have

$$\|F(u) - F(v)\|_{H^s} \le \int_0^1 \|F'(v + \theta(u - v))(u - v)\|_{H^s} \,\mathrm{d}\theta$$

$$\le C_1 \left(\max_{|w|\le \max(\|u\|_{L^\infty}, \|v\|_{L^\infty})} |F'(w)| \|u - v\|_{H^s} + \max_{\theta\in[0,1]} \|F'(v + \theta(u - v))\|_{H^s} \|u - v\|_{L^\infty} \right).$$

The first term in the parentheses is already in the wanted form, by the Sobolev Embedding $H^s \hookrightarrow L^\infty$. In the second term, we have

$$\|F'(v + \theta(u - v))\|_{H^s} \|u - v\|_{L^\infty} \le C_0(\|v + \theta(u - v)\|_{L^\infty}) \|v + \theta(u - v)\|_{H^s} \|u - v\|_{H^s}$$

by Theorem 3.2 and Sobolev Embedding $H^s \hookrightarrow L^\infty$, which yields the wanted inequality. ☐

4. Para-differential calculus

The tools introduced in the previous sections provide a basis for what is called para-differential calculus, involving operators whose "symbols" have a limited regularity in x. In particular, the operators T_a encountered in para-products are special cases of para-differential operators.

The purpose of this section is not to develop the whole theory but only present some major aspects. We shall use again the notation $\lambda^s(\xi) \doteq (1 + |\xi|^2)^{s/2}$ for all $s \in \mathbb{R}$.

4.1. **Construction of para-differential operators.**

4.1.1. *Definitions of symbols and associated para-differential operators.*

Definition 4.1. For any real number m and any nonnegative integer k, we define the set Γ^m_k of functions, also called *symbols*, $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}^{N\times N}$ such that

- for almost all $x \in \mathbb{R}^d$, the mapping $\xi \in \mathbb{R}^d \mapsto a(x,\xi)$ is $\mathscr{C}^\infty$,
- for all d-uples β and all $\xi \in \mathbb{R}^d$, the mapping $x \in \mathbb{R}^d \mapsto \partial^\beta_\xi a(x,\xi)$ belongs to $W^{k,\infty}$ and there exists $C_\beta > 0$ so that for all $\xi \in \mathbb{R}^d$,

$$\left\|\partial^\beta_\xi a(\cdot,\xi)\right\|_{W^{k,\infty}} \leq C_\beta \lambda^{m-|\beta|}(\xi). \tag{4.1}$$

Symbols belong to Γ^m_k are said to be of order m and regularity k.

Of course, we have $\mathbf{S}^m \subset \Gamma^m_k$ for all k. That is, any symbol of pseudo-differential operator of order m is a symbol in Γ^m_k, for any positive integer k. The novelty is that functions with rather poor regularity in x are allowed. In particular, $W^{k,\infty}$ functions of x only may be viewed as symbols in Γ^0_k.

Unlike infinitely smooth symbols in $\mathbf{S}^m$, functions in Γ^m_k are not naturally associated with bounded operators $H^s \to H^{s-m}$. But this will be the case for the subclass Σ^m_k of symbols in Γ^m_k satisfying the additional, spectral property:

$$\operatorname{supp}\left(\mathscr{F}(a(\cdot,\xi))\right) \subset B(0;\varepsilon\lambda^1(\xi)) \tag{4.2}$$

for some $\varepsilon \in (0,1)$ *independent* of ξ, see Theorem 4.1 below. Notice that here ξ is considered as a parameter, and the Fourier transform is performed for the first variables. One may argue that since their Fourier transform is compactly supported such symbols are necessarily $\mathscr{C}^\infty$ in x, while we want to handle non-smooth symbols. So where is the trick? In fact, it relies on a special smoothing procedure, associating any symbol $a \in \Gamma^m_k$ with a symbol $\sigma \in \Sigma^m_k$. We shall give more details below. Let us start with the study of operators associated with symbols in Σ^m_k, $k \geq 0$.

Theorem 4.1. *For all* $a \in \Gamma^m_0$ *satisfying* (4.7), *consider*

$$\mathrm{Op}(a) : \mathscr{F}^{-1}(\mathscr{E}') \to \mathscr{C}^\infty_b$$

$$u \mapsto \mathrm{Op}(a)u$$

with

$$(\mathrm{Op}(a)u)(x) = \frac{1}{(2\pi)^d}\langle \mathrm{e}^{\mathrm{i}x\cdot} a(x,\cdot), \hat{u}(\cdot)\rangle_{(\mathscr{C}^\infty,\mathscr{E}')},$$

where $\mathscr{E}'$ *denotes the space of tempered distributions having compact support (i.e., the dual space of* $\mathscr{C}^\infty$*),and the usual ordering* $(\mathscr{C}^\infty,\mathscr{E}')$ *is just meant to account for matrix-valued* a*. The definition of* $\mathrm{Op}(a)$ *coincides with the definition of corresponding pseudo-differential* *operators if $a \in \mathbf{S}^m$. Furthermore, for all $s \in \mathbb{R}$, $\mathrm{Op}(a)$ extends in a unique way into a bounded operator from H^s to H^{s-m}.*

This is a fundamental result, which we admit here.

4.1.2. *Smoothing procedure for symbols with limited regularity.* Let us now describe the smoothing procedure for symbols in Γ_k^m, which amounts to a frequency cut-off depending on the ξ-variables.

Definition 4.2. A $\mathscr{C}^\infty$ function $\chi : (\eta, \xi) \mapsto \chi(\eta, \xi) \in \mathbb{R}^+$ is called an *admissible frequency cut-off* if there exist $\varepsilon_1, \varepsilon_2$ with $0 < \varepsilon_1 < \varepsilon_2 < 1$ so that

$$\begin{cases} \chi(\eta,\xi) = 1, & \text{if } \ |\eta| \le \varepsilon_1 |\xi| \ \text{ and } \ |\xi| \ge 1, \\ \chi(\eta,\xi) = 0, & \text{if } \ |\eta| \ge \varepsilon_2 \lambda^1(\xi) \ \text{ or } \ |\xi| \le \varepsilon_2, \end{cases} \tag{4.3}$$

and if for all d-uples α and β there exists $C_{\alpha,\beta} > 0$ so that

$$|\partial_\eta^\alpha \partial_\xi^\beta \chi(\eta,\xi)| \le C_{\alpha,\beta} \lambda^{-|\alpha|-|\beta|}(\xi). \tag{4.4}$$

Example 4.1. If ψ and ϕ are as in the Littlewood-Paley decomposition, the function χ defined by

$$\boxed{\chi(\eta,\xi) \doteq \sum_{p\ge 0} \psi(2^{2-p}\eta)\phi(2^{-p}\xi) = \sum_{p \ge 0} \psi_{p-2}(\eta)\phi_p(\xi)}$$

is an admissible frequency cut-off.

Indeed, recall first that for given ξ, there are at most two indices p for which $\phi_p(\xi)$ is non-zero. So the sum is locally finite. Furthermore, recalling that

$$\operatorname{supp} \psi_{p-2} \subset \{\eta : |\eta| \le 2^{p-2}\}, \qquad \operatorname{supp} \phi_p \subset \{\xi : 2^{p-1} \le |\xi| \le 2^{p+1}\},$$

it is easy to check that χ vanishes as requested with $\varepsilon_2 = 1/2$. In fact, for $|\xi| \le 1/2$, $\phi_p(\xi) = 0$ for all $p \ge 0$, which implies $\chi(\eta, \xi) = 0$ wherever η is.

Additionally, for all (η, ξ) we have

$$\chi(\eta,\xi) = \sum_{p \ge 0 \quad \text{and} \quad |\eta| \le 2^{p-2}, 2^{p-1} \le |\xi| \le 2^{p+1}} \psi_{p-2}(\eta)\phi_p(\xi).$$

So if $|\xi| \le 2|\eta|$, there will be no index p available in this sum and hence we find $\chi(\eta, \xi) = 0$ whenever $|\xi| \le 2|\eta|$. Since $\lambda^1(\xi) = (1 + |\xi|^2)^{\frac{1}{2}} \ge |\xi|$, so if $|\eta| \ge \frac{1}{2}\lambda^1(\xi)$, then $|\eta| \ge \frac{1}{2}|\xi|$ and $\chi(\eta, \xi) = 0$. Thus we demonstrated the second line in (4.3).

We then show the first line in (4.3). Since $\psi \equiv 1$ on the sphere of radius $1/2$, so $\psi_{p-2}(\eta) = 1$ if $|\eta| < 2^{p-3}$. Suppose $|\xi| \ge 16|\eta|$, then for those indices p for which

$\phi_p(\xi) \neq 0$, we have $2^{p+1} \geq |\xi| \geq 16|\eta|$, hence $|\eta| \leq 2^{p-3}$, and $\psi_{p-2}(\eta) = 1$. So for $|\eta| \leq |\xi|/16$, we have

$$\chi(\eta, \xi) = \sum_{p \geq 0} \phi_p(\xi).$$

If, furthermore, $|\xi| \geq 1$, then $\psi(\xi) = 0$, and by partition of unity $1 = \psi + \sum_{p\geq 0} \phi_p(\xi)$, we get $\chi(\eta, \xi) = 1$ as required. So we may take $\varepsilon_1 = 1/16$.

The inequality (4.4), which means $\chi \in \mathbf{S}^0$ as a function of $2d$ variables, are trivially satisfied for $|\xi| \leq 1/2$ (as shown before, $\chi(\xi, \eta) \equiv 0$ for $|\xi| \leq 1/2$). Otherwise, for $|\xi| \geq 1/2$, let us write

$$\chi(\eta, \xi) = \sum_{p\geq 0:\ \ |\xi| \leq 2^{p+2}} \psi_{p-2}(\eta)\phi_p(\xi),$$

hence

$$\partial_\eta^\alpha \partial_\xi^\beta \chi(\eta, \xi) = \sum_{p\geq 0:\ \ |\xi|\leq 2^{p+2}} 2^{(2-p)|\alpha| - p|\beta|} \partial_\eta^\alpha \psi(2^{2-p}\eta) \partial_\xi^\beta \phi(2^{-p}\xi).$$

For $1 \leq 2|\xi| \leq 2^{p+3}$ we have

$$2^{-p} \leq \frac{4}{|\xi|} \leq \frac{9}{\lambda^1(\xi)},$$

so, recalling the sum is locally finite, we find that

$$|\partial_\eta^\alpha \partial_\xi^\beta \chi(\eta, \xi)| \leq C_0 2^{2|\alpha|} \left(\frac{9}{\lambda^1(\xi)}\right)^{|\alpha|+|\beta|} \left\| \partial_\eta^\alpha \psi \partial_\xi^\beta \phi \right\|_{L^\infty}.$$

This is (4.4) with $C_{\alpha,\beta} = C_0 2^{2|\alpha|} 9^{|\alpha|+|\beta|} \left\| \partial_\eta^\alpha \psi \partial_\xi^\beta \phi \right\|_{L^\infty}$, and we may take $C_0 = 2$ (only two nonzero terms in the sum, for fixed ξ).

Proposition 4.1. *Let χ be an admissible frequency cut-off according to Definition 4.2 and consider the operator*

$$R^\chi : \ a \in \Gamma_k^m \mapsto \sigma \in \mathscr{C}^\infty; \quad \sigma(\cdot, \xi) = K^\chi(\cdot, \xi) *_x a(\cdot, \xi),$$

where the kernel K^χ is defined by

$$K^\chi(\cdot, \xi) = \mathscr{F}^{-1}(\chi(\cdot, \xi)).$$

Then R^χ maps into

$$\Sigma_k^m = \{a \in \Gamma_k^m : \mathrm{supp}\,(\mathscr{F}(a(\cdot, \xi))) \subset B(0; \varepsilon_2 \lambda^1(\xi))\}.$$

Furthermore, if $k \geq 1$, for all $a \in \Gamma_k^m$, $a - R^\chi(a)$ belongs to Γ_{k-1}^{m-1}.

In other words, the symbol $\sigma = R^\chi(a)$ is related to a in Fourier space by

$$\boxed{\mathscr{F}(\sigma(\cdot, \xi)) = \chi(\cdot, \xi) \mathscr{F}(a(\cdot, \xi))}$$

for all $\xi \in \mathbb{R}^d$. In particular, if a is independent of x, then $\mathscr{F}(a(\cdot,\xi)) = a(\xi)\delta$, hence $\mathscr{F}(\sigma(\cdot,\xi)) = \chi(0,\xi)a(\xi)$. So we see if $\chi(0,\xi)$ were equal to 1 for all ξ, we would have $\sigma = a$. This is NOT exactly the case (by analysis before, only for $|\xi| \geq 1$), but σ and a differ by a compactly supported function of ξ (it is $1-\chi(0,\xi)$ and supported in $|\xi| \leq 1$). In terms of operators, this means that $\mathrm{Op}(\sigma)$ differs from the Fourier multiplier associated with a by an infinitely smoothing operator, which is harmless in terms of para-differential calculus.

Proof. **1.** Take $a \in \Gamma_k^m$ and consider $\sigma = R^\chi(a)$. Since $\operatorname{supp}\chi(\cdot,\xi) \subset B(0;\varepsilon_2\lambda^1(\xi))$ by our construction, we get

$$\operatorname{supp}(\mathscr{F}(\sigma(\cdot,\xi))) \subset B(0;\varepsilon_2\lambda^1(\xi)).$$

2. The fact that σ belongs to Γ_k^m requires L^1 estimates of kernel K^χ, namely

$$\boxed{\left\|\partial_\xi^\beta K^\chi(\cdot,\xi)\right\|_{L^1(\mathbb{R}^d)} \leq C_\beta \lambda^{-|\beta|}(\xi).} \tag{4.5}$$

To show this, using the proof of (4.4), we have obtained

$$\partial_\xi^\beta \chi(\eta,\xi) = \sum_{p\geq 0:|\xi|\leq 2^{p+2}} 2^{-p|\beta|}\psi(2^{2-p}\eta)\partial_\xi^\beta\phi(2^{-p}\xi),$$

hence by local finiteness of the sum (for fixed ξ),

$$\left\|\partial_\xi^\beta K^\chi(\cdot,\xi)\right\|_{L^1(\mathbb{R}^d)} \leq C\left\|\mathscr{F}^{-1}(\psi(2^{2-p}\eta))(\cdot)\right\|_{L^1}\left\|\partial_\xi^\beta\phi(\xi)\right\|_{L^\infty} 2^{-p|\beta|} \leq C_\beta\lambda^{-|\beta|}(\xi),$$

because $\|\mathscr{F}^{-1}(\psi(2^{2-p}\eta))(\cdot)\|_{L^1} = \left\|\check{\psi}\right\|_{L^1}$ and $\lambda(\xi) \leq C2^p$ as $|\xi| \leq 2^{p+2}$.

3. Now we show $\sigma(\cdot,\xi)$ satisfies (4.1). By Leibniz Rule and the fact

$$\partial_x^\alpha(R^\chi(a)) = R^\chi(\partial_x^\alpha a)$$

of convolution, there holds, for $|\alpha| \leq k$,

$$\begin{aligned}
|\partial_x^\alpha\partial_\xi^\beta\sigma(x,\xi)| &= |\partial_\xi^\beta(\partial_x^\alpha\sigma(x,\xi))| = |\partial_\xi^\beta(K^\gamma(\cdot,\xi) *_x \partial_x^\alpha a(\cdot,\xi))| \\
&\leq \sum_{|\delta|\leq|\beta|}\left|C_\beta^\delta\partial_\xi^\delta K^\chi(\cdot,\xi) *_x \partial_x^\alpha\partial_\xi^{\beta-\delta}a(\cdot,\xi)\right| \\
&\leq \sum_{|\delta|\leq|\beta|} C_\beta\left\|\partial_\xi^\delta K^\chi(\cdot,\xi)\right\|_{L^1}\left\|\partial_x^\alpha\partial_\xi^{\beta-\delta}a(\cdot,\xi)\right\|_{L^\infty} \\
&\leq C_\beta\lambda^{-|\delta|}(\xi)\lambda^{m-(|\beta|-|\delta|)}(\xi) = C_\beta\lambda^{m-|\beta|}(\xi).
\end{aligned} \tag{4.6}$$

Here we used $a \in \Gamma_k^m$ and (4.1) for a, and (4.5).

4. We now turn to the last claim of the Proposition 4.1. One observes that for $\sigma \in \Sigma_k^m$ with the number ε in (4.7), i.e.,

$$\operatorname{supp}(\mathscr{F}(a(\cdot,\xi))) \subset B(0;\varepsilon\lambda^1(\xi))$$

to be less than $\varepsilon_1/2$, $R^\chi(\sigma)$ is “almost” equal to σ. (Note that $\varepsilon_1, \varepsilon_2$ have been fixed.)

Indeed, observing for $|\xi| \geq 1$, $\lambda^1(\xi) \leq 2|\xi|$, so if $\varepsilon \leq \varepsilon_1/2$, then $|\eta| \leq \varepsilon\lambda^1(\xi) \leq \varepsilon_1|\xi|$. However, $\chi(\eta,\xi) = 1$ if $|\xi| \geq 1$ and $|\eta| \leq \varepsilon_1|\xi|$. So (4.7) with $\varepsilon \leq \varepsilon_1/2$ implies

$$\mathscr{F}(R^\chi(\sigma))(\cdot,\xi) = \mathscr{F}(\sigma(\cdot,\xi))$$

if $|\xi| \geq 1$. In terms of operators, it means that $\mathrm{Op}(R^\chi(\sigma) - \sigma)$ is infinitely smoothing.

5. For $a \in \Gamma_1^m$, let us show now that $a - R^\chi(a)$ belongs to Γ_0^{m-1}. We already know that, also thanks to (4.6),

$$\left\|\partial_\xi^\beta(a - R^\chi(a))\right\|_{W^{1,\infty}} \leq \left\|\partial_\xi^\beta a\right\|_{W^{1,\infty}} + \left\|\partial_\xi^\beta R^\chi(a)\right\|_{W^{1,\infty}} \leq C_\beta \lambda^{m-|\beta|}(\xi), \tag{4.7}$$

and we want to show that

$$\left\|\partial_\xi^\beta(a - R^\chi(a))\right\|_{L^\infty} \leq \tilde{C}_\beta \lambda^{m-1-|\beta|}(\xi).$$

For convenience, we shall denote $b = \partial_\xi^\beta(a - R^\chi(a))$.

There is nothing to prove for $|\xi| \leq 1$ since λ^1 is bounded on the unit ball — using (4.7), we just take $\tilde{C}_\beta = C_\beta \max_{|\xi|\leq 1} |\lambda^1(\xi)|$.

It is more delicate to obtain a bound of $\|b(\cdot,\xi)\|_{L^\infty}$ for $|\xi| > 1$. Littlewood -Paley decomposition would be of help again. Indeed, by definition of R^χ we have

$$\mathscr{F}(b(\cdot,\xi)) = \partial_\xi^\beta[(1-\chi(\cdot,\xi))\mathscr{F}(a(\cdot,\xi))],$$

which vanishes identically on $B(0;\varepsilon_1|\xi|)$ for $|\xi| \geq 1$. Therefore, recalling that $\triangle_q = \mathscr{F}^{-1}(\phi_q\mathscr{F})$ with $\operatorname{supp}\phi_q \subset B(0;2^{q+1})$, we find $\triangle_q(b(\cdot,\xi)) \equiv 0$ for q and ξ such that

$$\varepsilon_1|\xi| > 2^{q+1} \qquad \text{and} \qquad |\xi| \geq 1.$$

Consequently, when $|\xi| \geq 1$, the Littlewood-Paley decomposition of $b(\cdot,\xi)$ reads

$$\boxed{b(\cdot,\xi) = \sum_{q\geq -1:\ \varepsilon_1|\xi|\leq 2^{q+1}} \triangle_q(b(\cdot,\xi)).} \tag{4.8}$$

Furthermore, by Corollary 1.1,

$$\|\triangle_q(b(\cdot,\xi))\|_{L^\infty} \leq C2^{-q}\|b(\cdot,\xi)\|_{W^{1,\infty}},$$

and for $1 \leq |\xi| \leq 2^{q+5}$ (as $\varepsilon_1 = 2^{-4}$), we have

$$2^{-q} \leq \frac{2^5}{|\xi|} \leq \frac{2^6}{\lambda^1(\xi)}.$$

This implies

$$\begin{aligned}
\|b(\cdot,\xi)\|_{L^\infty} &\le \left(\sum_{q\ge\max\{-1,[\log_2(\varepsilon_1|\xi|)]-1\}} 2^{-q}\right)\left(C\,\|b(\cdot,\xi)\|_{W^{1,\infty}}\right)\\
&\le \frac{C}{|\xi|}\,\|b(\cdot,\xi)\|_{W^{1,\infty}} \le \frac{C}{\lambda(\xi)}\,\|b(\cdot,\xi)\|_{W^{1,\infty}}\\
&\le CC_\beta\lambda^{m-|\beta|-1}(\xi). \qquad (4.7)
\end{aligned}$$

6. Now for $k\ge 2$ and $a\in\Gamma_k^m$, we consider $\tilde a=\partial_x^\alpha a$ with $|\alpha|\le k-1$. Then $\tilde a\in\Gamma_1^m$, and $\tilde a-R^\chi(\tilde a)=\partial_x^\alpha(a-R^\chi(a))\in\Gamma_0^{m-1}$ as shown in Step 5. Since such d-uples α are arbitrary, we get $a-R^\chi(a)\in\Gamma_{k-1}^{m-1}$. □

A straightforward consequence of Proposition 4.1 is the following.

Corollary 4.1. *If χ_1 and χ_2 are two admissible cut-off functions, for all $a\in\Gamma_1^m$, $R^{\chi_1}(a)-R^{\chi_2}(a)$ belongs to Γ_0^{m-1}.*

4.1.3. *Para-differential operators and para-products.* Now we introduce the notion of para-differential operator.

Definition 4.3. Let χ be an admissible frequency cut-off according to Definition 4.2. To any symbol $a\in\Gamma_k^m$ we associate the so-called para-differential operator, said to be of order m,

$$T_a^\chi=\mathrm{Op}(R^\chi(a)).$$

In particular, Corollary 4.1 shows that for $a\in\Gamma_1^m$, T_a^χ are unique modulo operators of order $m-1$.

An interesting point concerning para-product is the following remark.

Remark 4.1. If χ is constructed through Littlewood-Paley decomposition as explained above and $k\ge 1$, for any function of x only, $a\in W^{k,\infty}$ viewed as a symbol in Γ_k^0, the operator T_a^χ coincide with para-product operator T_a up to an infinitely smoothing operator.

Indeed, if

$$\chi(\eta,\xi)=\sum_{p\ge0}\psi(2^{2-p}\eta)\phi(2^{-p}\xi)=\sum_{p\ge0}\psi_{p-2}(\eta)\phi_p(\xi),$$

then

$$\begin{aligned}
&\mathscr{F}(R^\chi(a)(\cdot,\xi))=\chi(\cdot,\xi)\mathscr{F}(a)(\cdot)\\
=&\sum_{p=0,1}\psi_{p-2}(\cdot)\phi_p(\xi)\mathscr{F}(a)(\cdot)+\sum_{p\ge2}\mathscr{F}(S_{p-2}(a))\phi_p(\xi).
\end{aligned}$$

Here we used $\mathscr{F}(S_{p-2}a) = \psi_{p-2}\hat{a}$. Hence we get

$$R^\chi(a)(x,\xi) = \sum_{|p|\le 1} \mathscr{F}^{-1}(\psi_{p-2}\mathscr{F}(a))(x)\phi_p(\xi) + \sum_{p\ge 2} S_{p-2}(a)(x)\phi_p(\xi).$$

In terms of operators, this means that for all $u \in \mathscr{F}^{-1}(\mathscr{E}')$,

$$T^\chi_a u = \mathrm{Op}(b)u + T_a u,$$

where, the last term is the usual para-product — Recall, by definition, $T_a u = \sum_{p\ge 2} S_{p-2}a\triangle_p u$, and we have here

$$\mathscr{F}^{-1}_{\xi\to x}(\sum_{p\ge 2} S_{p-2}(a)(x)\phi_p(\xi)\hat{u}(\xi)) = \sum_{p\ge 2} S_{p-2}(a)(x)\triangle_p u(x) = T_a u(x),$$

while

$$b(x,\xi) = \sum_{|p|\le 1} \mathscr{F}^{-1}(\psi_{p-2}\hat{a})(x)\phi_p(\xi)$$

satisfies $\mathscr{F}_{x\to\eta}(b(x,\xi)) = \sum_{|p|\le 1} \psi_{p-2}(\eta)\hat{a}(\eta)\phi_p(\xi)$. We note $\operatorname{supp}\psi_{p-2}(\eta) \subset B(0;2^{p-2}) \subset B(0;1/2)$ since $p = 0,1$, while $\lambda^1(\xi) \ge 1$, so we see $\operatorname{supp}\mathscr{F}_{x\to\eta}(b(x,\xi)) \subset B(0;\frac{1}{2}\lambda^1(\xi))$. This is (4.7) with $\varepsilon = 1/2$. Also, we see $\mathscr{F}_{x\to\eta}(b(x,\xi))$ is compactly supported in ξ. So $\mathrm{Op}(b)$ is an infinitely smoothing operator.

Remark 4.2. For $a \in W^{k,\infty}$ a function depending only on x, viewed as a symbol in Γ^0_k, there holds

$$T^\chi_a \partial_j u = T^\chi_{\mathrm{i}\xi_j a} u.$$

To see this, we first show $(\mathrm{i}\xi_j) \times (R^\chi(a)(x,\xi)) = R^\chi(\mathrm{i}\xi_j a)(x,\xi)$. Indeed, by definition,

$$R^\chi(\mathrm{i}\xi_j a)(x,\xi) = \mathscr{F}^{-1}_{\eta\to x}(\mathrm{i}\xi_j\hat{a}(\eta)\chi(\eta,\xi)) = \mathrm{i}\xi_j\mathscr{F}^{-1}_{\eta\to x}(\hat{a}(\eta)\chi(\eta,\xi)) = \mathrm{i}\xi_j R^\chi(a).$$

Then we get

$$\begin{aligned}(T^\chi_a \partial_j u)(x) &= \frac{1}{(2\pi)^d}\int \mathrm{e}^{\mathrm{i}x\cdot\xi}(R^\chi(a))(x,\xi)\mathrm{i}\xi_j\hat{u}(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^d}\int \mathrm{e}^{\mathrm{i}x\cdot\xi}(R^\chi(\mathrm{i}\xi_j a))(x,\xi)\hat{u}(\xi)\,\mathrm{d}\xi \\ &= (T^\chi_{\mathrm{i}\xi_j a}u)(x).\end{aligned}$$

4.2. **Basic results on para-differential calculus.** We omit below the superscript χ, as all results being valid for any admissible frequency cut-off.

Theorem 4.2. *For all $a \in \Gamma^m_1$, the adjoint operator $(T_a)^*$ is of order m and $(T_a)^* - T_{a^*}$ is of order $m-1$.*

Theorem 4.3. *For all $a \in \Gamma_1^m$ and $b \in \Gamma_1^n$, the product ab belongs to Γ_1^{m+n} and $T_a \circ T_b - T_{ab}$ is a para-differential operator of order $m+n-1$, associated with a symbol in Γ_0^{m+n-1}. In particular, if the symbols a and b commute — for example, if at least one of them is scalar-valued — the commutator $[T_a, T_b]$ is of order $m+n-1$.*

Proposition 4.2. *If $a \in \Gamma_1^{2m}$, there exists $C > 0$ such that for all $u \in H^m$,*

$$|\mathrm{Re}\langle T_a u, u\rangle| \leq C \|u\|_{H^m}^2 .$$

Proof. We have (recall Λ is the operator associated with the symbol $\lambda(\xi)$)

$$|\mathrm{Re}\langle T_a u, u\rangle| = |\mathrm{Re}\langle \Lambda^{-m} T_a u, \Lambda^m u\rangle| \leq \left\|\Lambda^{-m} \circ T_a(u)\right\|_{L^2} \|\Lambda^m u\|_{L^2} = \left\|\Lambda^{-m} \circ T_a(u)\right\|_{L^2} \|u\|_{H^m} .$$

Now note $\Lambda^{-m} - T_{\lambda^{-m}} \doteq T_1$ is an infinitely smoothing operator (see the remark following Proposition 4.1), and by Theorem 4.3, $T_{\lambda^{-m}a} - T_{\lambda^{-m}} \circ T_a \doteq T_2$ is of order $-m+2m-1 = m-1$, so we get $\Lambda^{-m} \circ T_a = T_1 \circ T_a + T_{\lambda^{-m}a} - T_2$, hence $\|\Lambda^{-m} \circ T_a u\|_{L^2} \leq \|T_1 \circ T_a u\|_{L^2} + \|T_{\lambda^{-m}a} u\|_{L^2} + \|T_2 u\|_{L^2} \leq C(\|u\|_{H^m} + \|u\|_{H^m} + \|u\|_{H^{m-1}}) \leq C \|u\|_{H^m}$. Here we also used $T_{\lambda^{-m}a}$ is of order m. □

Theorem 4.4 (Gårding Inequality)**.** *If $a \in \Gamma_1^{2m}$ is such that for some positive α,*

$$a(x,\xi) + a(x,\xi)^* \geq \alpha \lambda^{2m}(\xi) I_N$$

(in the sense of Hermitian matrices) for all $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$, then there exists $C > 0$ so that for all $u \in H^m$,

$$\mathrm{Re}\langle T_a u, u\rangle \geq \frac{\alpha}{4} \|u\|_{H^m}^2 - C \|u\|_{H^{m-\frac{1}{2}}}^2 . \tag{4.9}$$

One may also state a sharpened version of Gårding's inequality in this context, but for smoother symbols (at least $\mathscr{C}^2$ in x).

5. Para-differential calculus with a parameter

5.1. **Lower-regular symbols and Littlewood–Paley decomposition with a parameter.** The final refinement in this review of modern analysis tools concerns families of para-differential operators depending on one parameter, as extensions of pseudo-differential operators with parameters.

We denote

$$\lambda^{s,\gamma}(\xi) \doteq \lambda^s(\xi,\gamma) = (\gamma^2 + |\xi|^2)^{s/2},$$

and define parameter-dependent symbols of limited regularity as follows. Notice in particular that we require that $\gamma \geq 1$ in the following, which induces some differences from the standard para-differential calculus.

Definition 5.1. For any real number m and any nonnegative integer k, the set Γ_k^m consists of functions $a : \mathbb{R}^d \times \mathbb{R}^d \times [1,\infty) \to \mathbb{C}^{N\times N}$ that are $\mathscr{C}^\infty$ in ξ and such that for all d-uple β, there exists $C_\beta > 0$ so that for all $(\xi,\gamma) \in \mathbb{R}^d \times [1,\infty)$,

$$\left\| \partial_\xi^\beta a(\cdot,\xi,\gamma) \right\|_{W^{k,\infty}} \leq C_\beta \lambda^{m-|\beta|,\gamma}(\xi). \tag{5.1}$$

The subset Σ_k^m is made of symbols $a \in \Gamma_k^m$ satisfying the spectral requiement

$$\operatorname{supp}\left(\mathscr{F}(a(\cdot,\xi,\gamma))\right) \subset B(0;\varepsilon\lambda^{1,\gamma}(\xi)) \tag{5.2}$$

for some $\varepsilon \in (0,1)$ independent of (ξ,γ).

The analogous of Theorem 4.1 is the following fundamental result.

Theorem 5.1. *Any symbol $a \in \Sigma_0^m$ can be associated with a family of operators denoted by $\{\mathrm{Op}^\gamma(a)\}_{\gamma\geq 1}$, defined on temperate distributions with a compact spectrum by*

$$\mathrm{Op}^\gamma(a) : \mathscr{F}^{-1}(\mathscr{E}') \to \mathscr{C}_b^\infty$$

$$u \mapsto \mathrm{Op}^\gamma(a)u; \quad (\mathrm{Op}^\gamma(a)u)(x) = \frac{1}{(2\pi)^d}\langle \mathrm{e}^{\mathrm{i}x\cdot} a(x,\cdot,\gamma), \hat{u}(\cdot)\rangle_{(\mathscr{C}^\infty,\mathscr{E}')}.$$

This definition of $\mathrm{Op}^\gamma(a)$ coincides with that of pseudo-differential operators with a parameter, if $a \in \mathbf{S}^m$, Furthermore, for all $s \in \mathbb{R}$ and $\gamma \geq 1$, $\mathrm{Op}^\gamma(a)$ extends in a unique way into a bounded operator from H^s to H^{s-m}, and there exists $C_s > 0$ independent of γ and u so that

$$\left\|\mathrm{Op}^\gamma(a)u\right\|_{H_\gamma^{s-m}} \leq C_s \left\|u\right\|_{H_\gamma^s}.$$

The norm of H_γ^s is given by $\|u\|_{H_\gamma^s}^2 \doteq \|\lambda^{s,\gamma}(\xi)\hat{u}(\xi)\|_{L^2}^2$. We omit the proof of this theorem. It makes use of a parameter version of Littlewood–Paley decomposition, based on cut-off functions in the (ξ,γ)-space. Namely, taking $\psi \in \mathscr{D}(\mathbb{R}^d \times \mathbb{R})$ with $\psi(\xi,\gamma) = \Psi((\gamma^2+|\xi|^2)^{1/2})$ and Ψ monotonically decaying such that

$$\Psi(r) = 1 \quad \text{if} \quad r \leq 1/2, \qquad \Psi(r) = 0 \quad \text{if} \quad r \geq 1,$$

and denoting

$$\psi_q^\gamma(\xi) = \psi(2^{-q}\xi, 2^{-q}\gamma), \quad \phi(\xi,\gamma) = \psi(\xi/2,\gamma/2) - \psi(\xi,\gamma),$$
$$\phi_q^\gamma(\xi) = \phi(2^{-q}\xi, 2^{-q}\gamma),$$

we may define operators S_q^γ and $\triangle_q^\gamma$ of symbols, respectively, ψ_q^γ and ϕ_q^γ. Observing that $\triangle_q^\gamma = 0$ for $\gamma \geq 2^{q+1}$, and in particular $\triangle_{-1}^\gamma = 0$ for $\gamma \geq 1$, we easily check that

$$\sum_{p\geq 0} \triangle_p^\gamma = \mathrm{id}$$

in $\mathscr{S}'$. (The $p=-1$ term vanishes identically, and for fixed ξ,γ, it is a finite sum in frequency space.) Furthermore, the analogue of Proposition 1.2 for the standard H^s norm is the following for the H^s_γ norm.

Proposition 5.1. *For all $s\in\mathbb{R}$, $u\in H^s_\gamma(\mathbb{R}^d)$ if and only if*

$$\sum_{p\geq 0} 2^{2ps}\left\|\triangle_p^\gamma u\right\|_{L^2}^2<\infty$$

for all $\gamma\geq 1$. In addition, there exists $C_s>1$ so that

$$\frac{1}{C_s}\sum_{p\geq 0} 2^{2ps}\left\|\triangle_p^\gamma u\right\|_{L^2}^2\leq\|u\|_{H^s_\gamma}^2\doteq\|\lambda^{s,\gamma}(\xi)\hat{u}(\xi)\|_{L^2}^2\leq C_s\sum_{p\geq 0} 2^{2ps}\left\|\triangle_p^\gamma u\right\|_{L^2}^2$$

for all $\gamma\geq 1$.

5.2. **Smoothing procedure for lower-regular symbols with a parameter.** Knowing Theorem 5.1, it is then possible to define a family of operators associated with all symbols a in Γ^m_k. The procedure is the same as in standard (that is, without parameter) para-differential calculus. The basic tool is a so-called admissible cut-off function.

Definition 5.2. A $\mathscr{C}^\infty$ function $\chi:(\eta,\xi,\gamma)\in\mathbb{R}^d\times\mathbb{R}^d\times[1,\infty)\mapsto\chi(\eta,\xi,\gamma)\in\mathbb{R}^+$ is termed as an *admissible frequency cut-off function* if there exist $\varepsilon_{1,2}$ with $0<\varepsilon_1<\varepsilon_2<1$ so that

$$\begin{cases}\chi(\eta,\xi,\gamma)=1, & \text{if } \ |\eta|\leq\varepsilon_1\lambda^1(\xi,\gamma),\\ \chi(\eta,\xi,\gamma)=0, & \text{if } \ |\eta|\geq\varepsilon_2\lambda^1(\xi,\gamma),\end{cases}$$

and if for all d-uples α and β there exists $C_{\alpha,\beta}>0$ so that

$$|\partial_\eta^\alpha\partial_\xi^\beta\chi(\eta,\xi,\gamma)|\leq C_{\alpha,\beta}\lambda^{-|\alpha|-|\beta|}(\xi,\gamma). \tag{5.3}$$

Example 5.1. If ψ and ϕ are as in the Littlewood-Paley decomposition with parameter described above, the function χ defined by

$$\chi(\eta,\xi,\gamma)=\sum_{p\geq 0}\psi(2^{2-p}\eta,0)\phi(2^{-p}\xi,2^{-p}\gamma)$$

is an admissible frequency cut-off with $\varepsilon_1=1/16$ and $\varepsilon_2=1/2$.

The verification is easy. For $\phi(2^{-p}\xi,2^{-p}\gamma)\neq 0$, we need $2^{p-1}\leq\lambda^{1,\gamma}(\xi)\leq 2^{p+1}$, so $|\eta|\geq\frac{1}{2}\lambda^1(\xi,\gamma)$ implies, for the term to be nonzero, $|\eta|\geq 2^{p-2}$, while $\operatorname{supp}\psi(2^{2-p}\eta,0)\subset\{|\eta|\leq 2^{p-2}\}$, this means all terms are zero and hence $\chi(\eta,\xi,\gamma)=0$.

For fixed (ξ,γ), we have

$$\chi(\eta,\xi,\gamma)=\sum_{p:\ 2^{p-1}\leq\lambda^1(\xi,\gamma)\leq 2^{p+1}}\psi(2^{2-p}\eta,0)\phi(2^{-p}\xi,2^{-p}\gamma).$$

For such indices p, as $\varepsilon_1 = 2^{-4}$ if $|\eta| \leq 2^{-4}\lambda^1(\xi,\gamma) \leq 2^{-4+p+1}$, we get $2^{2-p}|\eta| \leq 2^{2-p-3+p} = 1/2$, so $\psi(2^{2-p}\eta, 0) \equiv 1$, and then we get

$$\chi(\eta,\xi,\gamma) = \sum_{p:\ 2^{p-1}\leq\lambda^1(\xi,\gamma)\leq 2^{p+1}} \phi(2^{-p}\xi, 2^{-p}\gamma) = \sum_{p\geq 0} \phi_p^\gamma(\xi) = 1.$$

Proposition 5.2. *Let χ be an admissible frequency cut-off according to Definition 5.2 and consider the operator*

$$R^\chi : a \in \Gamma_k^m \mapsto \sigma \in \mathscr{C}^\infty; \qquad \sigma(\cdot,\xi,\gamma) = K^\gamma(\cdot,\xi,\gamma) *_{\textbf{.}} a(\cdot,\xi,\gamma),$$

where the kernel K^γ is defined by

$$K^\gamma(\cdot,\xi,\gamma) = \mathscr{F}^{-1}(\chi(\cdot,\xi,\gamma)).$$

Then R^χ maps into

$$\Sigma_k^m \doteq \big\{a \in \Gamma_k^m : \ \operatorname{supp}(\mathscr{F}(a(\cdot,\xi,\gamma))) \subset B(0;\varepsilon_2\lambda^{1,\gamma}(\xi))\big\}.$$

Furthermore, if $k \geq 1$, for all $a \in \Gamma_k^m$, $a - R^\chi(a)$ belongs to Γ_{k-1}^{m-1}.

Remark 5.1. Since $\chi(0,\xi,\gamma) \equiv 1$, $R^\chi(a) = a$ for all symbols a depending only on (ξ,γ). [4]

5.3. **Para-differential operators with a parameter and para-product.**

Definition 5.3. If χ is an admissible frequency cut-off, to any symbol $a \in \Gamma_k^m$ we associate the family of para-differential operators $\{T_a^{\chi,\gamma}\}_{\gamma\geq 1}$ defined by

$$T_a^{\chi,\gamma} \doteq \operatorname{Op}^\gamma(R^\chi(a)).$$

Remark 5.2. If the symbol a is a function of x only, $a \in W^{k,\infty}$, it can be viewed as a symbol in Γ_k^0 and $T_a^{\chi,\gamma}u$ is a parameter version of the para-product of a and u. More precisely, if the cut-off function χ is based on the Littlewood-Paley decomposition with parameter in the way explained above, we have [5]

$$T_a^{\chi,\gamma}u = \sum_{p\geq 0} S_{p-2}^0 a \triangle_p^\gamma u, \tag{5.4}$$

where $S_p^0 \doteq \mathscr{F}^{-1}(\psi_p^0\mathscr{F})$ with $\psi_p^0(\xi) \doteq \psi(2^{-p}\xi, 0) = \Psi(2^{-p}|\xi|)$ (as in the standard Littlewood-Paley decomposition, except for the definitions of S_{-2}, S_{-1}, which were taken to be zero).

[4]Verification:

$$\mathscr{F}(R^\chi(a))(\eta,\xi,\gamma) = \chi(\eta,\xi,\gamma)a(\xi,\gamma)\delta_{\eta=0} = \chi(0,\xi,\gamma)a(\xi,\gamma)\delta_{\eta=0} = a(\xi,\gamma)\delta_{\eta=0} = \mathscr{F}_{x\to\eta}(1(x)a(\xi,\gamma))(\eta,\xi,\gamma).$$

So $R^\chi(a) = a$ whenever a is a Fourier multiplier.

[5]We have

$$(R^\chi a)(x,\xi,\gamma) = \mathscr{F}^{-1}_{\eta\to x}(\chi(\eta,\xi,\gamma)\hat{a}(\eta)) = \sum_{p\geq 0}(S_{p-2}^0 a)(x)\phi_p^\gamma(\xi).$$

So $T_a^{\chi,\gamma}u = \mathscr{F}^{-1}_{\xi\to x}\big[\sum_{p\geq 0}(S_{p-2}^0 a)(x)\phi_p^\gamma(\xi)\hat{u}(\xi)\big] = \sum_{p\geq 0}(S_{p-2}^0 a)(x)(\triangle_p^\gamma u)(x)$.

For simplicity, we shall now omit the dependence on χ and just denote T_a^γ.

Remark 5.3. For a symbol of the form

$$a(x,\xi,\gamma)=p(\xi)b(x,\gamma),$$

the regularized symbol is given by $R^\chi(a)(x,\xi,\gamma)=p(\xi)R^\chi(b)(x,\xi,\gamma)$. Applying this in particular to polynomials p, we see that for any d-uple α, [6]

$$T_b^\gamma\partial^a u=T^\gamma_{\mathrm{i}^{|\alpha|}\xi^\alpha b}u.$$

5.4. **Basic results on para-differential calculus.** It is important for the applications to be able to estimate the error when replacing products by para-products. Such estimates are given in Theorem 2.7 and its corollary for standard para-products. We have similar results for T_a^γ, when differential operator is replaced by para-differential operator, which shows that $a-T_a^\gamma$ is of order -1 as soon as a is Lipschitz.

Theorem 5.2. *There exists $C>0$ so that, for all $a\in W^{1,\infty}$ and $u\in L^2(\mathbb{R}^d)$, for all $\gamma\ge 1$,*

$$\begin{aligned}
\gamma\left\|au-T_a^\gamma u\right\|_{L^2} &\le C\left\|a\right\|_{W^{1,\infty}}\left\|u\right\|_{L^2}, &(5.5)\\
\left\|a\partial_j u-T_a^\gamma\partial_j u\right\|_{L^2}=\left\|a\partial_j u-T^\gamma_{i\xi_j a}u\right\|_{L^2} &\le C\left\|a\right\|_{W^{1,\infty}}\left\|u\right\|_{L^2}, &(5.6)\\
\left\|au-T_a^\gamma u\right\|_{H^1_\gamma} &\le C\left\|a\right\|_{W^{1,\infty}}\left\|u\right\|_{L^2}. &(5.7)
\end{aligned}$$

Proof. **1.** The first inequality is easy to show. The factor γ comes from the fact that $\triangle_q^\gamma u=0$ for $\gamma\ge 2^{q+1}$. Indeed, the fact that a is Lipschitz implies, by Corollary 1.1 for the standard Littlewood–Paley decomposition, that

$$\left\|\triangle_q^0 a\right\|_{L^\infty}\lesssim 2^{-q}\left\|a\right\|_{W^{1,\infty}},$$

[6] $T_b^\gamma(\partial^\alpha u)=\mathscr{F}^{-1}_{\xi\to x}\Big(R^\chi(b(x,\gamma))(\mathrm{i}\xi)^\alpha\hat u(\xi)\Big)=\mathscr{F}^{-1}_{\xi\to x}\Big(R^\chi((\mathrm{i}\xi)^\alpha b(x,\gamma))\hat u(\xi)\Big)=T^\gamma_{(\mathrm{i}\xi)^\alpha b}u.$

and therefore the series $\sum \triangle_q^0 a$ is normally convergent in L^∞. Take $u \in \mathscr{S}$. Then the series $\sum \triangle_p^\gamma u$ is normally convergent in L^2 and $u = \sum \triangle_p^\gamma u$. Therefore,

$$\begin{aligned}
au - T_a^\gamma u &= \sum_{q\geq -1}\sum_{p\geq 0} \triangle_q^0 a \triangle_p^\gamma u - \sum_{p\geq 0} S_{p-2}^0 a \triangle_p^\gamma u \\
&= \sum_{q\geq -1}\sum_{p\geq 0} \triangle_q^0 a \triangle_p^\gamma u - \sum_{p\geq 0}\sum_{q=-1}^{p-3} \triangle_q^0 a \triangle_p^\gamma u \\
&= \sum_{q\geq -1}\sum_{p\geq 0} \triangle_q^0 a \triangle_p^\gamma u - \sum_{q\geq -1}\sum_{p\geq q+3} \triangle_q^0 a \triangle_p^\gamma u \\
&= \sum_{q\geq -1} \triangle_q^0 a \sum_{p=0}^{q+2} \triangle_p^\gamma u = \sum_{q\geq -1} \triangle_q^0 a S_{q+3}^\gamma u \\
&= \sum_{2^{q+3}\geq \gamma} \triangle_q^0 a S_{q+3}^\gamma u.
\end{aligned}$$

For the last equality, we used the fact that $S_q^\gamma = 0$ for $\gamma \geq 2^q$. Hence

$$\|au - T_a^\gamma u\|_{L^2} \lesssim \left(\sum_{2^{q+3}\geq\gamma} 2^{-q}\right) \|a\|_{W^{1,\infty}} \|u\|_{L^2} \lesssim \frac{1}{\gamma} \|a\|_{W^{1,\infty}} \|u\|_{L^2}.$$

Here we have used the fact that

$$\left\|S_q^\gamma u\right\|_{L^2} \lesssim \|u\|_{L^2},$$

which comes from the definition of S_q^γ, a $L^1 * L^2$ convolution estimate and a uniform bound for $\left\|\mathscr{F}^{-1}(\psi_q^\gamma)\right\|_{L^1}$. The derivation of the latter bound comes from the observation that

$$\sup_{1\leq\gamma<\infty} \left\|\mathscr{F}_\xi^{-1}(\psi_q^\gamma)\right\|_{L^1(\mathbb{R}^d)} \leq \left\|\mathscr{F}_{(\gamma,\xi)}^{-1}(\psi_q)\right\|_{L^1(\mathbb{R}\times\mathbb{R}^d)} = \left\|\mathscr{F}_{(\gamma,\xi)}^{-1}(\psi)\right\|_{L^1(\mathbb{R}\times\mathbb{R}^d)}.$$

This is proved below.

1. We show the first inequality above holds. We note that (L_γ^∞ means the norm is taken with respect to the variable γ)

$$\int |(\mathscr{F}_{\xi\to\eta}^{-1}(\psi_q(\xi,\gamma)))(\eta,\gamma)|\,\mathrm{d}\eta \leq \int \left\|(\mathscr{F}_{\xi\to\eta}^{-1}(\psi_q(\xi,\gamma)))(\eta,\gamma)\right\|_{L_\gamma^\infty} \mathrm{d}\eta,$$

and the fact Fourier Transform is of type $(1,\infty)$ implies

$$\begin{aligned}
\left\|(\mathscr{F}_{\xi\to\eta}^{-1}(\psi_q(\xi,\gamma)))(\eta,\gamma)\right\|_{L_\gamma^\infty} &\leq \left\|\mathscr{F}_{\gamma\to\delta}^{-1}((\mathscr{F}_{\xi\to\eta}^{-1}(\psi_q(\xi,\gamma)))(\eta,\gamma))(\eta,\delta)\right\|_{L_\delta^1} \\
&= \left\|(\mathscr{F}_{\xi\to\eta,\gamma\to\delta}^{-1}(\psi_q(\xi,\gamma)))(\eta,\delta))\right\|_{L_\delta^1}.
\end{aligned}$$

So

$$\left\|\mathscr{F}_\xi^{-1}(\psi_q^\gamma)\right\|_{L^1(\mathbb{R}^d)} = \int |(\mathscr{F}_{\xi\to\eta}^{-1}(\psi_q(\xi,\gamma)))(\eta,\gamma)|\,\mathrm{d}\eta \leq \left\|(\mathscr{F}_{\xi\to\eta,\gamma\to\delta}^{-1}(\psi_q(\xi,\gamma)))(\eta,\delta))\right\|_{L_{\eta,\delta}^1},$$

and taking the supremum of the left hand side with respect to γ, we then get

$$\sup_{1\le\gamma<\infty}\left\|\mathscr{F}_\xi^{-1}(\psi_q^\gamma)\right\|_{L^1(\mathbb{R}^d)}\le\left\|\mathscr{F}_{(\gamma,\xi)}^{-1}(\psi_q)\right\|_{L^1(\mathbb{R}\times\mathbb{R}^d)}.$$

2. We omit the proof of the second inequality.

3. The third inequality is an easy consequence of the first two. First, by definition, we have (taking Fourier Transform, there holds $(\gamma^2+|\xi|^2)\hat{f}^2=\gamma^2\hat{f}^2+|\xi|^2\hat{f}^2$ and then taking L^2 norm and using Plancherel's Theorem)

$$\boxed{\|f\|_{H^1_\gamma}^2\le\gamma^2\|f\|_{L^2}^2+\|\nabla f\|_{L^2}^2}$$

for all $f\in H^1_\gamma$ (a kind of interpolation inequality). Secondly, we note, by (5.4), $\partial_j(T_a^\lambda u)=T_a^\lambda(\partial_j u)+T_{\partial_j a}^\lambda u$, as ∂_j commutes with S_{p-2}^0 and $\triangle_q^\gamma$.

So we get, by $\partial_j(au-T_a^\gamma u)=a\partial_j u-T_a^\gamma(\partial_j u)+(\partial_j a)u-T_{\partial_j a}^\gamma u$,

$$\begin{aligned}\left\|\partial_j(au-T_a^\lambda u)\right\|_{L^2}^2 &\le 3\left\|a\partial_j u-T_a^\gamma(\partial_j u)\right\|_{L^2}^2+3\left\|(\partial_j a)u\right\|_{L^2}^2+3\left\|T_{\partial_j a}^\gamma u\right\|_{L^2}^2\\ &\le 3C^2\|a\|_{W^{1,\infty}}^2\|u\|_{L^2}^2+3(1+C_0^2)\|\partial_j a\|_{L^\infty}^2\|u\|_{L^2}^2,\end{aligned}$$

where C_0 comes from the basic estimate

$$\|T_b^\gamma u\|_{L^2}\le C_0\|b\|_{L^\infty}\|u\|_{L^2}.$$

Therefore, using (5.5) to handle the first term,

$$\begin{aligned}\|au-T_a^\gamma u\|_{H^1_\gamma}^2 &\le \gamma^2\|au-T_a^\gamma u\|_{L^2}^2+\sum_j\|\partial_j(au-T_a^\gamma u)\|_{L^2}^2\\ &\le ((1+3d)C^2+3d(1+C_0^2))\|a\|_{W^{1,\infty}}^2\|u\|_{L^2}^2.\end{aligned}$$

□

Other basic results, similar to those in pseudo-differential calculus with parameter, are listed in the following.

Theorem 5.3. *For all $a\in\Gamma_1^m$, the family of adjoint operators $\{(T_a^\gamma)^*\}_{\gamma\ge1}$ is of order m and the family $\{(T_a)^*-T_{a^*}\}_{\gamma\ge1}$ is of order (less than or equal to) $m-1$.*

Theorem 5.4. *For all $a\in\Gamma_1^m$ and $b\in\Gamma_1^n$, the product ab belongs to Γ_1^{m+n} and the family $\{T_a^\gamma\circ T_b^\gamma-T_{ab}^\gamma\}_{\gamma\ge1}$ is of order (less than or equal to) $m+n-1$.*

Theorem 5.5 (Gårding Inequality)**.** *If $a\in\Gamma_1^{2m}$ is such that for some positive α,*

$$a(x,\xi,\gamma)+a(x,\xi,\gamma)^*\ge\alpha\lambda^{2m,\gamma}(\xi)I_N$$

(in the sense of Hermitian matrices) for all $(x, \xi, \gamma) \in \mathbb{R}^d \times \mathbb{R}^d \times [1, \infty)$*, then there exists* $\gamma_0 \geq 1$ *so that for all* $\gamma \geq \gamma_0$ *and all* $u \in H^m$,

$$\operatorname{Re}\langle T_a^\gamma u, u\rangle \geq \frac{\alpha}{4} \|u\|^2_{H^m_\gamma}. \tag{5.8}$$

We see the weight γ helps to absorb a lower order term which appeared in the standard Gårding Inequality.

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LECTURE NOTES 7: PERSISTENCE OF SHOCKS IN DUCTS

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In this note, we show how to prove local in time existence and stability of shock waves in non-isentropic compressible Euler flows in two-dimensional straight duct, provided that the shock satisfies the uniform stability condition, and the upcoming supersonic flow and the pressure at the exit of the duct, as well as the initial data satisfy certain orders of compatibility and symmetry conditions. The note is based upon my research paper: [Yuan, Hairong: Persistence of shocks in ducts. Nonlinear Anal. 75 (2012), no. 9, 3874–3894]. For the most recent developments on the subject, see [Fang, Beixiang; Xiang, Wei; Xiao, Feng: Persistence of the steady normal shock structure for the unsteady potential flow. SIAM J. Math. Anal. 52 (2020), no. 6, 6033–6104].

1. Introduction

In the past decade, following the work of Chen and Feldman [1], there has been an intensive study on transonic shocks in nozzles or ducts in the framework of steady potential flow equation or steady Euler system (see [2, 3, 4, 5, 6, 7] and references therein). Up to now, particularly in the two-space-dimensional case, rather complete knowledge on construction of exact solutions, stability under various boundary conditions on upstream and downstream flows, and uniqueness in the class of piecewise C^1 flow fields are available. With these achievements, it is natural to investigate such transonic shocks in nozzles for unsteady flows. A basic problem is whether such shocks are stable for a short time.

If there is no other boundaries, this problem on persistence of multidimensional shock waves for unsteady Euler system has been well studied for a longer time, since Majda [8], see also [9, 10] and references therein. It turns out the shock needs to satisfy a uniform stability condition to guarantee it is stable local in time in a strong sense. It is then of interests to know what happens if there are other boundaries in the flow field. This lecture is devoted to studying such a problem, where appear solid walls as well as entry and exit of a duct. We prove that, for a two-dimensional straight duct, if the reference normal shock satisfies the uniform stability condition, and the upcoming (unsteady) supersonic flow, the pressure at the exit, and the initial flow field satisfy $(m-1)$-order of compatibility and appropriate symmetry conditions, and the initial flow field is close to the reference normal

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shock in the L^∞ sense, then for a short time, there is one and uniquely one piecewise H^m flow field containing a shock-front of class H^{m+1}. Here H^m is the usual Sobolev space defined on appropriate domains, and $m \geq 3$ is a fixed integer.

One of the difficulty in the problem is the space domain – the duct – is non-smooth, and part of the boundary – the solid wall with slip condition – is characteristic, which also intersects with the shock-front. As mentioned in [11, Section E in p.60], this situation "gathers almost all difficulties encountered in the study of mixed problems". Up to now, there is no general theory on initial-boundary value problems of hyperbolic systems in non-smooth (space) domains, especially when there involve characteristic boundaries (see [12] for some developments). This is why in [13] Gazzola and Secchi need a reflection technique in the study of compressible isentropic flows (without shocks) in ducts – it bypasses the difficulty caused by the characteristic boundaries. As a first step on the analysis of persistence of shocks in general nozzles, we also employ a symmetry argument, as in [1, 3, 13] to handle the characteristic boundaries. The price is that we need to introduce Sobolev spaces of certain symmetric functions and take great care to make sure the constructed approximate solution still shares these symmetry properties, see section 3.

In the following section 2, we formulate the problem of persistence of shocks in duct as a nonclassical initial-boundary value problem in a rectangle, introduce some Sobolev spaces of functions that can be periodically extended in one variable, and then state the main result, Theorem 2.1. In section 3, we present compatibility conditions and construct an approximate solution to the nonlinear problem. Special attention is paid to make sure the approximate solution also enjoys symmetry properties of the initial-boundary data. In section 4 we study the linearized problem. The crucial point is by suitable localization and extension of operators to reduce the linearized problem to several classical problems in half-space, for which existence and regularity results are now directly available from [9] or [10]. The results on linear problem enable us to use Banach fixed-point theorem to prove Theorem 2.1 in Section 5.

2. Formulation of problem and main result

In the following we first formulate a free boundary problem. By fixing the free-boundary – the shock-front, we get a nonclassical initial-boundary value problem of the unsteady Euler equations in a rectangle. Finally, after introducing some function spaces, we give a precise statement of our main result.

2.1. **Shock wave in a duct.** The two-dimensional duct Ω we considered in this lecture is given by

$$\Omega \doteq \{(x,y)\in\mathbb{R}^2:\ -1<x<1,\ 0<y<1\}.$$

We use Γ_1,Γ_0 to denote the upper and lower wall of the duct:

$$\Gamma_k \doteq \{(x,y)\in\mathbb{R}^2:\ -1<x<1,\ y=k\},\quad k=0,\ 1.$$

The fluid is assumed to flow in Ω through the entry Σ_{-1}, and flow out through the exit Σ_1, where

$$\Sigma_s \doteq \{(x,y)\in\mathbb{R}^2:\ x=s,\ 0<y<1\},\quad s=-1,\ 1.$$

We consider compressible, inviscid and non-heat-conducting fluid flows in the duct. It is governed by the unsteady compressible complete Euler equations:

$$\partial_t\rho+\partial_x(\rho u)+\partial_y(\rho v) = 0, \tag{2.1}$$

$$\partial_t(\rho u)+\partial_x(\rho u^2+p)+\partial_y(\rho uv) = 0, \tag{2.2}$$

$$\partial_t(\rho v)+\partial_x(\rho vu)+\partial_y(\rho v^2+p) = 0, \tag{2.3}$$

$$\partial_t(\rho h)+\partial_x((\rho h+p)u)+\partial_y((\rho h+p)v) = 0, \tag{2.4}$$

where $h=\frac{1}{2}(u^2+v^2)+e$. As usual, ρ,p,u,v,e are, respectively, the (mass) density, (scalar) pressure, velocity component in x-direction and y-direction, and specific internal energy of the fluid. The equation of state is given by $p=p(\rho,e)$. Particularly, for polytropic gas, we have $p=(\gamma-1)\rho e$, with $\gamma>1$ the adiabatic exponent. We also use s to denote the specific entropy of the fluid. If p is expressed as a function of ρ,s, then the sound speed c is given by $c\doteq\sqrt{\partial p(\rho,s)/\partial\rho}$. For polytropic gas, we have the following relations:

$$p=(\gamma-1)\exp(s/c_\nu)\rho^\gamma,\quad c=\sqrt{\gamma p/\rho},\quad e=c_\nu\Theta.$$

Here $c_\nu>0$ is a constant, and Θ is the temperature.

Assuming the fluid is supersonic at the entry ($u>c$), and subsonic at the exit ($u<c$), we formulate a (general) initial-boundary value problem of Euler equations (2.1)–(2.4) in the time-space domain

$$\Omega^T \doteq \{(t,x,y):\ t\in[0,T],\ (x,y)\in\Omega\}\qquad (T>0)$$

as follows

$$\begin{cases} (2.1)-(2.4), & \text{in} \quad \Omega^T, \\ (u,\ v,\ p,\ \rho)|_{t=0} = (u_0,\ v_0,\ p_0,\ \rho_0), & \text{on} \quad \Omega, \\ (u,\ v,\ p,\ \rho) = (u_{\mathrm{n}},\ v_{\mathrm{n}},\ p_{\mathrm{n}},\ \rho_{\mathrm{n}}), & \text{on} \quad \Sigma_{-1}^T, \\ p = p_{\mathrm{out}}, & \text{on} \quad \Sigma_1^T, \\ v = 0, & \text{on} \quad \Gamma_k^T, \quad k = 0, 1. \end{cases} \tag{2.5}$$

Here we have set

$$\Gamma_k^T \doteq [0,T] \times \Gamma_k (k = 0, 1), \qquad \Sigma_s^T \doteq [0,T] \times \Sigma_s (s = -1, 1).$$

The second line in (2.5) is the initial data, with u_0, v_0, p_0, ρ_0 given functions of $(x, y) \in \Omega$. The third line is the boundary condition on the entry, with $u_{\mathrm{n}}, v_{\mathrm{n}}, p_{\mathrm{n}}, \rho_{\mathrm{n}}$ being given functions of (t, y). Since the flow is assumed to be supersonic there, as analyzed in [9, p.412], all the unknowns should be prescribed (see also Lecture 4). The forth line is the boundary condition on the exit. As shown in [9, p.411], one and only one boundary condition should be given. Although there are many choices, we prescribe pressure since it is physically more interesting [14, p.373, p.385] and in the framework of steady flow ill-posedness will occur. Here p_{out} is a given function of (t, y). Since $\Gamma_{0,1}$ are assumed to be solid walls, there should pose slip condition, that is the last line in (2.5).

Let us consider a special, while physically relevant case of (2.5): the flow is steady and piecewise constant, and there is a normal shock in the duct; the flow $\underline{U}_-$ ahead (in the left) of the shock-front

$$\underline{\Sigma} \doteq \{(t, x, y) :\ t > 0,\ x = \underline{\chi} \equiv 0,\ y \in [0, 1]\}$$

is supersonic, while the flow $\underline{U}_+$ behind of it (in the right) is subsonic. (We use temporarily $U = (u, v, p, \rho)$ to denote the unknowns.) Such a special solution $(\underline{U}_\pm; \underline{\chi})$ of (2.5) can be easily constructed ([7, Proposition 2.1 in p.1347]), and is called a *normal transonic shock in duct*. In the framework of steady flows, it has been shown to be globally unique modulo translation of the shock-front in x-direction, and unstable with respect to perturbation of back pressure p_{out} ([4, 6]). It is then of great interest to know whether such shocks are structurally stable in the sense of unsteady flow. That is, if the flow pattern persists to be piecewise smooth, and contains a transonic shock, for a short time, under small (unsteady) perturbations of the upcoming supersonic flow and back pressure, as well as small perturbation of the initial data?

It is now well-known the following Majda's uniform stability condition is necessary for a planar shock to be stable in a strong sense.

Definition 2.1. [9, p.437] A discontinuity that is a Lax shock in a solution of (2.1)–(2.4) satisfies the *uniform stability condition*, provided that

$$\boxed{(\Gamma+1)\left(\frac{u}{c}\right)^2\frac{\rho-\rho_0}{\rho_0}<1.} \tag{2.6}$$

Here u, c, ρ, Γ are respectively the normal velocity (respect to the discontinuity), sound speed, density, Grüneisen coefficient of the flow behind the shock-front, and ρ_0 is the density of the flow ahead of the shock-front. Suppose the state function of the fluid is given by $e=e(s,\tau)$, with $\tau=1/\rho$ the specific volume, Θ the temperature, then Grüneisen coefficient Γ is defined by $\Gamma:=-\frac{\tau}{\Theta}\frac{\partial^2 e}{\partial s\partial\tau}$.

We will show in this work that the reference normal shock in duct $(\underline{U}_\pm,\underline{\chi})$ is actually stable in the sense of unsteady flows, if it satisfies the uniform stability condition, and the given initial-boundary data also satisfy certain orders of compatibility and symmetry conditions. We then conclude that the instability of such shocks in the framework of steady flows is a result of long-time accumulation of effects of the back pressure, while not a consequence of the discontinuity itself.

So we are interested in those solutions of (2.5) in *the class of piecewise smooth functions containing a shock-front.* This shock-front Σ would be a free boundary to be solved simultaneously with the smooth flow fields U_- ahead and U_+ behind of it in the duct. It is well-known that (see [9, Proposition 10.2 in p.312]) such discontinuous flow fields $(U_\pm;\Sigma)$ are weak solutions to (2.5) if and only if $U_\pm$ satisfies the Euler equations in classical sense away from the shock-front, that is, in $\Omega_\pm^T$, with

$$\Omega_\pm \doteq \{(x,y)\in\Omega:\ x\gtrless\chi(t,y)\},\qquad \Omega_\pm^T=[0,T]\times\Omega_\pm,$$

and the following Rankine-Hugoniot (R-H) conditions

$$\partial_t\chi\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ \rho h\end{bmatrix}+\partial_y\chi\begin{bmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ (\rho h+p)v\end{bmatrix}-\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ (\rho h+p)u\end{bmatrix}=0 \tag{2.7}$$

hold across the shock-front given by

$$\Sigma\doteq\{(t,x,y):t\in[0,T],\ (x,y)\in\Omega,\ x=\chi(t,y)\}.$$

Here, as usual, $[\cdot]$ denotes the value of the quantity behind the shock-front minus its value ahead of the shock front. A shock-front should also satisfy the entropy criterion (for Smith fluid, which idea gas is a special case, see [9, p.402]): $[s]=[p]>0$.

With the above structure of solution in mind, we now specify (2.5) as an initial-boundary value problem with a free boundary Σ in Ω^T:

$$\begin{cases} U_\pm \ \text{ solve } \ (2.1)-(2.4) \ \text{ respectively in } \ \Omega^T_\pm, & \\ \text{R-H conditions } \ (2.7), & \text{on} \quad \Sigma, \\ \chi|_{t=0} = \chi_0(y), & y \in [0,1], \\ U_\pm|_{t=0} \doteq (u_\pm,\ v_\pm,\ p_\pm,\ \rho_\pm)|_{t=0} & \\ \qquad = U_0^\pm \doteq (u_0^\pm,\ v_0^\pm,\ p_0^\pm,\ \rho_0^\pm), & \text{on} \quad \Omega_\pm, \\ U_- \doteq (u_-,\ v_-,\ p_-,\ \rho_-) & \\ \qquad = U_\mathrm{n} \doteq (u_\mathrm{n},\ v_\mathrm{n},\ p_\mathrm{n},\ \rho_\mathrm{n}), & \text{on} \quad \Sigma^T_{-1}, \\ p_+ = p_\mathrm{out}, & \text{on} \quad \Sigma^T_1, \\ v_\pm = 0, & \text{on} \quad \Gamma^T_k \cap \overline{\Omega^T_\pm}, \quad k=0,1, \end{cases} \tag{2.8}$$

where $\{x = \chi_0(y)\}$ is the initial position of the shock-front.

2.2. **Reduction of free boundary problem.** We now rewrite the above free boundary problem to a nonclassical fixed boundary problem. Since the flow is smooth away from shock-front, we first write the Euler system in symmetric form, and recall some of its properties. Then we fix the shock-front as done in [9].

In the sequel, we denote $\mathscr{W}_\mu$ to be a ball in $\mathbb{R}^8$ centered at the reference state $(\underline{U}_-, \underline{U}_+)$ with radius μ, and $\mathscr{V}_\mu$ a ball in $\mathbb{R}^3$ centered at $(0,0,0)$ also with radius μ. We will choose μ to be small (depending only on the reference state $(\underline{U}_\pm, \underline{\chi})$) to ensure that once $(U_-, U_+) \in \mathscr{W}_\mu$ and $(\chi, \partial_t\chi, \partial_y\chi) \in \mathscr{V}_\mu$, then they share some fine properties of the reference state.

2.2.1. *Euler equations in symmetric form.* From now on, we use $U = (p,u,v,s)^\top$ as the unknown, and for $\rho > 0$, the Euler equations (2.1)–(2.4) can be written as a symmetric hyperbolic system [9, p.394]:

$$A^0(U)\partial_t U + A^1(U)\partial_x U + A^2(U)\partial_y U = 0, \tag{2.9}$$

where $A^0(U) = \mathrm{diag}((\rho c^2)^{-1}, \rho, \rho, 1)$ is positive-definite, and for $\mathbf{u} = (u,v)^\top$, $\mathbf{n} = (n_1, n_2)^\top$,

$$A(U;\mathbf{n}) = \sum_{j=1}^{2} A^j(U) n_j = \begin{pmatrix} \frac{\mathbf{u}\cdot\mathbf{n}}{\rho c^2} & \mathbf{n}^T & 0 \\ \mathbf{n} & \rho(\mathbf{u}\cdot\mathbf{n}) I_2 & 0 \\ 0 & 0 & \mathbf{u}\cdot\mathbf{n} \end{pmatrix}$$

is symmetric. It is well-known ([9, p.393]) that for $\gamma > 0$, the Euler system (2.9) is hyperbolic with characteristic fields of constant multiplicity (“constantly hyperbolic” for short): its eigenvalues in the direction $\mathbf{n}$ are

$$\lambda_1(U;\mathbf{n}) = \mathbf{u}\cdot\mathbf{n} - c|\mathbf{n}|, \quad \lambda_2(U;\mathbf{n}) = \mathbf{u}\cdot\mathbf{n}, \quad \lambda_3(U;\mathbf{n}) = \mathbf{u}\cdot\mathbf{n} + c|\mathbf{n}|;$$

$\lambda_{1,3}$ (resp. λ_2) has multiplicity one (resp. two) for all U with $p, \rho > 0$ and $\mathbf{n}$ with $|\mathbf{n}| = 1$.

2.2.2. *Fixing shock-front.* Let $\phi \in \mathscr{D}(\mathbb{R})$ be a nonnegative cut-off function equals to one on $[-\frac{1}{8}, \frac{1}{8}]$, and vanishes outside $(-\frac{5}{8}, \frac{5}{8})$, and satisfies $\|\phi'\|_{L^\infty(\mathbb{R})} \leq 4$. Then for those χ satisfying $\|\chi\|_{L^\infty([0,T]\times(0,1))} \leq 1/8$, (this requirement will be fulfilled later by seeing the solved χ satisfies $(\chi, \partial_t\chi, \partial_y\chi) \in \mathscr{V}_\mu$ for $\mu \leq 1/8$,) the mappings

$$\Psi_\pm: \ (t,\ z,\ y) \mapsto (t,\ x = \pm z + \phi(z)\chi(y,t),\ y)$$

are diffeomorphisms from $D^T = [0, T] \times D$ to $\Omega^T_\pm$, with

$$D \doteq \{(z,\ y):\ z \in (0,1),\ y \in (0,1)\}.$$

Actually, the Jacobian of $\Psi_\pm$ is

$$|\partial x/\partial z| = |\pm 1 + \phi'(z)\chi(y,t)| = |1 \pm \phi'(z)\chi(y,t)| \geq 1 - 4/8 \geq 1/2$$

for all $z \in (0, 1)$.

2.2.3. *Initial-boundary value problem in D^T.* We now transform the free boundary problem (2.8) to a initial-boundary value problem in D^T.

`The interior equations.` Denoting point in D^T by (t', z', y'), then from $\Psi_\pm$ we may solve

$$\partial_x = \frac{1}{\pm 1 + \phi'\chi}\partial_{z'}, \quad \partial_y = \partial_{y'} - \frac{\phi\partial_y\chi}{\pm 1 + \phi'\chi}\partial_{z'}, \quad \partial_t = \partial_{t'} - \frac{\phi\partial_t\chi}{\pm 1 + \phi'\chi}\partial_{z'}.$$

So (2.9), for $U = U_\pm$, becomes

$$A^0(U_\pm)\partial_{t'}U_\pm + A_1^\pm(U_\pm, \chi, \mathrm{d}\chi)\partial_{z'}U_\pm + A^2(U_\pm)\partial_{y'}U_\pm = 0, \tag{2.10}$$

where $\mathrm{d}\chi = (\partial_t\chi, \partial_y\chi)^\top$, and

$$A_1^\pm(U_\pm, \chi, \mathrm{d}\chi) = \frac{1}{\pm 1 + \phi'\chi}(A^1(U_\pm) - \phi\partial_t\chi A^0(U_\pm) - \phi\partial_y\chi A^2(U_\pm))$$

is also symmetric.

Set $\mathbf{U} = \begin{pmatrix} U_- \\ U_+ \end{pmatrix}$, and $\mathbb{A}^0(\mathbf{U}) = \mathrm{diag}(A^0(U_-), A^0(U_+))$,

$$\mathbb{A}^1(\mathbf{U}, \chi, \mathrm{d}\chi) = \mathrm{diag}(A_1^-(U_-, \chi, \mathrm{d}\chi), A_1^+(U_+, \chi, \mathrm{d}\chi)),$$
$$\mathbb{A}^2(\mathbf{U}) = \mathrm{diag}(A^2(U_-), A^2(U_+)).$$

Then, dropping apostrophes in (t', z', y'), (2.10) can be written altogether as

$$L(\mathbf{U}, \chi, \mathrm{d}\chi)\mathbf{U} \doteq \mathbb{A}^0(\mathbf{U})\partial_t\mathbf{U} + \mathbb{A}^1(\mathbf{U}, \chi, \mathrm{d}\chi)\partial_z\mathbf{U} + \mathbb{A}^2(\mathbf{U})\partial_y\mathbf{U} = 0 \text{ in } D^T. \tag{2.11}$$

This is a 8×8, symmetric and constantly hyperbolic system.

`The boundary conditions.` **a).** $\Sigma_0^T \doteq [0,T]\times\{z=0,\ y\in(0,1)\}$**.** By entropy criteria, for reference state, we have

$$[\rho][\rho v^2+p]-[\rho v]^2\big|_{\underline{\mathbf{U}}} = [\underline{\rho}][\underline{p}] \neq 0.$$

So this also holds for $\mathbf{U}\in\mathscr{W}_\mu$ if μ is small (depending only on $\underline{\mathbf{U}}$). Then we can solve from the first and the third R-H conditions $\mathrm{d}\chi$, and by substituting it to the second and forth R-H conditions, get the following equivalent form of (2.7):

$$\begin{pmatrix} \partial_t\chi \\ \partial_y\chi \\ 0_2 \end{pmatrix} = -Q(\mathbf{U}) \doteq \begin{pmatrix} q(\mathbf{U}) \\ r(\mathbf{U}) \\ s(\mathbf{U}) \end{pmatrix} \quad \text{on } \ \Sigma_0^T, \tag{2.12}$$

where

$$\begin{aligned} q(\mathbf{U}) &= \frac{[\rho u][\rho v^2+p]-[\rho uv][\rho v]}{[\rho][\rho v^2+p]-[\rho v]^2}, \quad r(\mathbf{U}) = \frac{[\rho][\rho uv]-[\rho u][\rho v]}{[\rho][\rho v^2+p]-[\rho v]^2}, \\ s(\mathbf{U}) &= \begin{pmatrix} q(\mathbf{U})[\rho u]+r(\mathbf{U})[\rho uv]-[\rho u^2+p] \\ q(\mathbf{U})[\rho h]+r(\mathbf{U})[(\rho h+p)v]-[(\rho h+p)u] \end{pmatrix}. \end{aligned}$$

Let $\mathbb{J}=(I_2, 0_{2\times 2})^\top$. Then (2.12) can be written simply as

$$\mathbb{J}\mathrm{d}\chi + Q(\mathbf{U}) = 0. \tag{2.13}$$

b). $\Sigma_1^T := [0,T]\times\{z=1,\ y\in(0,1)\}$**.** Now $\{z=1\}$ represents the entry for U_-, and exit for U_+. Here we have a linear boundary condition

$$M\mathbf{U} = \mathbf{g}, \quad \text{on } \ \Sigma_1^T, \tag{2.14}$$

with

$$M = \begin{pmatrix} I_5 & 0_{5\times 3} \end{pmatrix}, \mathbf{g} = (p_{\mathrm{n}},\ u_{\mathrm{n}},\ v_{\mathrm{n}},\ s_{\mathrm{n}},\ p_{\mathrm{out}})^\top.$$

By our assumption that $u_{\mathrm{n}} > c_{\mathrm{n}}$ and $0 < u_{\mathrm{out}} < c_{\mathrm{out}}$, Σ_1 is non-characteristic.

c). $\Gamma_k^T \doteq [0,T]\times\{z\in(0,1),\ y=k\}$ $(k=0,1)$**.** On the walls $\Gamma_{0,1}$, we still have the slip condition

$$M'\mathbf{U} = 0, \quad M' = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.15}$$

The walls are characteristic boundaries with constant rank 2, related to the eigenvalue λ_2.

2.2.4. *Initial conditions.* The initial data for χ is the same as in (2.8):

$$\chi|_{t=0} = \chi_0(y), \quad \text{for} \quad y \in (0,1). \tag{2.16}$$

For $\mathbf{U}$ it is

$$\mathbf{U}|_{t=0} = \mathbf{U}_0 = \begin{pmatrix} U_0^- \\ U_0^+ \end{pmatrix}, \quad \text{on} \quad D. \tag{2.17}$$

Note that since χ_0 is given, we can totally determine $\mathbf{U}_0$ here by using the transform $\Psi_\pm$ with $t = 0$ and $U_0^\pm$ in (2.8).

So for μ small, by the above reductions, we are led to solve regular functions ($\mathbf{U}(t,z,y) \in \mathbb{R}^8, \chi(t,y) \in \mathbb{R}$), for $(t,z,y) \in D^T$, that satisfy the interior equations (2.11), boundary conditions (2.13) (2.14) (2.15), and initial conditions (2.16)(2.17). We call this as *Problem (N)* in the following.

2.3. **Sobolev space of symmetric functions.** As mentioned before, we use a symmetric reflection technique to 'hide' the solid walls. This symmetry method depends on special structure of the Euler equations, as well as symmetry properties of given initial-boundary data. To make the latter clear, we introduce some function spaces.

For s a nonnegative number, we as usual use $H^s(D)$ to denote the standard Sobolev space $W^{s,2}(D)$. We then define for $s > 3/2$,

$$H^s_{\mathrm{e}}(D) \doteq \left\{ u \in H^s(D) : \quad \partial_y^{2j+1} u|_{y=0,1} = 0, \quad j = 0, \cdots, m \right\},$$

where

$$m = \begin{cases} k, & \frac{3}{2} < s - 2k < 2, \quad k \geq 0 \text{ an integer}, \\ k-1, & 0 \leq s - 2k \leq \frac{3}{2}, \quad k \geq 1 \text{ an integer}, \end{cases}$$

and $H^s_{\mathrm{e}}(D) = H^s(\Omega)$ for $0 \leq s \leq \frac{3}{2}$. These are sets of functions that can be extended periodically to $y \in \mathbb{R}$ with period 2 by even reflection and still belong to H^s_{loc} (see Lemma 3.3). Also, for $s > \frac{1}{2}$, we define

$$H^s_{\mathrm{o}}(D) \doteq \{ u \in H^s(D) : \quad \partial_y^{2j} u|_{y=0,1} = 0, \quad j = 0, \cdots, m \}$$

with

$$m = \begin{cases} k, & \frac{1}{2} < s - 2k < 2, \quad k \geq 0 \text{ an integer}, \\ k-1, & 0 \leq s - 2k \leq \frac{1}{2}, \quad k \geq 1 \text{ an integer}, \end{cases}$$

and $H^s_{\mathrm{o}}(D) = H^s(D)$ for $0 \leq s \leq \frac{1}{2}$. These are sets of functions that can be extended periodically to $y \in \mathbb{R}$ by odd reflection and still belong to H^s_{loc}. Both $H^s_{\mathrm{e}}(D)$ and $H^s_{\mathrm{o}}(D)$ are closed subspaces of $H^s(D)$. They inherit the norm of $H^s(D)$. In the above definitions, we note that traces are well-defined even for polygonal domains (see [15, p.42]). For $I = [0,1]$

(or $I^T = [0,T] \times I$, D^T), we can define $H^s_{\mathrm{o}}(I)$ and $H^s_{\mathrm{e}}(I)$ (or $H^s_{\mathrm{o}}(I^T)$, $H^s_{\mathrm{e}}(I^T)$, $H^s_{\mathrm{o}}(D^T)$, $H^s_{\mathrm{e}}(D^T)$) just by replacing D above with I (or I^T, D^T). Finally, we set

$$\mathcal{H}^s(D) \doteq (H^s_{\mathrm{e}}(D)^2 \times H^s_{\mathrm{o}}(D) \times H^s_{\mathrm{e}}(D))^2.$$

2.4. **Main results.** We can now state main result of this lecture.

Theorem 2.1 (Main result)**.** *For given reference state* $(\underline{\mathbf{U}}, \underline{\chi} = 0)$ *satisfying uniform stability condition (2.6),* $m \geq 3$ *a fixed integer, and* $T > 0$*, suppose*

a) $\mathbf{U}_0 \in \mathcal{H}^{m+\frac{1}{2}}(D)$*,* $\chi_0 \in H_{\mathrm{e}}^{m+\frac{1}{2}}(I)$*,* $\mathbf{g} \in H^m_{\mathrm{e}}(I^T)^2 \times H^m_{\mathrm{o}}(I^T) \times H^m_{\mathrm{e}}(I^T)^2$*;*
b) $\mathbf{U}_0, \chi_0$ *and* $\mathbf{g}$ *satisfy compatibility conditions up to order* $m-1$*;*
c) *for all* $z, y \in [0,1]$*,* $\mathbf{U}_0(z,y) \in \mathscr{W}_{\mu/3}$*,* $(\chi_0(y), q(\mathbf{U}_0(0,y)), \chi_0'(y)) \in \mathscr{V}_{\mu/3}$*, here* $\mu \leq 1/8$ *is determined by* $(\underline{\mathbf{U}}, \underline{\chi})$*;*
d) $\mathbf{g}$ *satisfies* $u_{\mathrm{n}} > c_{\mathrm{n}}$ *at* Σ_0^T*, and* $\mathbf{U}_0$ *satisfies* $0 < u_0^+ < c_0^+$ *at* Σ_1*.*

Then problem (N) has uniquely one solution $(\mathbf{U}, \chi) \in \mathcal{H}^m(D^{\bar{T}}) \times H^{m+1}_{\mathrm{e}}(I^{\bar{T}})$*, with* $\bar{T} \in (0, T]$ *depending only on* $\underline{\mathbf{U}}$*,* μ *and initial-boundary data* $\mathbf{U}_0, \chi_0$ *and* $\mathbf{g}$*. In addition,* $\mathbf{U}$ *takes value in* $\mathscr{W}_\mu$*,* $(\chi, \mathrm{d}\chi)$ *take values in* $\mathscr{V}_\mu$*; hence the flow is supersonic ahead of shock-front and subsonic behind of it as* μ *small.*

The compatibility conditions required in this theorem is given in Section 3.1 below.

3. Compatibility conditions and approximate solutions

In this section we give the compatibility conditions assumed in Theorem 2.1. These conditions are necessary for resolution of Problem (N) in the class of regular functions. Moreover, it provides an approximate solution to the initial-boundary value problem. Fine properties of approximate solutions will greatly simplify the study of linearized problem. So we devote the second part of this section to the construction of suitable approximate solutions.

3.1. **Compatibility conditions.** The basic idea of compatibility conditions is as follows. Since $\{t = 0\}$ is non-characteristic (i.e. $\mathbb{A}^0(\mathbf{U})$ is always invertible), for all integer $j \geq 0$, we can solve from the equations and initial data $(\mathbf{U}_0, \chi_0)$ all the partial derivatives with respect to time ($\mathbf{U}_j = \partial_t^j \mathbf{U}|_{t=0}, \chi_j = \partial_t^j \chi|_{t=0}$), valued at $t = 0$. We also act ∂_t^j on the boundary conditions, taking value at $t = 0$, and obtain a relation, say $b(\cdot) = 0$, involving $(\partial_t^j \mathbf{U}|_{t=0}, \partial_t^j \chi|_{t=0})$. Obviously, to ensure the solution to be in $H^m(D^T)$, the $\mathbf{U}_j$ obtained above, when restricted to boundary, and χ_j, should satisfy $b(\cdot) = 0$ where $\partial_t^j \mathbf{U}|_{t=0}$ and $\partial_t^j \chi|_{t=0}$ are replaced by $\mathbf{U}_j$ and χ_j, for all $j = 0, 1, \cdots, m-1$. These are the compatibility conditions up to order $m-1$.

To make the above description rigorous, we first calculate the sequence $\{(\mathbf{U}_j, \chi_j)\}_{j=0}^m$. This is the same as in [9, p.370]. Let

$$\mathbb{B}_1(\mathbf{U}, \chi, \mathrm{d}\chi) \doteq \mathbb{A}^0(\mathbf{U})^{-1}\mathbb{A}^1(\mathbf{U}, \chi, \mathrm{d}\chi),$$

and $\mathbb{B}_2(\mathbf{U}) \doteq \mathbb{A}^0(\mathbf{U})^{-1}\mathbb{A}^2(\mathbf{U})$. Then using Faá di Bruno's formula, $(\mathbf{U}_j, \chi_j)$ can be obtained inductively, starting from $(\mathbf{U}_0, \chi_0)$, by

$$\begin{cases} \chi_1 = q(\mathbf{U}_0)|_{z=0}, \quad \mathbf{U}_1 = -\mathbb{B}_1(\mathbf{U}_0, \chi_0, (\chi_1, \chi_0'))\partial_z \mathbf{U}_0 - \mathbb{B}_2(\mathbf{U}_0)\partial_y \mathbf{U}_0; \\ \chi_{j+1} = \sum_{m=1}^j \sum_{\ell_1+\cdots+\ell_m=j} c_{\ell_1\cdots\ell_m}(\mathrm{d}^m q \circ (\mathbf{U}_0)|_{z=0}) \cdot (\mathbf{U}_{\ell_1}, \mathbf{U}_{\ell_2}, \cdots, \mathbf{U}_{\ell_m})|_{z=0}, \\ \mathbf{U}_{j+1} = -\mathbb{B}_1(\mathbf{U}_0, \chi_0, (\chi_1, \chi_0'))\partial_z \mathbf{U}_j - \mathbb{B}_2(\mathbf{U}_0)\partial_y \mathbf{U}_j \\ \qquad - \sum_{\ell=1}^j \begin{pmatrix} j \\ \ell \end{pmatrix} \sum_{k=1}^\ell \sum_{\ell_1+\cdots+\ell_k=\ell} c_{\ell_1\cdots\ell_k} \mathrm{d}^k\mathbb{B}_1(\mathbf{U}_0, \chi_0, (\chi_1, \chi_0')) \\ \qquad\qquad \cdot\Big((\mathbf{U}_{\ell_1}, \chi_{\ell_1}, (\chi_{\ell_1+1}, \chi_{\ell_1}'))\cdots, (\mathbf{U}_{\ell_k}, \chi_{\ell_k}, (\chi_{\ell_k+1}, \chi_{\ell_k}'))\Big)\partial_z \mathbf{U}_{j-\ell} \\ \qquad - \sum_{\ell=1}^j \begin{pmatrix} j \\ \ell \end{pmatrix} \sum_{k=1}^\ell \sum_{\ell_1+\cdots+\ell_k=\ell} c_{\ell_1\cdots\ell_k} \mathrm{d}^k\mathbb{B}_2(\mathbf{U}_0) \cdot \Big(\mathbf{U}_{\ell_1}\cdots, \mathbf{U}_{\ell_k}\Big)\partial_y \mathbf{U}_{j-\ell}. \end{cases} \tag{3.1}$$

Definition 3.1 (Compatibility conditions)**.** We say Problem (N) satisfies compatibility conditions up to order m, if $\mathbf{U}_0, \chi_0, \mathbf{g}$ satisfy

$$\chi_0' = r(\mathbf{U}_0)|_{z=0}, \quad 0 = s(\mathbf{U}_0)|_{z=0}, \quad M\mathbf{U}_0|_{z=1} = \mathbf{g}|_{t=0}, \quad M'\mathbf{U}_0|_{\Gamma_{0,1}} = 0, \tag{3.2}$$

and furthermore, for $p = 1, \cdots, m$,

$$\begin{cases} \chi_p' = \mathrm{d}\chi_p/\mathrm{d}y = \sum_{k=1}^p \sum_{\ell_1+\cdots+\ell_k=p} c_{\ell_1\cdots\ell_k}(\mathrm{d}^k r \circ (\mathbf{U}_0)|_{z=0}) \cdot (\mathbf{U}_{\ell_1}, \cdots, \mathbf{U}_{\ell_k})|_{z=0}, \\ 0 = \sum_{k=1}^p \sum_{\ell_1+\cdots+\ell_k=p} c_{\ell_1\cdots\ell_k}(\mathrm{d}^k s \circ (\mathbf{U}_0)|_{z=0}) \cdot (\mathbf{U}_{\ell_1}, \cdots, \mathbf{U}_{\ell_k})|_{z=0}, \\ M\mathbf{U}_p|_{z=1} = \partial_t^p \mathbf{g}|_{t=0}, \qquad M'\mathbf{U}_p|_{\Gamma_{0,1}} = 0. \end{cases} \tag{3.3}$$

3.2. **Properties of spaces H_{o}^s and H_{e}^s.** We need to derive many symmetry properties of $(\mathbf{U}_j, \chi_j)$ once (3.2) holds. These symmetry properties are important for us to construct approximate solutions. To this end, we list some properties of the spaces H_{o}^s and H_{e}^s here. The following three facts on general Sobolev functions are well-known.

Proposition 3.1 ([9, p.469])**.** *For all $s > 0$, there is a constant $C > 0$ so that for all $u, v \in L^\infty \cap H^s$, the product uv also belongs to H^s and $\|uv\|_{H^s} \le C(\|u\|_{L^\infty}\|v\|_{H^s} + \|v\|_{L^\infty}\|u\|_{H^s})$.*

Proposition 3.2 ([9, p.472])**.** *For all s and t with $s + t > 0$, if $u \in H^s$ and $v \in H^t$, then the product uv belongs to H^r for all $r \le \min(s, t)$ such that $r < s + t - \frac{d}{2}$. (d is the dimension of the domain where u, v are defined.) Furthermore, there exists C (depending only on r, s, t and d) such that $\|uv\|_{H^r} \le C\|u\|_{H^s}\|v\|_{H^t}$.*

Proposition 3.3 ([9, p.474]). *If $F \in \mathscr{C}^\infty$, $F(0) = 0$, and $s > \frac{d}{2}$, then there is a continuous function $C : [0,\infty) \to [0,\infty)$ such that for all $u \in H^s(\mathbb{R}^d)$, there holds $\|F(u)\|_{H^s} \le C(\|u\|_{L^\infty}) \|u\|_{H^s}$.*

Lemma 3.1. *Let $d = 3$ for $\Omega = D^T$, $d = 2$ for $\Omega = D, I^T$, and $d = 1$ for $\Omega = I$, and $s > \frac{d}{2}$. Suppose $F(u)$ is a C^∞ smooth function of u. Then the following hold:*

i) *If $u \in H^s_{\mathrm{e}}(\Omega)$, then $F(u) \in H^s_{\mathrm{e}}(\Omega)$ and*

$$\|F(u)\|_{H^s_{\mathrm{e}}(\Omega)} \le C(\|u\|_{L^\infty(\Omega)}) \|u\|_{H^s_{\mathrm{e}}(\Omega)} + |F(0)||\Omega|^{\frac{1}{2}};$$

ii) *For $u \in H^s_{\mathrm{e}}(\Omega)$ that is bounded away from zero in Ω, its inverse $1/u$ also belongs to $H^s_{\mathrm{e}}(\Omega)$; moreover, there holds*

$$\|1/u\|_{H^s(\Omega)} \le c_0(1 + \|u\|_{H^s(\Omega)}),$$

with c_0 depending only on and continuously on lower and upper bounds of $|u|$ in Ω;

iii) *If $u, v \in H^s_{\mathrm{o}}(\Omega)$, then $uv \in H^s_{\mathrm{e}}(\Omega)$ and*

$$\|uv\|_{H^s_{\mathrm{e}}(\Omega)} \le C(\|u\|_{L^\infty(\Omega)} \|v\|_{H^s_{\mathrm{o}}(\Omega)} + \|v\|_{L^\infty(\Omega)} \|u\|_{H^s_{\mathrm{o}}(\Omega)});$$

iv) *If $u \in H^s_{\mathrm{o}}(\Omega)$, $v \in H^s_{\mathrm{e}}(\Omega)$ then $uv \in H^s_{\mathrm{o}}(\Omega)$ and similar inequality as in iii) holds;*

v) *For $u \in H^s_{\mathrm{e}}(\Omega)$ (resp. $H^s_{\mathrm{o}}(\Omega)$), $\partial_z u$ belongs to $H^{s-1}_{\mathrm{e}}(\Omega)$ (resp. $H^{s-1}_{\mathrm{o}}(\Omega)$), and $\partial_y u$ belongs to $H^{s-1}_{\mathrm{o}}(\Omega)$ (resp. $H^{s-1}_{\mathrm{e}}(\Omega)$).*

Proof. 1. By applying Proposition 3.3 to $F(u) - F(0)$ (after extending u to be defined in $\mathbb{R}^d$), we only need to show $\partial_y^{2j+1} F(u)|_{y=0,1} = 0$ for $j = 0, \cdots, m$ (we use here notation in Section 2.3. m is determined by s). As a matter of fact,

$$\partial_y^{2j+1} F(u)|_{y=0,1} = \sum_{q=1}^{2j+1} \sum_{\ell_1+\cdots+\ell_q=2j+1} c_{\ell_1\cdots\ell_q} \mathrm{d}^q F \circ (u)|_{y=0,1} \cdot ((\partial_y^{\ell_1} u)|_{y=0,1}, \cdots, (\partial_y^{\ell_q} u)|_{y=0,1}).$$

Each term here makes sense by trace theorem. We note there should be at least one ℓ_p which is an odd number in each term $c_{\ell_1\cdots\ell_q} \mathrm{d}^q F \circ (u)|_{y=0,1} \cdot ((\partial_y^{\ell_1} u)|_{y=0,1}, \cdots, (\partial_y^{\ell_q} u)|_{y=0,1})$, which are products of many factors; while by definition, the factor $\partial_y^{\ell_p} u|_{y=0,1} = 0$. This proves i).

2. Without loss of generality, suppose $u > u_0 > 0$ in Ω, where u_0 is a number. Let $\tilde{u} = u - u_0$. Then by step 1, $F(\tilde{u}) = \frac{\tilde{u}}{u_0(u_0+\tilde{u})} = \frac{1}{u_0} - \frac{1}{u} \in H^s_{\mathrm{e}}(\Omega)$. Hence $1/u \in H^s_{\mathrm{e}}(\Omega)$, and

$$\left\|\frac{1}{u}\right\|_{H^s(\Omega)} \le c\left(\frac{1}{u_0} + c(\|\tilde{u}_{L^\infty}\|)\Big(\|u\|_{H^s} + |u_0|\Big)\right) \le c_0(1 + \|u\|_{H^s}).$$

3. By Proposition 3.1, for iii), we only need show $\partial_y^{2j+1}(uv)|_{y=0,1}=0$ for $j=0,\cdots,m$. This follows from

$$\partial_y^{2j+1}(uv)|_{y=0,1}=\sum_{q=0}^{2j+1}\begin{pmatrix}2j+1\\ q\end{pmatrix}\partial_y^q u|_{y=0,1}\partial_y^{2j+1-q}v|_{y=0,1},$$

since either q or $2j+1-q$ should be odd, each term in the sum is zero. Claim iv) can be proved similarly. Finally, v) follows directly from definition of $H^s_{\mathrm{e,o}}(D)$. $\square$

We remark that results similar to iii)–iv) hold if we apply Proposition 3.2 instead of Proposition 3.1 in the proof when u,v have different index s.

3.3. **Symmetry of $(\mathbf{U}_j,\chi_j)$.** We now give symmetry properties of $(\mathbf{U}_j,\chi_j)$ once (3.2) holds. Note that such symmetry properties depend heavily on the special structure of Euler equations.

Lemma 3.2. *For $m\geq 3$ a fixed integer, suppose $\mathbf{U}_0\in\mathcal{H}^{m+\frac12}(D)$ and (3.2) holds. Then $(\mathbf{U}_j,\chi_j)$ belongs to $\mathcal{H}^{m+\frac12-j}(D)\times H_{\mathrm{e}}^{m+1-j}(I)$ $(j=0,1,\cdots,m)$. Moreover, there holds*

$$\begin{aligned}&\|\chi_0-\chi_0(0)\|_{H^{m+1}(I)}+\sum_{j=1}^m\left(\|\mathbf{U}_j\|_{H^{m+\frac12-j}(D)}+\|\chi_j\|_{H^{m+1-j}(I)}\right)\\ \leq\ & c_0\left\|\mathbf{U}_0-\underline{\mathbf{U}}\right\|_{H^{m+\frac12}(D)},\end{aligned}\tag{3.4}$$

with c_0 a nondecreasing function depending only on $\|\mathbf{U}_0\|_{H^{m+\frac12}(D)}$ and $|\chi_0(0)|$.

Proof. 1. By i) in Lemma 3.1, if $p,s\in H_{\mathrm{e}}^{m+\frac12}$, then express ρ as a function of (p,s), we see $\rho\in H_{\mathrm{e}}^{m+\frac12}$.

2. We now use the first condition in (3.2) to show, once $\mathbf{U}_0\in\mathcal{H}^{m+\frac12}(D)$, then $\chi_0\in H_{\mathrm{e}}^{m+1}(I)$. By trace theorem, $\mathbf{U}_0|_{z=0}$ actually belongs to $\mathcal{H}^m(I)$. We write $r(\mathbf{U})=r_1(\mathbf{U})/r_2(\mathbf{U})$, with $r_1(\mathbf{U})=[\rho][\rho uv]-[\rho u][\rho v]$, $r_2(\mathbf{U})=[\rho][\rho v^2+p]-[\rho v]^2$. Therefore $r_2(\mathbf{U})\chi_0'=r_1(\mathbf{U})$. For simplicity, here and below in step 2 and step 3, we write $\mathbf{U}_0|_{z=0}$ as $\mathbf{U}$.

By Lemma 3.1 iii)–iv), we infer $r_2(\mathbf{U})\in H_{\mathrm{e}}^m(I)$, while $r_1(\mathbf{U})\in H_{\mathrm{o}}^m(I)$. Note by smallness of μ assumed before, $r_2(\mathbf{U})\neq 0$. So by ii), $1/r_2(\mathbf{U})\in H_{\mathrm{e}}^m(I)$ and hence by iv), $\chi_0'\in H_{\mathrm{o}}^m(I)$. This implies $\chi_0-\chi_0(0)\in H_{\mathrm{e}}^{m+1}(I)$. Furthermore, using $r_1(\underline{\mathbf{U}})=0$, we have

$$\|\chi_0-\chi_0(0)\|_{H^{m+1}(I)}\leq c_0\left\|\mathbf{U}_0-\underline{\mathbf{U}}\right\|_{H^{m+\frac12}(D)}.$$

3. The next is to show $\chi_1\in H_{\mathrm{e}}^m(I)$. Set $r_0(\mathbf{U})=[\rho u][\rho v^2+p]-[\rho uv][\rho v]$, which belongs to $H_{\mathrm{e}}^m(I)$. Then $r_2(\mathbf{U})\chi_1=r_0(\mathbf{U})$. So as in step 2, we infer $\chi_1\in H_{\mathrm{e}}^m(I)$. Furthermore,

we have

$$\begin{aligned}\|\chi_1\|_{H^m(I)} &\le c(1+\|\mathbf{U}_0|_{z=0}\|_{H^m(I)})\,\|r_0(\mathbf{U}_0|_{z=0})-r_0(\underline{\mathbf{U}})\|_{H^m(I)}\\ &\le c(\|\mathbf{U}_0\|_{H^{m+\frac12}(D)})\,\|\mathbf{U}_0-\underline{\mathbf{U}}\|_{H^{m+\frac12}(D)}\,. \end{aligned}\tag{3.5}$$

Here $c(\cdot)$ is a non-decreasing function.

4. We can prove by induction that $\mathbf{U}_j \in (H^{m+\frac12-j}(D))^8$ for $j=1,\cdots,m$, and $\chi_j \in H^{m+1-j}(I)$ for $j=2,\cdots,m$, and obtain corresponding estimates in (3.4). The analysis is similar to that in [9, pp.322-323] and hence omitted.

5. We now prove symmetry properties of $\mathbf{U}_j, \chi_j$ for $j=0,1,\cdots,m-1$ (there is nothing to prove for $j=m$). We have shown the case of $\mathbf{U}_0$ and χ_0, χ_1. For $0\le k\le m-2$, suppose $\mathbf{U}_j\in\mathcal{H}^{m+\frac12-j}(D)$, $\chi_{j+1}\in H_{\mathrm{e}}^{m-j}(I)$ hold for $j=0,\cdots,k$, we verify it for $j=k+1$. There are four steps.

5.1. For $\vartheta\in\mathbb{R}$, set $\mathbf{U}(\vartheta)=\sum_{p=0}^k\frac{\vartheta^p}{p!}\mathbf{U}_p$ and

$$\breve{\chi}(\vartheta)=\sum_{p=0}^k\frac{\vartheta^p}{p!}\chi_{p+1},\qquad \chi(\vartheta)=\sum_{p=0}^k\frac{\vartheta^p}{p!}\chi_p.$$

They are respectively $\mathscr{C}^\infty$ curves in $\mathcal{H}^{m+\frac12-k}(D)$ and $H_{\mathrm{e}}^{m-k}(I)$, $H_{\mathrm{e}}^{m+1-k}(I)$. We may check that $\mathbf{U}_{k+1}$ is also given by (cf. (3.1))

$$\begin{aligned}\mathbf{U}_{k+1} \;=\; &-\left(\frac{\mathrm{d}}{\mathrm{d}\vartheta}\right)^k\Bigg|_{\vartheta=0}\Big(\mathbb{B}_1\Big(\mathbf{U}(\vartheta),\chi(\vartheta),(\breve{\chi}(\vartheta),\partial_y\chi(\vartheta))\Big)\partial_z\mathbf{U}(\vartheta)\Big)\\ &-\left(\frac{\mathrm{d}}{\mathrm{d}\vartheta}\right)^k\Bigg|_{\vartheta=0}(\mathbb{B}_2(\mathbf{U}(\vartheta))\partial_y\mathbf{U}(\vartheta))\,.\end{aligned}\tag{3.6}$$

5.2. We now claim that both $\mathbb{B}_1\Big(\mathbf{U}(\vartheta),\chi(\vartheta),(\breve{\chi}(\vartheta),\partial_y\chi(\vartheta))\Big)\partial_z\mathbf{U}(\vartheta)$ and $\mathbb{B}_2(\mathbf{U}(\vartheta))\partial_y\mathbf{U}(\vartheta)$ are of class $\mathscr{C}^\infty(\mathbb{R};\mathcal{H}^{m-\frac12-k}(D))$. If this is true, then clearly $\mathbf{U}_{k+1}$ belongs to $\mathcal{H}^{m-\frac12-k}(D)$.

Since both $\mathbb{B}_1,\mathbb{B}_2$ are $\mathscr{C}^\infty$ with respect to their arguments, and noting $\mathcal{H}^{m-\frac12-k}(D)$ is a Banach space, to prove the claim, we only need show that for fixed $\vartheta\in\mathbb{R}$, both $\mathbb{B}_1\Big(\mathbf{U}(\vartheta),\chi(\vartheta),(\breve{\chi}(\vartheta),\partial_y\chi(\vartheta))\Big)\partial_z\mathbf{U}(\vartheta)$ and $\mathbb{B}_2(\mathbf{U}(\vartheta))\partial_y\mathbf{U}(\vartheta)$ belong to $\mathcal{H}^{m-\frac12-k}(D)$. We prove this only for $\mathbb{B}_1$, since the treatment of $\mathbb{B}_2$ is similar and simpler.

5.3. We easily see $\partial_z\mathbf{U}(\vartheta)\in\mathcal{H}^{m-\frac12-k}(D)$. Direct computation yields

$$\mathbb{B}_1\Big(\mathbf{U}(\vartheta),\chi(\vartheta),(\breve{\chi}(\vartheta),\partial_y\chi(\vartheta))\Big)=\mathrm{diag}(\mathbb{B}_1^-,\mathbb{B}_1^+),$$

where

$$\mathbb{B}_1^{\pm} = \frac{1}{\pm 1 + \phi' \chi(\vartheta)} \times \begin{pmatrix} b_\pm(\vartheta) & \rho_\pm(\vartheta)(c_\pm(\vartheta))^2 & -\phi\rho_\pm(\vartheta)(c_\pm(\vartheta))^2\partial_y\chi(\vartheta) & 0 \\ \frac{1}{\rho_\pm(\vartheta)} & b_\pm(\vartheta) & 0 & 0 \\ -\phi\frac{1}{\rho_\pm(\vartheta)}\partial_y\chi(\vartheta) & 0 & b_\pm(\vartheta) & 0 \\ 0 & 0 & 0 & b_\pm(\vartheta) \end{pmatrix},$$

and $b_\pm(\vartheta) \doteq u_\pm(\vartheta) - \phi\breve{\chi}(\vartheta) - \phi\partial_y\chi(\vartheta)v_\pm(\vartheta)$. Recall here $u_\pm(\vartheta) \in H_{\mathrm{e}}^{m+\frac{1}{2}-k}(D)$,

$$\breve{\chi}(\vartheta) \in H_{\mathrm{e}}^{m-k}(I) \subset H_{\mathrm{e}}^{m-k}(D),\ \partial_y\chi(\vartheta) \in H_{\mathrm{o}}^{m-k}(I) \subset H_{\mathrm{o}}^{m-k}(D),\ v_\pm(\vartheta) \in H_{\mathrm{o}}^{m+\frac{1}{2}-k}(D).$$

Applying Proposition 3.2 and iii) of Lemma 3.1 to the last product, we have $\phi\partial_y\chi(\vartheta)v_\pm(\vartheta) \in H_{\mathrm{e}}^{m-k}(D)$, hence $b_\pm \in H_{\mathrm{e}}^{m-k}(D)$. So we can check that each row of $\mathbb{B}_1$ belongs to $\mathcal{H}^{m-k}(D)$. Applying again Proposition 3.2 (with $r = t = m-k-\frac{1}{2}$ and $s = m-k$, and note $m-k \geq 2 > 1 = d/2$) and iii) of Lemma 3.1 to the product of matrix and vector, we readily get $\mathbb{B}_1\partial_z\mathbf{U}(\vartheta) \in \mathcal{H}^{m-k-\frac{1}{2}}(D)$. This proves $\mathbf{U}_{k+1} \in \mathcal{H}^{m-k-\frac{1}{2}}(D)$.

5.4. Now for $k \leq m-3$, we prove $\chi_{k+2} \in H_{\mathrm{e}}^{m-k-1}(I)$. Set $\breve{\mathbf{U}}(\vartheta) = \sum_{p=1}^{k+1} \frac{\vartheta^p}{p!}\mathbf{U}_p|_{z=0}$. By trace theorem and induction hypotheses, this is a $\mathscr{C}^\infty$ curve in $\mathcal{H}^{m-k-1}(I)$. We may check that

$$\chi_{k+2} = \left(\frac{\mathrm{d}}{\mathrm{d}\vartheta}\right)^{k+1}\Bigg|_{\vartheta=0} q(\breve{\mathbf{U}}(\vartheta)). \tag{3.7}$$

Note that, as analysis in step 3 and step 4, we infer $q(\breve{\mathbf{U}}(\vartheta))$ is a $\mathscr{C}^\infty$ curve in the Banach space $H_{\mathrm{e}}^{m-k-1}(I)$. So χ_{k+2} still lies in $H_{\mathrm{e}}^{m-k-1}(I)$. □

3.4. **Extension of symmetric functions.** We now consider how to extend $\mathbf{U}_j$ (or χ_j) to be H^s_{loc} functions defined in the whole plane (line) with period 2 in y-variable and share some symmetry property.

Definition 3.2 (Space for extended symmetric functions)**.** We define $H_{\mathrm{e}}^s(\mathcal{I}\times\mathbb{S})$ (resp. $H_{\mathrm{o}}^s(\mathcal{I}\times\mathbb{S})$) to be the Banach space of those even (resp. odd) symmetric functions $u \in H^s_{\mathrm{loc}}(\mathcal{I}\times\mathbb{R})$ which are periodic in y-variable, with period 2, (here even or odd is with respect to the line $y = 0$,) with norm $\|u\|_{H^s_{\mathrm{e,o}}(\mathcal{I}\times\mathbb{S})} = \|u\|_{H^s(\mathcal{I}\times[0,1])}$. Here $\mathcal{I}$ might be $\mathbb{R}$, $[0,1]$ etc.

Lemma 3.3 (Extension in space)**.** *Any $\chi \in H_{\mathrm{e}}^s(I)$ has an extension $\tilde{\chi} \in H_{\mathrm{e}}^s(\mathbb{S})$ with $\|\tilde{\chi}\|_{H^s(\mathbb{S})} = \|\chi\|_{H^s(I)}$, and any $u \in H^s_{\mathrm{e,o}}(D)$ has an extension $\tilde{u} \in H^s_{\mathrm{e,o}}(\mathbb{R}\times\mathbb{S})$ with $\|\tilde{u}\|_{H^s(\mathbb{R}\times\mathbb{S})} \leq C\,\|u\|_{H^s(D)}$ and $C > 0$ depending only on s.*

Proof. 1. For $k \in \mathbb{Z}$ an integer and $\tau \in [0,1]$, define

$$\tilde{\chi}(y) \doteq \begin{cases} \chi(\tau) & \text{if } y = 2k+\tau, \\ \chi(\tau) & \text{if } y = 2k-\tau. \end{cases}$$

Obviously $\tilde{\chi}$ is of period 2, even symmetric with respect to $y=0$ (hence also even symmetric with respect to $y=1$). It can also be easily checked by using definition and trace theorem that $\tilde{\chi} \in H^s(\mathbb{S})$.

2. We then consider extension of a function $u \in H^s_{\mathrm{e}}(D)$. For $k \in \mathbb{Z}$ and $\tau \in [0,1]$, we set

$$\tilde{u}(z,y) = \begin{cases} u(z,\tau), & \text{if } \ y = 2k+\tau, \\ u(z,\tau), & \text{if } \ y = 2k-\tau, \end{cases}$$

which belongs to $H^s_{\mathrm{e}}([0,1]\times\mathbb{S})$. Then for $\theta(z)$ a $\mathscr{C}^\infty(\mathbb{R}^+)$ function with values in $[0,1]$, and equals 1 for $z \in [0,\frac{1}{2}]$ and vanishes for $z \geq \frac{3}{4}$, we set $\tilde{u}^{\mathrm{e}}(z,y) \doteq \tilde{u}_0(z,y)\theta(z)$. By this we regard $\tilde{u}^{\mathrm{e}}$ as defined on $[0,\infty)\times\mathbb{S}$. Then for $\phi(t) \in \mathscr{C}^\infty(\mathbb{R}^+)$ so that $\int_0^\infty t^k\phi(t)\,\mathrm{d}t = (-1)^k, \quad k = 0,1,2,\cdots$, (see [16, p.138],) we set

$$\breve{u}^{\mathrm{e}}(z,y) \doteq \begin{cases} \int_0^\infty \tilde{u}^{\mathrm{e}}(-zs,y)\phi(s)\,\mathrm{d}s, & \text{if } \ z<0, \\ \tilde{u}(z,y), & \text{if } \ 0 \leq z \leq 1. \end{cases}$$

By this way $\breve{u}^{\mathrm{e}}_0$ is an extension of $\tilde{u}$ and belongs to $H^s_{\mathrm{e}}((-\infty,1]\times\mathbb{S})$. Finally, define

$$\tilde{u}^\sharp(z,y) \doteq \begin{cases} \int_0^\infty \breve{u}^{\mathrm{e}}(1+s(1-z),y)\phi(s)\,\mathrm{d}s, & \text{if } \ z \geq 1, \\ \breve{u}^{\mathrm{e}}(z,y), & \text{if } \ z \leq 1, \end{cases}$$

which is an extension of $u \in H^s_{\mathrm{e}}(D)$ to $H^s_{\mathrm{e}}(\mathbb{R}\times\mathbb{S})$ as desired.

3. We then consider extension of a function $v \in H^s_{\mathrm{o}}(D)$. For $k \in \mathbb{Z}$ and $\tau \in [0,1]$, we set

$$\tilde{v}(z,y) \doteq \begin{cases} v(z,\tau), & \text{if } \ y = 2k+\tau, \\ -v(z,\tau), & \text{if } \ y = 2k-\tau, \end{cases}$$

which belongs to $H^s_{\mathrm{o}}([0,1]\times\mathbb{S})$. Then totally the same as in step 2, we can extend v to a $\tilde{v}^\sharp \in H^s_{\mathrm{o}}(\mathbb{R}\times\mathbb{S})$. □

The next lemma concerns trace-lift and is essential for existence of an approximate solution. For simplicity, as before, we write the space

$$(H^s_{\mathrm{e}}(\mathbb{R}\times[0,1]\times\mathbb{S})^2 \times H^s_{\mathrm{o}}(\mathbb{R}\times[0,1]\times\mathbb{S}) \times H^s_{\mathrm{e}}(\mathbb{R}\times[0,1]\times\mathbb{S}))^2$$

as $\mathcal{H}^s(\mathbb{R}\times[0,1]\times\mathbb{S})$.

Lemma 3.4 (Extension in time). *For $\{(\mathbf{U}_j, \chi_j)\}_{j=0}^m$ given by (3.1), there is a pair $(\tilde{\mathbf{U}}^a, \tilde{\chi}^a) \in \mathcal{H}^{m+1}(\mathbb{R} \times [0,1] \times \mathbb{S}) \times H_{\mathrm{e}}^{m+\frac{3}{2}}(\mathbb{R} \times \mathbb{S})$ with the following properties:*

i) $\partial_t^j \tilde{\mathbf{U}}^a|_{t=0} = \mathbf{U}_j, \quad \partial_t^j \tilde{\chi}^a|_{t=0} = \chi_j, \quad$ *for* $\quad j = 0, \cdots, m;$

ii) *there is a constant C depending only on $\|\mathbf{U}_0\|_{H^{m+\frac{1}{2}}(D)}, |\chi_0(0)|$ and m so that for any $T > 0$,*

$$\left\|\tilde{\mathbf{U}}^a - \underline{\mathbf{U}}\right\|_{\mathcal{H}^{m+1}([0,T]\times[0,1]\times\mathbb{S})} + \|\chi^a - \chi_0(0)\|_{H^{m+\frac{3}{2}}([0,T]\times\mathbb{S})} \le C \|\mathbf{U}_0 - \underline{\mathbf{U}}\|_{H^{m+\frac{1}{2}}(D)}. \quad (3.8)$$

Proof. The proof resembles that in [17, p.82], just replacing Fourier transform used there by Fourier series in y-variable. So the detail is omitted. □

Lemma 3.5 (Restriction). *The restriction mapping*

$$\mathcal{R} : H_{\mathrm{e}}^s(\mathbb{R}\times[0,1]\times\mathbb{S})\times H_{\mathrm{o}}^{s'}(\mathbb{R}\times[0,1]\times\mathbb{S})\times\times H_{\mathrm{e}}^{s''}(\mathbb{R}\times\mathbb{S}) \to (H_{\mathrm{e}}^s(D^T)\times H_{\mathrm{o}}^{s'}(D^T)\times H_{\mathrm{e}}^{s''}(I^T)$$

given by

$$\mathcal{R}(u,\ v,\ \chi) \doteq (u|_{D^T},\ v|_{D^T},\ \chi|_{I^T})$$

is one-to-one and onto. In addition, both itself and its inverse are continuous.

Proof. The mapping is well-defined. For instance, by definition, for $u \in H_{\mathrm{e}}^s(\mathbb{R}\times[0,1]\times\mathbb{S})$, as it satisfies $u(t,z,y) - u(t,z,-y) = 0$, definitely $\partial_y^{2j+1} u(t,z,0) = 0$ as long as this makes sense. Since it is periodic with period 2 in y, we also have

$$u(t,z,1-y) = u(t,z,-1-y) = u(t,z,1+y),$$

hence $\partial_y^{2j+1} u(t,z,1) = 0$. This shows u belongs to $H_{\mathrm{e}}^s(D^T)$. By the proof of Lemma 3.3, we know $\mathcal{R}$ is onto. The continuity of $\mathcal{R}$ and $\mathcal{R}^{-1}$ are clear. □

3.5. **Approximate solutions.** Now we use $\{(\mathbf{U}_j, \chi_j)\}_{j=0}^m$ to construct approximate solutions $(\mathbf{U}^a, \chi^a)$ to the nonlinear problem. The k-th order compatibility conditions ensure the accuracy of $(\mathbf{U}^a, \chi^a)$ is $O(t^k)$ at $t = 0$.

Lemma 3.6 (Existence of approximate solution $(\mathbf{U}^a, \chi^a)$). *Suppose a)–c) in Theorem 2.1 hold. Then there exist $T_0 > 0$ and $\mathbf{U}^a \in \underline{\mathbf{U}} + \mathcal{H}^{m+1}(\mathbb{R} \times D)$, $\chi^a \in H_{\mathrm{e}}^{m+\frac{3}{2}}(\mathbb{R} \times I)$ so that*

i) $\mathbf{U}^a - \underline{\mathbf{U}}$ *and* χ^a *both vanish for* $|t| \ge 2T_0$;

ii) $\mathbf{U}^a|_{t=0} = \mathbf{U}_0, \quad \chi^a|_{t=0} = \chi_0$;

iii) *there is a constant c depending only on and non-decreasingly on $\|\mathbf{U}_0\|_{H^{m+\frac{1}{2}}(D)}$ and $|\chi_0(0)|$ so that*

$$\|\mathbf{U}^a - \underline{\mathbf{U}}\|_{H^{m+1}(\mathbb{R}\times D)} + \|\chi^a\|_{H^{m+\frac{3}{2}}(\mathbb{R}\times I)} \le c(\|\mathbf{U}_0 - \underline{\mathbf{U}}\|_{H^{m+\frac{1}{2}}(D)} + |\chi_0(0)|).$$

iv) *for all* $t \in [-T_0, T_0]$, $(z, y) \in D$, *one has* $\mathbf{U}^a(t, z, y) \in \mathscr{W}_{2\mu/3}$, *as well as*

$$(\chi^a(t, z, y), \mathrm{d}\chi^a(t, z, y)) \in \mathscr{V}_{2\mu/3};$$

v) *for* $f_0 \doteq -L(\mathbf{U}^a, \chi^a, \mathrm{d}\chi^a)\mathbf{U}^a, h_0 \doteq -\mathbb{J}\mathrm{d}\chi^a - Q(\mathbf{U}^a|_{z=0}), g_0 \doteq M\mathbf{U}^a - \tilde{\mathbf{g}}$, *where* $\tilde{\mathbf{g}}(t) = \begin{cases} \mathbf{g}(t) & \text{for } |t| \leq T_0, \\ \underline{\mathbf{g}} & \text{for } |t| \geq 2T_0, \end{cases}$ *there hold, at* $t = 0$, *that*

$$\partial_t^p f_0 \equiv 0, \quad \partial_t^p h_0 \equiv 0, \quad \partial_t^p g_0 \equiv 0, \qquad \text{for } p = 0, 1, \cdots, m-1;$$

vi) *furthermore,* $f_0 \in \mathcal{H}^m(\mathbb{R} \times D)$, $g_0 \in H_{\mathrm{e}}^m(\mathbb{R} \times I)^2 \times H_{\mathrm{o}}^m(\mathbb{R} \times I) \times H_{\mathrm{e}}^m(\mathbb{R} \times I)^2$, $h_0 \in H_{\mathrm{e}}^m(\mathbb{R} \times I) \times H_{\mathrm{o}}^m(\mathbb{R} \times I) \times H_{\mathrm{e}}^m(\mathbb{R} \times I)^2$ *and all vanish for* $|t| > 2T_0$;

vii) *there are the following estimates:*

$$\|f_0\|_{H^m(D^T)} + \|h_0\|_{H^m(I^T)} \leq c_1 \|\mathbf{U}_0 - \underline{\mathbf{U}}\|_{H^{m+\frac{1}{2}}(D)}, \tag{3.9}$$

$$\|g_0\|_{H^m(I^T)} \leq c(\|\mathbf{U}_0 - \underline{\mathbf{U}}\|_{H^{m+\frac{1}{2}}(D)} + \|M\underline{\mathbf{U}} - \mathbf{g}\|_{H^m(I^T)}), \tag{3.10}$$

$$\|f_0\|_{H^m(D^T)} + \|g_0\|_{H^m(I^T)} + \|h_0\|_{H^m(I^T)} = O(T), \quad \text{as } T \to 0. \tag{3.11}$$

Here c_1 *is a constant depending increasingly on* $\|\mathbf{U}_0\|_{H^{m+\frac{1}{2}}(D)}$ *and* $\underline{\mathbf{U}}$, $|\chi_0(0)|$;

viii) *finally, there holds* $M'\mathbf{U}^a|_{\Gamma_{0,1}} = 0$ *for all* $t \in \mathbb{R}$.

Remark 3.1. In v) we introduced $\tilde{\mathbf{g}} \in H_{\mathrm{e}}^m(I^T)^2 \times H_{\mathrm{o}}^m(I^T) \times H_{\mathrm{e}}^m(I^T)^2$, which is a cut-off of $\mathbf{g}$, to fulfill the technical assumption that $g_0 = 0$ for $t > 2|T_0|$ in vi), which means $\mathbf{g} = \underline{\mathbf{g}}$ — the value of reference state — for $t > 2T_0$. To prove Theorem 2.1, later we will choose the time period $[0, \bar{T}]$ with $\bar{T} \leq T_0$, so this cut-off can be easily removed.

Proof. 1. Using Lemma 3.4 and Lemma 3.5, we have already found $\tilde{\mathbf{U}}^a \in \underline{\mathbf{U}} + \mathcal{H}^{m+1}(\mathbb{R} \times D)$ and $\tilde{\chi}^a \in H_{\mathrm{e}}^{m+\frac{3}{2}}(\mathbb{R} \times I)$ such that $\partial_t^k(\tilde{\mathbf{U}}^a)|_{t=0} = \mathbf{U}_k, \partial_t^k(\tilde{\chi}^a)|_{t=0} = \chi_k$ for $k = 0, 1, \cdots, m$ and

$$\left\|\tilde{\mathbf{U}}^a - \underline{\mathbf{U}}\right\|_{H^{m+1}(\mathbb{R} \times D)} + \|\tilde{\chi}^a - \chi_0(0)\|_{H^{m+\frac{3}{2}}(\mathbb{R} \times I)} \leq c \|\mathbf{U}_0 - \underline{\mathbf{U}}\|_{H^{m+\frac{1}{2}}(D)}.$$

2. Since

$$H^{m+1}(\mathbb{R} \times D) \hookrightarrow H^1(\mathbb{R}; H^m(D)) \hookrightarrow \mathscr{C}(\mathbb{R}; H^m(D)) \hookrightarrow \mathscr{C}(\mathbb{R}; \mathscr{C}(D)),$$

there is a $T_0 > 0$ so that $\left\|\tilde{\mathbf{U}}^a(t) - \mathbf{U}_0\right\|_{\mathscr{C}(D)} \leq \frac{\mu}{3}$, for any $|t| < 2T_0$. Since

$$H^{m+\frac{3}{2}}(\mathbb{R} \times I) \hookrightarrow H^2(\mathbb{R}; H^{m-1}(I)) \hookrightarrow \mathscr{C}^1(\mathbb{R}; \mathscr{C}^1(I)),$$

we also have

$$\|\tilde{\chi}^a(t) - \chi_0\|_{\mathscr{C}([0,1])} + \|\partial_t\tilde{\chi}^a(t) - \chi_1\|_{\mathscr{C}([0,1])} + \|\partial_y\tilde{\chi}^a(t) - \chi_0'\|_{\mathscr{C}([0,1])} \leq \mu/3$$

for any $|t| < 2T_0$, by taking T_0 smaller if necessary.

3. Let $\phi_0 \in \mathscr{D}(\mathbb{R})$ be a cut-off function so that $\phi_0(t) = 1$ for $|t| \le T_0$ and $\phi_0(t) = 0$ for $|t| > 2T_0$, and $0 \le \phi(t) \le 1$ for $t \in \mathbb{R}$. We now define

$$\mathbf{U}^a \doteq \phi_0(t)\tilde{\mathbf{U}}^a + (1-\phi_0(t))\underline{\mathbf{U}}, \quad \chi^a \doteq \phi_0(t)\tilde{\chi}^a.$$

Then claims i)–iii) follow easily.

4. For claims in iv), by step 2, we infer

$$|\mathbf{U}^a(t,z,y) - \underline{\mathbf{U}}| \le \phi_0(t)(|\tilde{\mathbf{U}}^a(t,z,y) - \mathbf{U}_0(z,y)| + |\mathbf{U}_0(z,y) - \underline{\mathbf{U}}|) \le 2\mu/3,$$

and, note $\phi_0(t) \equiv 1$ for $|t| \le T_0$,

$$|(\chi^a, \mathrm{d}\chi^a)| = |(\tilde{\chi^a}, \mathrm{d}\tilde{\chi}^a)| \le |(\tilde{\chi}^a - \chi_0, \mathrm{d}\tilde{\chi^a} - (\chi_1, \chi_0'))| + |(\chi_0, \chi_1, \chi_0')| \le \frac{2\mu}{3}.$$

5. $\partial_t^k f_0|_{t=0} = 0$ $(k = 0, 1, \cdots, m-1)$ follow from the definitions of $\mathbf{U}_{k+1}$, while $\partial_t^k g_0|_{t=0} = 0$ and $\partial_t^k h_0|_{t=0}$ $(k = 0, 1, \cdots, m-1)$ follow from compatibility conditions of order $m-1$ and definitions of χ_{k+1}.

6. It is easy to check that f_0, h_0, g_0 vanish for $|t| > 2T_0$.

7. The regularity and symmetry

$$g_0 \in H_{\mathrm{e}}^m(\mathbb{R}\times I)^2 \times H_{\mathrm{o}}^m(\mathbb{R}\times I) \times H_{\mathrm{e}}^m(\mathbb{R}\times I)^2$$

follows from $\mathbf{U}^a|_{z=1} \in \mathcal{H}^{m+1}(\mathbb{R}\times I)$ and the assumption

$$\mathbf{g} \in H_{\mathrm{e}}^m(\mathbb{R}\times I)^2 \times H_{\mathrm{o}}^m(\mathbb{R}\times I) \times H_{\mathrm{e}}^m(\mathbb{R}\times I)^2$$

assumed in Theorem 2.1. The estimate (3.10) is simple, since

$$\begin{aligned} \|g_0\|_{H^m(I^T)} &\le \|(\mathbf{U}^a - \underline{\mathbf{U}})|_{z=1}\|_{H^m(I^T)} + \|\tilde{\mathbf{g}} - M\underline{\mathbf{U}}\|_{H^m(I^T)} \\ &\le c(\|\mathbf{U}_0 - \underline{\mathbf{U}}\|_{H^{m+\frac12}(D)} + \|M\underline{\mathbf{U}} - \mathbf{g}\|_{H^m(I^T)}). \end{aligned}$$

8. The regularity $f_0 \in H^m(\mathbb{R}\times D)^8$ is a consequence of Propositions 3.1 and 3.3:

$$\begin{aligned} \|f_0\|_{H^m(D^T)} &= \|L(\mathbf{U}^a, \chi^a, \mathrm{d}\chi^a)(\mathbf{U}^a - \underline{\mathbf{U}})\|_{H^m(D^T)} \\ &\le C(\|\mathbf{U}^a\|_{H^m(D^T)} + \|\chi^a\|_{H^{m+1}(I^T)})\,\|\mathbf{U}^a - \underline{\mathbf{U}}\|_{H^{m+1}(D^T)} \\ &\le c_1\,\|\mathbf{U}_0 - \underline{\mathbf{U}}\|_{H^{m+\frac12}(D)}. \end{aligned}$$

Here c_1 is a constant depending increasingly on $\|\mathbf{U}_0 - \underline{\mathbf{U}}\|_{H^{m+\frac12}(D)}$ and $|\chi_0(0)|$.

We easily see that f_0 is periodic with period 2 in y-variable. By using the special structure of the operator L, one can directly show that (cf. Section 4.7)

$$\begin{aligned} f_0(t,z,-y) &= L(\mathbf{U}^a(t,z,-y), \chi^a(t,-y), \mathrm{d}\chi^a(t,-y))\mathbf{U}^a(t,z,-y) \\ &= \mathrm{diag}(1,1,-1,1,1,1,-1,1)f_0(t,z,y). \end{aligned}$$

Lemma 3.5 and this imply that $f_0 \in \mathcal{H}^m(\mathbb{R}\times D)$.

9. Since $\chi^a \in H^{m+\frac{3}{2}}(\mathbb{R}\times I)$, also recall that $Q(\underline{\mathbf{U}})=0$, we have

$$\begin{aligned}
\|h_0\|_{H^m(I^T)} &\leq \|\chi^a-\chi_0(0)\|_{H^{m+1}(I^T)} + \|Q(\mathbf{U}^a|_{z=0})-Q(\underline{\mathbf{U}})\|_{H^m(I^T)} \\
&\leq \|\mathbf{U}_0-\underline{\mathbf{U}}\|_{H^{m+\frac{1}{2}}(D)} + c_1\|\mathbf{U}^a-\underline{\mathbf{U}}\|_{H^m(I^T)} \\
&\leq \|\mathbf{U}_0-\underline{\mathbf{U}}\|_{H^{m+\frac{1}{2}}(D)} + c_1\|\mathbf{U}^a-\underline{\mathbf{U}}\|_{H^{m+1}(D^T)} \\
&\leq c_1\|\mathbf{U}_0-\underline{\mathbf{U}}\|_{H^{m+\frac{1}{2}}(D)}.
\end{aligned}$$

To show $h_0\in H^m_{\mathrm{e}}(\mathbb{R}\times I)\times H^m_{\mathrm{o}}(\mathbb{R}\times I)\times H^m_{\mathrm{e}}(\mathbb{R}\times I)^2$, we use again Lemma 3.5 and only need to verify $h_0(t,-y)=\mathrm{diag}(1,-1,1,1)h_0(t,y)$. This follows from simple calculations. For example, we have

$$(h_0)_2(t,-y) = -(\partial_y\chi^a)(t,-y)) - r(\mathbf{U}^a(t,-y)) = \partial_y\chi^a(t,y)+r(\mathbf{U}^a(t,y)) = -(h_0)_2(t,y).$$

10. The estimate (3.11) follows from absolute continuity of integrals. Claim viii) is a direct consequence of the symmetry property of $\mathbf{U}^a$, that is, $v^a_\pm$ is odd symmetric with respect to $y=0,1$. □

4. Linearized problem

In this section we linearize the nonlinear problem around a state $(\mathbf{U},\chi)$ quite close (in the sense of $\mathscr{W}_\mu$ and $\mathscr{V}_\mu$) to the reference state $(\underline{\mathbf{U}},0)$. By symmetry properties, the characteristic boundaries become periodic boundaries and hence "disappear". The linearized problem is reduced to the case with purely non-characteristic boundaries. A careful partition of unity is used to obtain well-posedness and regularity, as well as estimates of the linearized problem.

4.1. **The linearized problem.** For our purpose, we may use simply $L(\mathbf{U},\chi,\mathrm{d}\chi)\dot{\mathbf{U}}=F$ as the linearized interior equation, where $\dot{\mathbf{U}}$ is the unknown, F is a given nonhomogeneous term. Substitute $\mathbf{U}+\epsilon\dot{\mathbf{U}}$ for $\mathbf{U}$ and $\chi+\epsilon\dot{\chi}$ for χ into R-H conditions, differentiating it with respect to ϵ and then taking $\epsilon=0$, we find the linearized version to be $\mathbb{J}\mathrm{d}\dot{\chi}+\nabla Q(\mathbf{U})\dot{\mathbf{U}}=0$. This more accurate linearization will give us second order accuracy on controlling boundary terms. The other boundary conditions are linear, so the linearized version is simple. They are $M\dot{\mathbf{U}}=0$ on Σ_1 and $M'\dot{\mathbf{U}}=0$ on $\Gamma_{0,1}$. The initial data for $\dot{\mathbf{U}}$ and $\dot{\chi}$ are given by

$$\dot{\mathbf{U}}|_{t=0}=0 \quad \text{on} \quad D; \qquad \dot{\chi}|_{t=0}=0 \quad \text{on} \quad \Sigma_0. \tag{4.1}$$

We then have the linearized problem in D^T:

$$\begin{cases} L(\mathbf{U},\chi,\mathrm{d}\chi)\dot{\mathbf{U}}=F, & \text{in}\ \ D^T,\\ \dot{\mathbf{U}}=\dot{\mathbf{U}}_0, & \text{on}\ \ \{0\}\times D,\\ \mathbb{J}\mathrm{d}\dot{\chi}+\nabla Q(\mathbf{U})\dot{\mathbf{U}}=G, & \text{on}\ \ \Sigma_0^T,\\ \dot{\chi}=\dot{\chi}_0, & \text{on}\ \ \{0\}\times[0,1],\\ M\dot{\mathbf{U}}=g, & \text{on}\ \ \Sigma_1^T,\\ M'\dot{\mathbf{U}}=0, & \text{on}\ \ \Gamma_{0,1}^T. \end{cases} \tag{4.2}$$

We remark that in the application to study nonlinear problem, one only needs take zero initial data $\dot{\mathbf{U}}_0=0,\ \dot{\chi}_0=0$, while to solve the linear problem for a longer time, we need general initial data as assumed above. A weak solution of this problem can be defined by using adjoint problem and a related Green formula (cf. [9, p.357]). However, as we will finally consider (classical) solutions in H^m with $m\geq 3$, we omit the definition of weak solutions, although it is essential for a rigorous understanding of L^2 well-posedness.

With Lemma 3.5 in mind, we give the following definition, for the convenience of statement of results on linear problems.

Definition 4.1. For $s\geq 0$, we say $\mathbf{U},F\in H^s(D^T)^8,\chi\in H^{s+1}(I^T),g\in H^s(I^T)^5$, $G\in H^s(I^T)^4$ and $\dot{\mathbf{U}}_0\in H^{s+\frac12}(D)^8$, $\dot{\chi}_0\in H^{s+\frac12}(I)$ are *properly symmetric,* if they can be extended periodically in y-variable with periodic 2, to functions $\tilde{\mathbf{U}},\tilde{F}\in H^s([0,T]\times[0,1]\times\mathbb{S})^8,\tilde{\chi}\in H^{s+1}([0,T]\times\mathbb{S}),\tilde{g}\in H^s([0,T]\times\mathbb{S})^5$, $\tilde{G}\in H^s([0,T]\times\mathbb{S})^4$, and $\tilde{\dot{\mathbf{U}}}_0\in H^{s+\frac12}([0,1]\times\mathbb{S})^8$, $\tilde{\dot{\chi}}_0\in H^{s+\frac12}(\mathbb{S})$, and there hold

$$\begin{aligned} &\tilde{\mathbf{U}}(t,x,-y)=\mathrm{diag}(1,1,-1,1,1,1,-1,1)\tilde{\mathbf{U}}(t,x,y),\\ &\tilde{F}(t,x,-y)=\mathrm{diag}(1,1,-1,1,1,1,-1,1)\tilde{F}(t,x,y),\\ &\tilde{\chi}(t,y)=\tilde{\chi}(t,-y),\quad \tilde{g}(t,-y)=\mathrm{diag}(1,1,-1,1,1)\tilde{g}(t,y),\\ &\tilde{G}(t,-y)=\mathrm{diag}(1,-1,1,1)\tilde{G}(t,y),\quad \tilde{\dot{\chi}}_0(t,-y)=\tilde{\dot{\chi}}_0(t,y),\\ &\tilde{\dot{\mathbf{U}}}_0(t,x,-y)=\mathrm{diag}(1,1,-1,1,1,1,-1,1)\tilde{\dot{\mathbf{U}}}_0(t,x,y). \end{aligned}$$

As before, this definition means for any integer k, all quantities except v, such as u,p,ρ,s,e,χ, are extended by using even reflection with respect to $y=k$, while v is through odd reflection. In the following, we also drop tildes in the notations of the extended functions for simplicity of writing.

We will prove the following two theorems concerning well-posedness of the linear problem (4.2) at the end of this section.

Theorem 4.1 (L^2 well-posedness)**.** *For a reference state $(\underline{\mathbf{U}}, \underline{\chi})$ satisfying the uniform stability condition and $T > 0$, there is a number $\mu > 0$. Suppose*

a) $\mathbf{U} \in W^{1,\infty}(D^T)$, *taking values in* $\mathscr{W}_\mu$, *and* $\chi \in W^{2,\infty}(I^T)$, *with* $(\chi, \mathrm{d}\chi)$ *taking values in* $\mathscr{V}_\mu$;
b) $\mathbf{U}, \chi, F, G, g, \dot{\mathbf{U}}_0$ *and* $\dot{\chi}_0$ *are all properly symmetric for* $s = 0$;
c) $F \in L^2(D^T)$, $G \in L^2(I^T)$, $g \in L^2(I^T)$, $\dot{\mathbf{U}}_0 \in L^2(D)$, *and* $\dot{\chi}_0 \in H^{\frac{1}{2}}(I)$.

Then Problem (4.2) *has a unique solution* $(\dot{\mathbf{U}}, \dot{\chi}) \in L^2(D^T) \times H^1(I^T)$ *and* $\dot{\mathbf{U}}$ *belongs to* $\mathscr{C}([0,T]; L^2(D))$, *as well as* $\dot{\mathbf{U}}|_{z=0,1}$ *belong to* $L^2(I^T)$. *The solution is also properly symmetric. Moreover, for all real number* K, *there are constants* c *and* $T_1 > 0$ *such that, if* $\|\mathbf{U}, \chi, \mathrm{d}\chi\|_{W^{1,\infty}} \leq K$, *the solutions satisfy the estimate for any* $T \leq T_1$:

$$\begin{aligned} &\left\|\dot{\mathbf{U}}(t)\right\|^2_{\mathscr{C}([0,T];L^2(D))} + \frac{1}{T}\left\|\dot{\mathbf{U}}\right\|^2_{L^2(D^T)} + \left\|\dot{\mathbf{U}}|_{z=0,1}\right\|^2_{L^2(I^T)} + \|\dot{\chi}\|^2_{H^1(I^T)} \\ \leq\ & c\left(T\|F\|^2_{L^2(D^T)} + \left\|\dot{\mathbf{U}}_0\right\|^2_{L^2(D)} + \|G\|^2_{L^2(I^T)} + \|\dot{\chi}_0\|^2_{H^{\frac{1}{2}}(I)} + \|g\|^2_{L^2(I^T)}\right). \end{aligned} \tag{4.3}$$

Theorem 4.2 (H^m regularity)**.** *For* m *an integer larger than* $5/2$, *and a number* $T > 0$, *suppose*

a) *the reference state* $(\underline{\mathbf{U}}, \underline{\chi})$ *satisfies the uniform stability condition;*
b) $\mathbf{U}, \chi, F, G, g$ *are all properly symmetric for* $s = m$;
c) $\mathbf{U}$ *can be extended to be a function in* $H^m((-\infty, T] \times D)$, *and* χ *can be extended to be a function in* $H^{m+1}((-\infty, T] \times I)$, *and there hold* $\mathbf{U} - \underline{\mathbf{U}} \in H^m((-\infty, T] \times D; \mathbb{R}^8)$, $(\mathbf{U} - \underline{\mathbf{U}})|_{z=0} \in H^m((-\infty, T] \times I)$, *and* $\chi, \mathrm{d}\chi \in H^m((-\infty, T] \times I)$;
d) $(\mathbf{U} - \underline{\mathbf{U}})_{t<\tau} \equiv 0$ *and* $(\chi,\ \mathrm{d}\chi)|_{t<\tau} \equiv 0$ *for some* $\tau < T$;
e) *for some* $K > 0$, $\|\mathbf{U} - \underline{\mathbf{U}}\|_{H^m((-\infty,T]\times D)} \leq K$, $\|(\mathbf{U} - \underline{\mathbf{U}})|_{z=0}\|_{H^m((-\infty,T]\times I} \leq K$, *and* $\|\chi, \mathrm{d}\chi\|_{H^m((-\infty,T]\times I)} \leq K$; *moreover* $\mathbf{U} \in \mathscr{W}_\mu$, $(\chi,\ \mathrm{d}\chi) \in \mathscr{V}_\mu$;
f) $F \in H^m((-\infty, T] \times D)$, $G \in H^m((-\infty, T] \times I)$, *and* $g \in H^m([0, T] \times I)$;
g) $F|_{t<0} = 0$, $G|_{t<0} = 0$, *and* $g|_{t<0} = 0$;
h) $\dot{\mathbf{U}}|_{t<0} = 0$, *and* $\dot{\chi}|_{t<0} = 0$.

Then the solution $(\dot{\mathbf{U}}, \dot{\chi})$ *of problem* (4.2) *is also properly symmetric for* $s = m$, *and satisfies*

i) $\dot{\mathbf{U}} \in H^m((-\infty, T] \times D; \mathbb{R}^8)$, $\dot{\mathbf{U}}|_{z=0,1} \in H^m((-\infty, T] \times I)$, *and* $\dot{\chi} \in H^{m+1}((-\infty, T] \times I)$;

ii) *the estimate for any $T < T_1$ with the constants T_1 and c depending only on K and m:*

$$\begin{aligned}&\frac{1}{T}\left\|\dot{\mathbf{U}}\right\|^2_{H^m(D^T)} + \left\|\dot{\mathbf{U}}|_{z=0,1}\right\|^2_{H^m(I^T)} + \|\dot{\chi}\|^2_{H^{m+1}(I^T)}\\ \leq\ & c\left(T\|F\|^2_{H^m(D^T)} + \|g\|^2_{H^m(I^T)} + \|G\|^2_{H^m(I^T)}\right). \qquad (4.4)\end{aligned}$$

4.2. **Periodic extension and linear problem in a strip.** By proper symmetry of $\mathbf{U}, \chi$ and the data $F, G, g, \dot{\mathbf{U}}_0, \dot{\chi}_0$, we can readily extend them suitably to be defined for $y \in \mathbb{S}$ (we recall this means periodic in y with period 2, cf. Lemma 3.3). Dropping the walls $\Gamma_{0,1}$ and the boundary conditions on them, we formulate the following problem:

$$\begin{cases} L(\mathbf{U}, \chi, \mathrm{d}\chi)\dot{\mathbf{U}} = F, & t \in [0,T],\ z \in [0,1],\ y \in \mathbb{S},\\ \dot{\mathbf{U}} = \dot{\mathbf{U}}_0, & t = 0,\ z \in [0,1],\ y \in \mathbb{S},\\ \mathbb{J}\mathrm{d}\dot{\chi} + \nabla Q(\mathbf{U})\dot{\mathbf{U}} = G, & t \in [0,T],\ z = 0,\ y \in \mathbb{S},\\ \dot{\chi} = \dot{\chi}_0, & t \in [0,T],\ y \in \mathbb{S},\\ M\dot{\mathbf{U}} = g, & t \in [0,T],\ z = 1,\ y \in \mathbb{S}. \end{cases} \qquad (4.5)$$

We have the following L^2 well-posedness and H^m regularity results. They will be proved in Section 4.6. In this paper we always assume the reference state $(\underline{\mathbf{U}}, \underline{\chi})$ satisfies the uniform stability condition, so sometimes it will not be repeated.

Proposition 4.1. *Under the assumptions a), c) of Theorem 4.1 (with D replaced by $[0,1]\times\mathbb{S}$, I replaced by $\mathbb{S}$), Problem (4.5) has a unique solution $(\dot{\mathbf{U}}, \dot{\chi}) \in L^2([0,T]\times[0,1]\times\mathbb{S}) \times H^1([0,T]\times\mathbb{S})$ and $\dot{\mathbf{U}}$ belongs to $\mathscr{C}([0,T]; L^2([0,1]\times\mathbb{S}))$, as well as $\dot{\mathbf{U}}|_{z=0,1}$ belong to $L^2([0,T]\times\mathbb{S})$. Moreover, for all real number K, there are constants c and T_1 such that, if $\|\mathbf{U}, \chi, \mathrm{d}\chi\|_{W^{1,\infty}} \leq K$, then the solutions satisfy the estimate for any $T \leq T_1$:*

$$\begin{aligned}&\left\|\dot{\mathbf{U}}(t)\right\|^2_{\mathscr{C}([0,T];L^2([0,1]\times\mathbb{S}))} + \frac{1}{T}\left\|\dot{\mathbf{U}}\right\|^2_{L^2([0,T]\times[0,1]\times\mathbb{S})} + \left\|\dot{\mathbf{U}}|_{z=0,1}\right\|^2_{L^2([0,T]\times\mathbb{S})}\\ &+ \|\dot{\chi}\|^2_{H^1([0,T]\times\mathbb{S})} \leq c\Bigg(T\|F\|^2_{L^2[0,T]\times[0,1]\times\mathbb{S}} + \left\|\dot{\mathbf{U}}_0\right\|^2_{L^2([0,1]\times\mathbb{S})} + \|G\|^2_{L^2([0,T]\times\mathbb{S})}\\ &+ \|\dot{\chi}_0\|^2_{H^{\frac12}(\mathbb{S})} + \|g\|^2_{L^2([0,T]\times\mathbb{S})}\Bigg). \qquad (4.6)\end{aligned}$$

Proposition 4.2. *Under the assumptions a), c)–h) in Theorem 4.2 (with D replaced by $[0,1]\times\mathbb{S}$, I replaced by $\mathbb{S}$), the solution $(\dot{\mathbf{U}}, \dot{\chi})$ of Problem (4.5) satisfies*

i) $\dot{\mathbf{U}} \in H^m((-\infty,T]\times[0,1]\times\mathbb{S};\ \mathbb{R}^8)$, $\dot{\mathbf{U}}|_{z=0,1} \in H^m((-\infty,T]\times\mathbb{S})$, *and*

$$\dot{\chi} \in H^{m+1}((-\infty,T]\times\mathbb{S});$$

ii) *the following estimate for any $T \le T_1$ with constants c, T_1 depending only on K and m:*

$$\begin{aligned} & \frac{1}{T}\left\|\dot{\mathbf{U}}\right\|^2_{H^m([0,T]\times[0,1]\times\mathbb{S})} + \left\|\dot{\mathbf{U}}|_{z=0,1}\right\|^2_{H^m([0,T]\times\mathbb{S})} + \|\dot{\chi}\|^2_{H^{m+1}([0,T]\times\mathbb{S})} \\ \le\ & c\left(T\,\|F\|^2_{H^m([0,T]\times[0,1]\times\mathbb{S})} + \|g\|^2_{H^m([0,T]\times\mathbb{S})} + \|G\|^2_{H^m([0,T]\times\mathbb{S})}\right). \end{aligned} \tag{4.7}$$

4.3. **Localization.** We further decompose Problem (4.5) to two problems in "half-space", for which standard results are available now. Now introduce four cut-off functions $\psi_{1,2}, \varphi_{1,2} \in \mathscr{D}(\mathbb{R})$, all with values in $[0,1]$, and share the following properties:

- $\psi_1(z) = \begin{cases} 1, & 0 \le z \le \frac{1}{2}, \\ 0, & z \ge \frac{3}{4} \end{cases}$ and $\psi_2(z) = \begin{cases} 0, & 0 \le z \le \frac{1}{2}, \\ 1-\psi_1(z), & \frac{1}{2} \le z \le \frac{3}{4}, \\ 1, & \frac{3}{4} \le z \le 1; \end{cases}$

  so $\psi_1 + \psi_2 \equiv 1$ for $z \in [0,1]$;
- $\varphi_1(z) = \begin{cases} 1, & 0 \le z \le \frac{7}{8}, \\ 0, & z \ge \frac{15}{16}, \end{cases}$ and $\varphi_2(z) = \begin{cases} 0, & z \le \frac{1}{8}, \\ 1, & \frac{1}{4} \le z \le 1. \end{cases}$

We note that $\operatorname{supp}\psi_j \cap [0,1]$ is a proper subset of $\{\varphi_j \equiv 1\} \cap [0,1]$ $(j = 1,2)$. This is crucial for the later application of finite speed of propagation property of hyperbolic equations.

We then write down two problems in half-space.

$$\text{Problem A):} \quad \begin{cases} L(\varphi_1(z)\mathbf{U} + (1-\varphi_1(z))\underline{\mathbf{U}}), \chi, \mathrm{d}\chi)\dot{\mathbf{V}} = F_1, & t\in[0,T],\ z>0,\ y\in\mathbb{S}, \\ \dot{\mathbf{V}} = \dot{\mathbf{V}}_0^1, & t=0,\ z>0,\ y\in\mathbb{S}, \\ \mathbb{J}\mathrm{d}\dot{\chi} + \nabla Q(\mathbf{U})\dot{\mathbf{V}} = G, & t\in[0,T],\ z=0,\ y\in\mathbb{S}, \\ \dot{\chi} = \dot{\chi}_0, & t\in[0,T],\ y\in\mathbb{S}; \end{cases}$$

$$\text{Problem B):} \quad \begin{cases} L(\varphi_2(z)\mathbf{U} + (1-\varphi_2(z))\underline{\mathbf{U}}, \chi, \mathrm{d}\chi)\dot{\mathbf{V}} = F_2, & t\in[0,T],\ z<1,\ y\in\mathbb{S}, \\ \dot{\mathbf{V}} = \dot{\mathbf{V}}_0^2, & t=0,\ z<1,\ y\in\mathbb{S}, \\ M\dot{\mathbf{V}} = g, & t\in[0,T],\ z=1,\ y\in\mathbb{S}. \end{cases}$$

It is important to note that due to properties of Euler equations, this modification of the operator L is still symmetric and constantly hyperbolic (just replace the point $\mathbf{U}$ by $\varphi_{1,2}(z)\mathbf{U} + (1-\varphi_{1,2}(z))\underline{\mathbf{U}} \in \mathscr{W}_\mu$).

To present estimates concerning these two problems, we need the weighted Sobolev space $\mathscr{H}^m_\gamma$ (see [9, Remark 9.9 in p.240], or [10, (4.3.2) in p.74]). Let $\mathcal{O}$ be a domain of the space-time $\mathbb{R}\times\mathbb{R}^n$, m a nonnegative integer, and $\gamma \ge 1$ a parameter. Then $\mathscr{H}^m_\gamma(\mathcal{O})$

is the set of those distributions $u \in \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)$ such that

$$\boxed{\|u\|^2_{\mathscr{H}^m_\gamma(\mathcal{O})} \doteq \sum_{|\alpha|\le m} \gamma^{2(m-|\alpha|)} \left\|\mathrm{e}^{-\gamma t}\partial^\alpha u(t,x)\right\|^2_{L^2(\mathcal{O})} < \infty.}$$

It is a Hilbert space. The space $\mathscr{H}^0_\gamma(\mathcal{O})$ is usually written as $L^2_\gamma(\mathcal{O})$ (see [9, p.122]).

4.4. **Problem A).** The basic result concerning Problem A) is the following L^2 well-posedness proved by Métivier.

Lemma 4.1. *Suppose a) in Theorem 4.1 (but $z\in\mathbb{R}^+, y\in\mathbb{S}$), and $\dot{F}_1 \in L^2([0,T]\times\mathbb{R}^+\times\mathbb{S})$, $G\in L^2([0,T]\times\mathbb{S})$, $\dot{\mathbf{V}}^1_0 \in L^2(\mathbb{R}^+\times\mathbb{S})$, and $\dot\chi_0 \in H^{\frac12}(\mathbb{S})$. Then Problem A) has a unique solution $(\dot{\mathbf{V}},\dot\chi)\in L^2([0,T]\times\mathbb{R}^+\times\mathbb{S})\times H^1([0,T]\times\mathbb{S})$ and $\dot{\mathbf{V}}$ belongs to $\mathscr{C}([0,T];L^2(\mathbb{R}^+\times\mathbb{S}))$. Moreover, for all real number K, there are constants C and γ_0 such that if $\|\mathbf{U},\chi,\mathrm{d}\chi\|_{W^{1,\infty}}\le K$, the solutions satisfy for all $\gamma\ge\gamma_0$ and all $t\in[0,T]$ that*

$$\begin{aligned}
&\mathrm{e}^{-2\gamma t}\left\|\dot{\mathbf{V}}(t)\right\|^2_{L^2(\mathbb{R}^+\times\mathbb{S})} + \gamma\left\|\dot{\mathbf{V}}\right\|^2_{L^2_\gamma([0,t]\times\mathbb{R}^+\times\mathbb{S})} + \left\|\dot{\mathbf{V}}|_{z=0}\right\|^2_{L^2_\gamma([0,t]\times\mathbb{S})} + \|\dot\chi\|^2_{\mathscr{H}^1_\gamma([0,t]\times\mathbb{S})} \\
\le\ & C\left(\frac{1}{\gamma}\|F_1\|^2_{L^2_\gamma([0,t]\times\mathbb{R}^+\times\mathbb{S})} + \|G\|^2_{L^2_\gamma([0,t]\times\mathbb{S})} + \left\|\dot{\mathbf{V}}^1_0\right\|^2_{L^2(\mathbb{R}^+\times\mathbb{S})} + \|\dot\chi_0\|^2_{H^{\frac12}(\mathbb{S})}\right).
\end{aligned} \tag{4.8}$$

Proof. Our operator L is symmetric and constantly hyperbolic, so "block structure condition" ([10, Assumption 2.3.3 and Proposition 2.3.4 in pp.44–45]) holds. The reference state $(\underline{\mathbf{U}},\underline{\chi}\equiv 0)$ satisfies uniform stability condition, then for $(\mathbf{U},\chi)$ satisfies a), the uniform Kreiss-Lopatinskiĭ condition holds [10, Assumption 2.1.1 in p.40], by taking μ small. So this lemma follows directly from [10, Theorem 3.1.1 in p.59].

We note the term $\|\dot\chi\|^2_{\mathscr{H}^1_\gamma([0,t]\times\mathbb{S})}$ is written as $\|\dot\chi\|^2_{H^1_\gamma([0,t]\times\mathbb{S})} \doteq \|\mathrm{e}^{-\gamma t}\dot\chi\|^2_{H^1([0,t]\times\mathbb{S})}$ in the above cited theorem. It can be easily checked that these two norms are equivalent (independent of γ and $\dot\chi$). □

The next lemma concerns further regularity of Problem A), but with zero initial data.

Lemma 4.2. *Under the assumptions a), c)–h) in Theorem 4.2 (but $z\in\mathbb{R}^+, y\in\mathbb{S}$ and F is replaced by F_1, $\dot{\mathbf{U}}$ is replaced by $\dot{\mathbf{V}}$, and ignore g)), the solution $(\dot{\mathbf{V}},\dot\chi)$ of Problem A) satisfies*

i) $\dot{\mathbf{V}}\in H^m((-\infty,T]\times\mathbb{R}^+\times\mathbb{S};\mathbb{R}^8)$, $\dot{\mathbf{V}}|_{z=0}\in H^m((-\infty,T]\times\mathbb{S})$, *and*

$$\dot\chi\in H^{m+1}((-\infty,T]\times\mathbb{S});$$

ii) *the following estimate for all $\gamma \geq \gamma_0$, with $C_m > 0$ and $\gamma_0 \geq 1$ depending only on and continuously on m and K:*

$$\begin{aligned} &\gamma \left\|\dot{\mathbf{V}}\right\|^2_{\mathscr{H}^m_\gamma([0,T]\times\mathbb{R}^+\times\mathbb{S})} + \left\|\dot{\mathbf{V}}|_{z=0}\right\|^2_{\mathscr{H}^m_\gamma([0,T]\times\mathbb{S})} + \|\dot{\chi}\|^2_{\mathscr{H}^{m+1}_\gamma([0,T]\times\mathbb{S})} \\ \leq\ & C_m\left(\frac{1}{\gamma}\|F_1\|^2_{\mathscr{H}^m_\gamma([0,T]\times\mathbb{R}^+\times\mathbb{S})} + \|G\|^2_{\mathscr{H}^m_\gamma([0,T]\times\mathbb{S})}\right). \end{aligned} \tag{4.9}$$

Proof. This follows from Theorem 12.8 in [9, p.367]. As explained in the proof of Lemma 4.1, all the requirements of this theorem are fulfilled in our situation.

A difference is that the estimate (4.9) is not the same one as listed there. By checking the proof of that theorem, estimate (4.9) actually holds (cf. Theorem 12.5 in [9, p.364]), and the one in Theorem 12.8 in [9, p.367] follows by choosing $\gamma = \frac{1}{T}$ in (4.9). □

4.5. **Problem B).** The basic result concerning Problem B) is the following L^2 well-posedness.

Lemma 4.3. *Under assumptions a), c) in Theorem 4.1 (but $z \in (-\infty, 1]$ and $y \in \mathbb{S}$, and $F, \dot{\mathbf{U}}_0$ are replaced respectively by $F_2, \dot{\mathbf{V}}_0^2$, and ignore $\dot{\chi}_0$ and G), Problem B) admits a unique solution $\dot{\mathbf{V}} \in L^2([0,T]\times(-\infty,1]\times\mathbb{S})$, which is such that $\dot{\mathbf{V}}|_{[0,T]\times\{z=1\}\times\mathbb{S}} \in L^2([0,T]\times\{z=1\}\times\mathbb{S})$. Furthermore, $\dot{\mathbf{V}}$ belongs to $\mathscr{C}([0,T];L^2((-\infty,1]\times\mathbb{S}))$ and satisfies the estimate*

$$\begin{aligned} &\left\|\mathrm{e}^{-\gamma t}\dot{\mathbf{V}}(t)\right\|^2_{\mathscr{C}([0,T];L^2((-\infty,1]\times\mathbb{S}))} + \gamma\left\|\dot{\mathbf{V}}\right\|^2_{L^2_\gamma([0,T]\times(-\infty,1]\times\mathbb{S})} + \left\|\dot{\mathbf{V}}|_{z=1}\right\|^2_{L^2_\gamma([0,T]\times\mathbb{S})} \\ \leq\ & c\left(\left\|\dot{\mathbf{V}}_0^2\right\|^2_{L^2((-\infty,1]\times\mathbb{S})} + \frac{1}{\gamma}\|F_2\|^2_{L^2_\gamma([0,T]\times(-\infty,1]\times\mathbb{S})} + \|g\|^2_{L^2_\gamma([0,T]\times\mathbb{S})}\right) \end{aligned} \tag{4.10}$$

for all $\gamma \geq \gamma_0 \geq 1$ with the constants c, γ_0 depending only on and continuously on

$$K \doteq \max\left\{\|\mathbf{U}\|_{W^{1,\infty}([0,T]\times(-\infty,1]\times\mathbb{S})}, \|\chi\|_{W^{2,\infty}([0,T]\times[0,1])}\right\}.$$

Proof. We apply Theorem 9.19 in [9, p.275], due to Métivier [10, Proposition 3.5.2 in p.67], to prove this lemma. We remark although there is no decay with respect to y-variable, but it is periodic, so integration by parts worked and these theorems still valid in this situation.

1. It is easy to see that our operator $L(\varphi_2\mathbf{U}+(1-\varphi_2)\underline{\mathbf{U}}, \chi, \mathrm{d}\chi)$ is symmetric, constantly hyperbolic, and the boundary $[0,T]\times\{z=1\}\times\mathbb{R}$ is non-characteristic, by smallness of μ. Furthermore, $\mathrm{rank}M = 5$, which is exactly the number of incoming characteristics of the domain $(-\infty,1]\times\mathbb{S}$.

2. We then need verify the uniform Kreiss-Lopatinskiĭ condition. By Proposition 4.4 in [9, p.113], it is sufficient to show the boundary condition is strictly dissipative. However, that is demonstrated in [9, p.413]. See also [11] or Lecture 4 for direct verification.

3. Although the estimate (4.10) is different from the one listed in the above cited theorem, however, by checking the proof (see the end of step 2 in [9, p.280]), it actually holds. □

The following lemma concerns further regularity of the solution provided the data are more regular, and initial data vanishes.

Lemma 4.4. *Under the assumptions c), d), e) in Theorem 4.2 (but $z \in (-\infty, 1],\ y \in \mathbb{S}$), and suppose $F_2 \in H^m([0,T]\times(-\infty,1]\times\mathbb{S})$, $g \in H^m([0,T]\times\mathbb{S})$ satisfy, for $k = 0, 1, \cdots, m-1$, that $\partial_t^k F_2 = 0, \quad \partial_t^k g = 0$ at $t = 0$, and the initial data $\dot{V}_0^2 \equiv 0$. Then the solution $\dot{\mathbf{V}}$ of Problem B) is also in H^m, as well as its trace on the boundary $[0,T] \times \{z = 1\} \times \mathbb{S}$, and satisfies $\partial_t^j \dot{\mathbf{V}} = 0$ for $j = 0, 1, \cdots, m-1$ at $t = 0$, as well as the estimates*

$$\begin{aligned} & \gamma \left\| \dot{\mathbf{V}} \right\|^2_{\mathscr{H}_\gamma^m([0,T]\times(-\infty,1]\times\mathbb{S})} + \left\| \dot{\mathbf{V}}|_{z=1} \right\|^2_{\mathscr{H}_\gamma^m([0,T]\times\mathbb{S})} \\ \leq\ & C_m \left(\frac{1}{\gamma} \left\| F_2 \right\|^2_{\mathscr{H}_\gamma^m([0,T]\times(-\infty,1]\times\mathbb{S})} + \left\| g \right\|^2_{\mathscr{H}_\gamma^m([0,T]\times\mathbb{S})} \right), \end{aligned} \tag{4.11}$$

for all $\gamma \geq \gamma_0 > 1$, with positive constants C_m and γ_0 depending only on K and m.

Proof. This is an application of Theorem 9.21 in [9, p.282] to our situation (as shown in the proof of Lemma 4.3). The estimate follows from (9.2.58) in [9, p.283]. □

4.6. **Proof of Propositions 4.1 and 4.2.** We first derive the estimate satisfied by any solution $(\dot{\mathbf{U}}, \dot{\psi}) \in L^2([0,T] \times [0,1] \times \mathbb{S}) \times H^1([0,T] \times \mathbb{S})$ to Problem (4.5). Set $\dot{\mathbf{V}}_j = \psi_j \dot{\mathbf{U}}$ ($j = 1, 2$). Then $\dot{\mathbf{U}} = \dot{\mathbf{V}}_1 + \dot{\mathbf{V}}_2$. By multiplying ψ_j to the equations in (4.5), we get that $\dot{\mathbf{V}}_{1,2}$ satisfies (recall our special choice on support of $\varphi_{1,2}$), respectively, Problem A) and B), with $F_j = \psi_j F + (L(\mathbf{U}, \chi, \mathrm{d}\chi)\psi_j)\dot{\mathbf{U}}$ and $\dot{\mathbf{V}}_0^j = \psi_j \dot{\mathbf{U}}_0$. It is obvious that $\left\| \dot{\mathbf{V}}_0^j \right\|_{L^2([0,1]\times\mathbb{S})} \leq \left\| \dot{\mathbf{U}}_0 \right\|_{L^2([0,1]\times\mathbb{S})}$, and we also have

$$\begin{aligned} & \left\| F_j \right\|^2_{L^2_\gamma([0,T]\times[0,1]\times\mathbb{S})} = \left\| e^{-\gamma t} F_j \right\|^2_{L^2([0,T]\times[0,1]\times\mathbb{S})} \\ \leq\ & 2 \left\| e^{-\gamma t} F \right\|^2_{L^2([0,T]\times[0,1]\times\mathbb{S})} + 2 \left\| e^{-\gamma t} \mathbb{A}^1(\mathbf{U}, \chi, \mathrm{d}\chi) \psi_j'(z) \dot{\mathbf{U}} \right\|^2_{L^2([0,T]\times[0,1]\times\mathbb{S})} \\ \leq\ & c \left(\left\| F \right\|^2_{L^2_\gamma([0,T]\times[0,1]\times\mathbb{S})} + \left\| \dot{\mathbf{U}} \right\|^2_{L^2_\gamma([0,T]\times[0,1]\times\mathbb{S})} \right). \end{aligned}$$

Then applying estimates in Lemmas 4.1, 4.3 for $\dot{\mathbf{V}}_{1,2}$ respectively, and adding together, we find that, there are constants c and γ_0, depending only on $W^{1,\infty}$ norm of $\mathbf{U}$ and $W^{2,\infty}$

norm of χ, so that for all $\gamma \geq \gamma_0$,

$$\begin{aligned}
&\left\|\mathrm{e}^{-\gamma t}\dot{\mathbf{U}}(t)\right\|^2_{\mathscr{C}([0,T];L^2([0,1]\times\mathbb{S}))} + \gamma\left\|\dot{\mathbf{U}}\right\|^2_{L^2_\gamma([0,T]\times[0,1]\times\mathbb{S})} + \left\|\dot{\mathbf{U}}|_{z=0,1}\right\|^2_{L^2_\gamma([0,T]\times\mathbb{S})} \\
&+ \|\dot{\chi}\|^2_{\mathscr{H}^1_\gamma([0,T]\times\mathbb{S})} \\
\leq\ & 2\left(\sum_{j=1}^{2}(\left\|\mathrm{e}^{-\gamma t}\dot{\mathbf{V}}_j(t)\right\|^2_{\mathscr{C}([0,T];L^2([0,1]\times\mathbb{S}))} + \gamma\left\|\dot{\mathbf{V}}_j\right\|^2_{L^2_\gamma([0,T]\times[0,1]\times\mathbb{S})})\right. \\
&\left. + \left\|\dot{\mathbf{V}}_1|_{z=0}\right\|^2_{L^2_\gamma([0,T]\times\mathbb{S})} + \left\|\dot{\mathbf{V}}_2|_{z=1}\right\|^2_{L^2_\gamma([0,T]\times\mathbb{S})} + \|\dot{\chi}\|^2_{\mathscr{H}^1_\gamma([0,T]\times\mathbb{S})}\right) \\
\leq\ & c\left(\sum_{j=1}^{2}(\frac{1}{\gamma}\|F_j\|^2_{L^2_\gamma([0,T]\times[0,1]\times\mathbb{S})} + \left\|\dot{\mathbf{V}}^j_0\right\|^2_{L^2([0,1]\times\mathbb{S})}) + \|G\|^2_{L^2_\gamma([0,T]\times\mathbb{S})}\right. \\
&\left. + \|\dot{\chi}_0\|^2_{H^{\frac{1}{2}}(\mathbb{S})} + \|g\|^2_{L^2_\gamma([0,T]\times\mathbb{S})}\right) \\
\leq\ & c\left(\frac{1}{\gamma}\|F\|^2_{L^2_\gamma([0,T]\times[0,1]\times\mathbb{S})} + \frac{1}{\gamma}\left\|\dot{\mathbf{U}}\right\|^2_{L^2_\gamma([0,T]\times[0,1]\times\mathbb{S})} + \left\|\dot{\mathbf{U}}_0\right\|^2_{L^2([0,1]\times\mathbb{S})}\right. \\
&\qquad \left. + \|G\|^2_{L^2_\gamma([0,T]\times\mathbb{S})} + \|\dot{\chi}_0\|^2_{H^{\frac{1}{2}}(\mathbb{S})} + \|g\|^2_{L^2_\gamma([0,T]\times\mathbb{S})}\right).
\end{aligned}$$

Taking $\gamma_0 \geq \sqrt{2c}$ further larger, we conclude that

$$\begin{aligned}
&\left\|\mathrm{e}^{-\gamma t}\dot{\mathbf{U}}(t)\right\|^2_{\mathscr{C}([0,T];L^2([0,1]\times\mathbb{S}))} + \gamma\left\|\dot{\mathbf{U}}\right\|^2_{L^2_\gamma([0,T]\times[0,1]\times\mathbb{S})} + \left\|\dot{\mathbf{U}}|_{z=0,1}\right\|^2_{L^2_\gamma([0,T]\times\mathbb{S})} \\
&+ \|\dot{\chi}\|^2_{\mathscr{H}^1_\gamma([0,T]\times\mathbb{S})} \\
\leq\ & c\left(\frac{1}{\gamma}\|F\|^2_{L^2_\gamma([0,T]\times[0,1]\times\mathbb{S})} + \left\|\dot{\mathbf{U}}_0\right\|^2_{L^2([0,1]\times\mathbb{S})} + \|G\|^2_{L^2_\gamma([0,T]\times\mathbb{S})}\right. \\
&\left. + \|\dot{\chi}_0\|^2_{H^{\frac{1}{2}}(\mathbb{S})} + \|g\|^2_{L^2_\gamma([0,T]\times\mathbb{S})}\right).
\end{aligned}$$

Finally, for $T_1 = 1/\gamma_0 > 0$ and $T < T_1$, we choose $\gamma = 1/T$ to drop the weight. Then we have

$$\begin{aligned}
&\mathrm{e}^{-2\gamma T}\left(\left\|\dot{\mathbf{U}}(t)\right\|^2_{\mathscr{C}([0,T];L^2([0,1]\times\mathbb{S}))} + \frac{1}{T}\left\|\dot{\mathbf{U}}\right\|^2_{L^2([0,T]\times[0,1]\times\mathbb{S})}\right. \\
&\qquad\qquad \left. + \left\|\dot{\mathbf{U}}|_{z=0,1}\right\|^2_{L^2([0,T]\times\mathbb{S})} + \|\dot{\chi}\|^2_{\mathscr{H}^1([0,T]\times\mathbb{S})}\right) \\
\leq\ & c\left(T\|F\|^2_{L^2([0,T]\times[0,1]\times\mathbb{S})} + \left\|\dot{\mathbf{U}}_0\right\|^2_{L^2([0,1]\times\mathbb{S})} + \|G\|^2_{L^2([0,T]\times\mathbb{S})}\right. \\
&\qquad \left. + \|\dot{\chi}_0\|^2_{H^{\frac{1}{2}}(\mathbb{S})} + \|g\|^2_{L^2([0,T]\times\mathbb{S})}\right).
\end{aligned}$$

This gives the desired L^2 estimate, and in particular implies uniqueness of the solution (for any $T > 0$, using it repeatedly).

Now we show existence part of Proposition 4.1. We solve Problems A) and B) in L^2 with $F_j = \psi_j(z)F$ and $\dot{\mathbf{V}}_0^j = \psi_j(z)\dot{\mathbf{U}}_0$ $(j=1,2)$. By Lemmas 4.1 and 4.3, there are L^2 solutions. Denote the obtained solution to be $(\dot{\mathbf{V}}_1, \dot{\chi})$, $\dot{\mathbf{V}}_2$ respectively. Note both nonhomogeneous terms $F_{1,2}$ and initial datum $\dot{\mathbf{V}}_{1,2}$ have compact support on $\{z \in [0, \frac{3}{4}], y \in \mathbb{S}\}$ (resp. $\{z \in [\frac{1}{2}, 1],\ y \in \mathbb{S}\}$), by finite speed of propagation for hyperbolic operators [9, p.73 and p.78], we conclude that there is a $T' > 0$ (depending only on the maximal characteristic speed, or, $\|\mathbf{U} - \underline{\mathbf{U}}\|_{L^\infty}$ and $\|\chi\|_{W^{1,\infty}}$), so that $\dot{\mathbf{V}}_1 = 0$ for $z \geq \frac{7}{8}$ and $\dot{\mathbf{V}}_2 = 0$ for $z \leq \frac{1}{8}$, as long as $0 \leq t \leq T'$. By our choice of $\varphi_{1,2}$, this means there hold

$$
\begin{cases}
L(\mathbf{U}, \chi, \mathrm{d}\chi)\dot{\mathbf{V}}_1 = \psi_1 F, & t \in [0, T'],\ z > 0,\ y \in \mathbb{S}, \\
\dot{\mathbf{V}}_1 = \psi_1 \dot{\mathbf{U}}_0, & t = 0,\ z > 0,\ y \in \mathbb{S}, \\
\mathbb{J}\mathrm{d}\dot{\chi} + \nabla Q(\mathbf{U})\dot{\mathbf{V}}_1 = G, & t \in [0, T'],\ z = 0,\ y \in \mathbb{S}, \\
\dot{\chi} = \dot{\chi}_0, & t \in [0, T'],\ y \in \mathbb{S}, \\
M\dot{\mathbf{V}}_1 = 0, & t \in [0, T'],\ z = 1,\ y \in \mathbb{S}
\end{cases}
$$

and

$$
\begin{cases}
L(\mathbf{U}, \chi, \mathrm{d}\chi)\dot{\mathbf{V}}_2 = \psi_2 F, & t \in [0, T'],\ z < 1,\ y \in \mathbb{S}, \\
\dot{\mathbf{V}}_2 = \psi_2 \dot{\mathbf{U}}_0, & t = 0,\ z < 1,\ y \in \mathbb{S}, \\
\nabla Q(\mathbf{U})\dot{\mathbf{V}}_2 = 0, & t \in [0, T'],\ z = 0,\ y \in \mathbb{S}, \\
M\dot{\mathbf{V}}_2 = g, & t \in [0, T],\ z = 1,\ y \in \mathbb{S}.
\end{cases}
$$

Since these are linear problems, obviously $\dot{\mathbf{U}} = \dot{\mathbf{V}}_1 + \dot{\mathbf{V}}_2$ and $\dot{\chi}$ solve Problem (4.5), for the time interval $[0, T']$.

Since T' depends only on $\|\mathbf{U} - \underline{\mathbf{U}}\|_{L^\infty}$ and $\|\chi\|_{W^{1,\infty}}$, that is, μ, but not on initial or boundary data, the existence of a L^2 solution for $t \in [0, T]$ can be obtained by a simple continuation method. Indeed, for $t = T'$, we use $\dot{\mathbf{U}}(T')$ and $\dot{\chi}(T')$ as initial data and solve the corresponding problem (4.5). We then extend the solution to $[T', 2T']$. Similarly we can extend it to $[3T', 4T']$. etc. Then by finite many steps, we get a solution $\dot{\mathbf{U}}$ in $L^2([0,T] \times [0,1] \times \mathbb{S})$ and $\dot{\chi} \in H^1([0,T] \times \mathbb{S})$. This finishes proof of Proposition 4.1.

Now we prove Proposition 4.2. We still use the decomposition $\dot{\mathbf{U}} = \dot{\mathbf{V}}_1 + \dot{\mathbf{V}}_2$. with $\dot{\mathbf{V}}_j = \psi_j \dot{\mathbf{U}}$ $(j = 1, 2)$. We see $(\dot{\mathbf{V}}_1, \dot{\chi})$, $\dot{\mathbf{V}}_2$ satisfies Problem A) and B) respectively, but with zero initial data, and $F_j = \psi_j F + (L(\mathbf{U}, \chi, \mathrm{d}\chi)\psi_j)\dot{\mathbf{U}}$. So by Lemmas 4.2 and 4.4, $\dot{\mathbf{V}}_{1,2}$ are in H^m, and $\dot{\chi}$ belongs to H^{m+1}, so we get $\dot{\mathbf{U}} \in H^m$.

Now we derive the estimate. By definition of weighted norms, it holds

$$
\|F_j\|^2_{\mathscr{H}^m_\gamma([0,T]\times[0,1]\times\mathbb{S})} \leq c\,\|F\|^2_{\mathscr{H}^m_\gamma([0,T]\times[0,1]\times\mathbb{S})} + c\left\|\dot{\mathbf{U}}\right\|^2_{\mathscr{H}^m_\gamma([0,T]\times[0,1]\times\mathbb{S})},
$$

with c depending on $\|\mathbf{U}\|_{H^m}$ and $\|\chi\|_{H^{m+1}}$. It is here we need a fact that $\|au\|_{\mathscr{H}^q_\gamma} \leq C\|a\|_{H^r}\|u\|_{\mathscr{H}^s_\gamma}$ provided that $a \in H^r$, $u \in \mathscr{H}^s_\gamma$ and $r+s>0, q \leq \min(r,s)$, and $q < r+s-d/2$, with d the dimension of the space-time where a, u are defined, see [9, Lemma 9.3 in p.251]. So totally similar as before, we get

$$\begin{aligned} & \gamma\left\|\dot{\mathbf{U}}\right\|^2_{\mathscr{H}^m_\gamma([0,T]\times[0,1]\times\mathbb{S})} + \left\|\dot{\mathbf{U}}|_{z=0,1}\right\|^2_{\mathscr{H}^m_\gamma([0,T]\times\mathbb{S})} + \|\dot{\chi}\|^2_{\mathscr{H}^{m+1}_\gamma([0,T]\times\mathbb{S})} \\ \leq\ & c\left(\frac{1}{\gamma}\|F\|^2_{\mathscr{H}^m_\gamma([0,T]\times[0,1]\times\mathbb{S})} + \frac{1}{\gamma}\left\|\dot{\mathbf{U}}\right\|^2_{\mathscr{H}^m_\gamma([0,T]\times[0,1]\times\mathbb{S})}\right. \\ & \left. + \|G\|^2_{\mathscr{H}^m_\gamma([0,T]\times\mathbb{S})} + \|g\|^2_{\mathscr{H}^m_\gamma([0,T]\times\mathbb{S})}\right) \end{aligned}$$

for all $\gamma \geq \gamma_0$, with c and γ_0 depending only on K and m. Now choosing $\gamma_0 \geq \sqrt{2c}$, it follows that

$$\begin{aligned} & \gamma\left\|\dot{\mathbf{U}}\right\|^2_{\mathscr{H}^m_\gamma([0,T]\times[0,1]\times\mathbb{S})} + \left\|\dot{\mathbf{U}}|_{z=0,1}\right\|^2_{\mathscr{H}^m_\gamma([0,T]\times\mathbb{S})} + \|\dot{\chi}\|^2_{\mathscr{H}^{m+1}_\gamma([0,T]\times\mathbb{S})} \\ \leq\ & c\left(\frac{1}{\gamma}\|F\|^2_{\mathscr{H}^m_\gamma([0,T]\times[0,1]\times\mathbb{S})} + \|G\|^2_{\mathscr{H}^m_\gamma([0,T]\times\mathbb{S})} + \|g\|^2_{\mathscr{H}^m_\gamma([0,T]\times\mathbb{S})}\right). \end{aligned}$$

The final step is for $T_1 = 1/\gamma_0$ and $T \leq T_1$, taking $\gamma = \frac{1}{T}$ to drop the weights in the above inequality. Since $\gamma \geq 1$, by the definition of $\mathscr{H}^m_\gamma$ norm, the left-hand side obviously controls

$$\frac{1}{T}\left\|\dot{\mathbf{U}}\right\|^2_{H^m([0,T]\times[0,1]\times\mathbb{S})} + \left\|\dot{\mathbf{U}}|_{z=0,1}\right\|^2_{H^m([0,T]\times\mathbb{S})} + \|\dot{\chi}\|^2_{H^{m+1}([0,T]\times\mathbb{S})}.$$

However, to bound the right-hand side, there is a trick.

Suppose $w \in H^1([0,T];L^2(\Omega))$ and $w|_{t=0} = 0$. Then integration by parts and Cauchy-Schwarz inequality imply

$$\begin{aligned} & \gamma\int_0^T\int_\Omega |\mathrm{e}^{-\gamma t}w(t,x)|^2\,\mathrm{d}x\mathrm{d}t \\ =\ & -\frac{1}{2}\left(\mathrm{e}^{-2\gamma t}\int_\Omega |w(t,x)|^2\,\mathrm{d}x\Big|_0^T - 2\int_0^T\int_\Omega |\mathrm{e}^{-\gamma t}w(t,x)||\mathrm{e}^{-\gamma t}\partial_t w(t,x)|\,\mathrm{d}x\,\mathrm{d}t\right) \\ \leq\ & \int_0^T\int_\Omega |\sqrt{\gamma}\mathrm{e}^{-\gamma t}w(t,x)||\frac{1}{\sqrt{\gamma}}\mathrm{e}^{-\gamma t}\partial_t w(t,x)|\,\mathrm{d}x\,\mathrm{d}t \\ \leq\ & \left(\gamma\int_0^T\int_\Omega |\mathrm{e}^{-\gamma t}w(t,x)|^2\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}\left(\frac{1}{\gamma}\int_0^T\int_\Omega |\mathrm{e}^{-\gamma t}\partial_t w(t,x)|^2\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}. \end{aligned}$$

Therefore we proved $\boxed{\left\|\mathrm{e}^{-\gamma t}w\right\|_{L^2([0,T]\times\Omega)} \leq \frac{1}{\gamma}\left\|\mathrm{e}^{-\gamma t}\partial_t w\right\|_{L^2([0,T]\times\Omega)}.}$ Furthermore, suppose $w \in H^k([0,T];L^2(\Omega))$ so that $\partial_t^j w|_{t=0} = 0$ for $j = 0, 1, \cdots, k-1$, then using the above inequality repeatedly, we get

$$\boxed{\left\|\mathrm{e}^{-\gamma t}w\right\|_{L^2([0,T]\times\Omega)} \leq \frac{1}{\gamma^k}\left\|\mathrm{e}^{-\gamma t}\partial_t^k w\right\|_{L^2([0,T]\times\Omega)}.}$$

For the term

$$\|F\|^2_{\mathscr{H}^m_\gamma([0,T]\times[0,1]\times\mathbb{S})} = \sum_{|\alpha|\leq m}\gamma^{2(|m|-|\alpha|)}\left\|\mathrm{e}^{-\gamma t}\partial^\alpha F\right\|^2_{L^2([0,T]\times[0,1]\times\mathbb{S})},$$

set $w=\partial^\alpha F$. By our assumption on F, w satisfies the above requirements for $k=m-|\alpha|$. Therefore

$$\begin{aligned}\|F\|^2_{\mathscr{H}^m_\gamma([0,T]\times[0,1]\times\mathbb{S})} &\leq C_m\left\|\mathrm{e}^{-\gamma t}D^m F\right\|^2_{L^2([0,T]\times[0,1]\times\mathbb{S})}\\ &\leq C_m\left\|D^m F\right\|^2_{L^2([0,T]\times[0,1]\times\mathbb{S})}\\ &\leq \|F\|^2_{H^m([0,T]\times[0,1]\times\mathbb{S})}.\end{aligned}$$

Here D^mF represents all m-th order partial derivatives of F. Similar inequalities hold for g and G. So finally, recall $\gamma=\frac{1}{T}$, we get the desired estimate.

4.7. **Proof of Theorems 4.1 and 4.2.** We now prove Theorem 4.1. By the assumption of proper symmetry, we can extend Problem (4.2) to formulate Problem (4.5). The latter has a solution $(\dot{\mathbf{U}},\dot{\chi})$ by Proposition 4.1. We now show that $M'\mathbf{U}=0$ on $y=0,1$. Let $\bar{\dot{\mathbf{U}}}(t,z,y)=\mathbb{E}\dot{\mathbf{U}}(t,z,-y)$, with $\mathbb{E}=\operatorname{diag}(1,1,-1,1,1,1,-1,1)$ and $\bar{\dot{\chi}}(t,y)=\dot{\chi}(t,-y)$. Then $\mathrm{d}\bar{\dot{\chi}}(t,y)=\operatorname{diag}(1,-1)\mathrm{d}\dot{\chi}(t,-y)$, and

$$\begin{aligned}&\Big(L(\mathbf{U},\chi,\mathrm{d}\chi)\bar{\dot{\mathbf{U}}}\Big)|_{(t,z,y)}\\ =\;&\mathbb{A}^0(\mathbf{U})|_{t,z,y}\mathbb{E}(\partial_t\dot{\mathbf{U}})|_{(t,z,-y)}+\mathbb{A}^1(\mathbf{U},\chi,\mathrm{d}\chi)|_{(t,z,y)}\mathbb{E}(\partial_z\dot{\mathbf{U}})|_{(t,z,-y)}\\ &\quad-\mathbb{A}^2(\mathbf{U})|_{(t,z,y)}\mathbb{E}(\partial_y\dot{\mathbf{U}})|_{(t,z,-y)}\\ =\;&\Big(\mathbb{A}^0(\mathbf{U})\mathbb{E}(\partial_t\dot{\mathbf{U}})+\mathbb{E}\mathbb{A}^1(\mathbf{U},\chi,\mathrm{d}\chi)(\partial_z\dot{\mathbf{U}})-(\mathbb{E}\mathbb{A}^2(\mathbf{U})(-I_8\mathbb{E}))\mathbb{E}(\partial_y\dot{\mathbf{U}})\Big)|_{(t,z,-y)}\\ =\;&\mathbb{E}L(\mathbf{U},\chi,\mathrm{d}\chi)\dot{\mathbf{U}}|_{(t,z,-y)}=\mathbb{E}F(t,z,-y)=F(t,z,y).\end{aligned}$$

In the last equality, we used the proper symmetry of F, and for the third equality, by proper symmetry of $\mathbf{U},\chi$ (note especially $\partial_y\chi$ is odd symmetric), we used that

$$\begin{aligned}&\mathbb{A}^1(\mathbf{U},\chi,\mathrm{d}\chi)(t,z,-y)\mathbb{E}\\ =\;&\begin{pmatrix}\frac{1}{-1+\phi'\chi}I_4 & 0\\ 0 & \frac{1}{1+\phi'\chi}I_4\end{pmatrix}\\ &\quad\times\left(\begin{pmatrix}A_1(U_-) & \\ & A_1(U_+)\end{pmatrix}\mathbb{E}-\phi\partial_t\chi\mathbb{A}^0(\mathbf{U})\mathbb{E}-\phi\partial_y\chi\mathbb{E}\mathbb{A}^2(\mathbf{U})\mathbb{E}^2\right)|_{(t,z,y)}\\ =\;&\mathbb{E}\mathbb{A}^1(\mathbf{U},\chi,\mathrm{d}\chi)(t,z,y),\end{aligned}$$

because $\mathbb{E}$ commutes with $\mathbb{A}^0(\mathbf{U})$ and $\mathrm{diag}(A_1(U_-), A_1(U_+))$.

Also, for the boundary condition on Σ_0, we have

$$\begin{aligned}
\mathbb{J}\mathrm{d}\bar{\dot{\chi}} + \nabla Q(\mathbf{U})\bar{\dot{\mathbf{U}}}|_{(t,z=0,y)} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}^{\top} \mathrm{d}\dot{\chi}(t,-y) + \nabla Q(\mathbf{U})\mathbb{E}\dot{\mathbf{U}}(t,0,-y) \\
&= \mathbb{E}(\mathbb{J}\mathrm{d}\dot{\chi} + \nabla Q(\mathbf{U})\dot{\mathbf{U}})|_{(t,0,-y)} = \mathbb{E}G(t,-y) \\
&= \mathbb{E}^2 G(t,y) = G(t,y).
\end{aligned}$$

Similarly we can check that $\bar{\dot{\mathbf{U}}}$ actually solves Problem (4.5). By uniqueness claimed in Proposition 4.1, one gets $\bar{\dot{\mathbf{U}}} = \dot{\mathbf{U}}$ and $\bar{\dot{\chi}} = \dot{\chi}$. Therefore the solution $(\dot{\mathbf{U}}, \dot{\chi})$ itself is also properly symmetric. This implies particularly that $v_\pm(t,z,y) = -v_\pm(t,z,-y)$. So $v_\pm = 0$ on $y = 0$. Since the solution $\dot{\mathbf{U}}$ is also periodic with respect to y with period 2, so $v_\pm(t,z,1) = v_\pm(t,z,-1) = -v_\pm(t,z,1)$, therefore $v_\pm = 0$ on $y = 1$.

Therefore, by restriction of the solution to $[0,T]\times[0,1]\times[0,1]$, we may get a solution $(\dot{\mathbf{U}}, \dot{\chi})$ to Problem (4.2) claimed by Theorem 4.1. The corresponding estimate follows directly from that of Proposition 4.2. The proof of Theorem 4.2 is similar by using Proposition 4.2.

5. Solution of nonlinear problem

This section is devoted to proving Theorem 2.1 by using linear theory and Banach fixed-point theorem. Consider the following linear problem:

$$\begin{cases}
L(\mathbf{U}^a + \dot{\mathbf{U}}, \chi^a + \dot{\chi}, \mathrm{d}(\chi^a + \dot{\chi}))\dot{\mathbf{V}} = F \\
\qquad \doteq -L(\mathbf{U}^a + \dot{\mathbf{U}}, \chi^a + \dot{\chi}, \mathrm{d}(\chi^a + \dot{\chi}))\mathbf{U}^a, & \text{in } D^T, \\
\dot{\mathbf{V}} = 0, & \text{on } \{0\}\times D, \\
\mathbb{J}\mathrm{d}\dot{\psi} + \nabla Q((\mathbf{U}^a + \dot{\mathbf{U}})|_{z=0})\dot{\mathbf{V}}_{|z=0} = G \\
\qquad \doteq -\mathbb{J}\mathrm{d}\chi^a + \nabla Q((\mathbf{U}^a + \dot{\mathbf{U}})|_{z=0})\dot{\mathbf{U}}|_{z=0} \\
\qquad -Q((\mathbf{U}^a + \dot{\mathbf{U}})|_{z=0}), & \text{on } \Sigma_0^T, \\
\dot{\psi} = 0, & \text{at } \{0\}\times[0,1], \\
M\dot{\mathbf{V}} = g \doteq -M\mathbf{U}^a + \tilde{\mathbf{g}}, & \text{on } \Sigma_1^T, \\
M'\dot{\mathbf{V}} = 0, & \text{on } \Gamma_{0,1}^T.
\end{cases} \tag{5.1}$$

Here $(\mathbf{U}^a, \chi^a)$ is the approximate solution constructed in Lemma 3.6. We show below (5.1) defines a mapping $\mathfrak{N} : (\dot{\mathbf{U}}, \dot{\chi}) \mapsto (\dot{\mathbf{V}}, \dot{\psi})$. It is clear that if $(\dot{\mathbf{U}}, \dot{\chi})$ is a fixed-point of $\mathfrak{N}$, then $(\dot{\mathbf{U}} + \mathbf{U}^a, \dot{\chi} + \chi^a)$ solves problem (N), due to the properties of the approximate solution $(\mathbf{U}^a, \chi^a)$.

Let $m \geq 3$ be a fixed integer. We define $\mathcal{S}_{T,M}$ being the set of pair $(\dot{\mathbf{U}}, \dot{\chi}) \in H^m(D^T; \mathbb{R}^8) \times H^{m+1}(I_T)$, where T and M are parameters to be fixed later, such that the following three hold:

1) $\left\|\dot{\mathbf{U}}\right\|_{H^m(D^T)} + \left\|\dot{\mathbf{U}}|_{z=0}\right\|_{H^m(I^T)} + \|\dot{\chi}\|_{H^{m+1}(I^T)} \leq M$;
2) $\dot{\mathbf{U}}$ and $\dot{\chi}$ are properly symmetric with order m;
3) there hold $\partial_t^j \dot{\mathbf{U}}|_{t=0} = 0$ for $j = 0, 1, \cdots, m-1$ and $\partial_t^j \dot{\chi}|_{t=0} = 0$ for $j = 0, 1, \cdots, m$.

We note $\mathcal{S}_{T,M}$ is nonempty since $(0,0)$ lies in it for any $T > 0, M > 0$. In the following, we will show that

a): $\mathfrak{N}$ is a mapping on $\mathcal{S}_{T,M}$;

b): it contracts under a $L^2(D^T) \times H^1(I^T)$ topology, by choosing carefully T and M.

5.1. **Well-definition of $\mathfrak{N}$ on $\mathcal{S}_{T,M}$.** Suppose $(\dot{\mathbf{U}}, \dot{\chi}) \in \mathcal{S}_{T,M}$, we show $(\dot{\mathbf{V}}, \dot{\psi}) \in \mathcal{S}_{T,M}$ for suitably chosen T and M by applying Theorem 4.1 and Theorem 4.2 to (5.1). The following first three subsections are devoted to verifying the assumptions in these theorems.

5.1.1. *H^m estimates of nonlinear terms.* We first show $F \in H^m(D^T)$ and $G \in H^m(I^T)$. In the following, we always use C to denote constants that are independent of T, M. To estimate F, we first recall that $\mathbf{U}^a \in H^{m+1}(D^T)$, $\chi \in H^{m+1}(I^T)$, and $f_0 \doteq -L(\mathbf{U}^a, \chi^a, \mathrm{d}\chi^a)\mathbf{U}^a$. Therefore, by using Sobolev embedding $H^m(D^T) \hookrightarrow \mathscr{C}^1(D^T)$, and Propositions 3.1 and 3.3, it follows

$$
\begin{aligned}
\|F\|_{H^m(D^T)} \leq\ & \left\|L(\mathbf{U}^a + \dot{\mathbf{U}}, \chi^a + \dot{\chi}, \mathrm{d}(\chi^a + \dot{\chi}))\mathbf{U}^a - f_0\right\|_{H^m(D^T)} + \|f_0\|_{H^m(D^T)} \\
\leq\ & \Big(\left\|\mathbb{A}^0(\mathbf{U}^a + \dot{\mathbf{U}}) - \mathbb{A}^0(\mathbf{U}^a)\right\|_{H^m(D^T)} + \left\|\mathbb{A}^2(\mathbf{U}^a + \dot{\mathbf{U}}) - \mathbb{A}^2(\mathbf{U}^a)\right\|_{H^m(D^T)} \\
& \left\|\mathbb{A}^1(\mathbf{U}^a + \dot{\mathbf{U}}, \chi^a + \dot{\chi}, \mathrm{d}(\chi^a + \dot{\chi})) - \mathbb{A}^1(\mathbf{U}^a, \chi^a, \mathrm{d}\chi^a)\right\|_{H^m(D^T)} \Big) \\
& \times \|\mathbf{U}^a\|_{H^{m+1}(D^T)} + \|f_0\|_{H^m(D^T)} \\
\leq\ & C_1(\left\|\dot{\mathbf{U}}\right\|_{L^\infty}, \|\dot{\chi}\|_{W^{1,\infty}}) \left(\left\|\dot{\mathbf{U}}\right\|_{H^m(D^T)} + \|\dot{\chi}\|_{H^{m+1}(I^T)} \right) + \|f_0\|_{H^m(D^T)} \\
\leq\ & C_1(M)M + \|f_0\|_{H^m(D^T)}\,.
\end{aligned} \tag{5.2}
$$

In the second last inequality we have assimilated the number $\|\mathbf{U}^a\|_{H^{m+1}(D^T)}$ into the nondecreasing function $C_1(\cdot)$, and in the last one we used Sobolev embedding theorem and nondecreasing of $C_1(\cdot)$.

Similarly, recalling $h_0 \doteq -\mathbb{J}\mathrm{d}\chi^a - Q(\mathbf{U}^a|_{z=0})$, we have

$$\begin{aligned}
\|G\|_{H^m(I^T)} &\leq \left\|\nabla Q((\mathbf{U}^a + \dot{\mathbf{U}})|_{z=0})\dot{\mathbf{U}}|_{z=0} - Q((\mathbf{U}^a + \dot{\mathbf{U}})|_{z=0}) + Q(\mathbf{U}^a|_{z=0})\right\|_{H^m(I^T)} \\
&\quad + \|h_0\|_{H^m(I_T)} \\
&= \left\|\int_0^1 \theta(\nabla^2 Q((\mathbf{U}^a + \theta\dot{\mathbf{U}})|_{z=0})\dot{\mathbf{U}}|_{z=0}, \dot{\mathbf{U}}|_{z=0})\,\mathrm{d}\theta\right\|_{H^m(I^T)} + \|h_0\|_{H^m(I^T)} \\
&\leq C\left\|\nabla^2 Q(\mathbf{U}^a + \dot{\mathbf{U}}|_{z=0})\right\|_{H^m(I^T)} \left\|\dot{\mathbf{U}}|_{z=0}\right\|_{L^\infty(I^T)} \left\|\dot{\mathbf{U}}|_{z=0}\right\|_{H^m(I^T)} \\
&\quad + \|h_0\|_{H^m(I^T)} \\
&\leq C_2(M)M^2 + \|h_0\|_{H^m(I^T)}. \qquad (5.3)
\end{aligned}$$

Here $C_2(M)$ is again a positive non-decreasing function.

5.1.2. *Proper symmetry.* By construction of $(\mathbf{U}^a, \chi^a)$, it is straightforward to see that $\mathbf{U}^a + \dot{\mathbf{U}}$ and $\chi^a + \dot{\chi}$ are properly symmetric with $s = m$. The proofs of proper symmetry of F and G are similar to those performed in the proof of Theorem 4.1 and Theorem 4.2. For example, remember now

$$\mathbf{U}^a(t, z, y) = \bar{\mathbf{U}}^a(t, z, y) = \mathbb{E}\mathbf{U}^a(t, z, -y),$$

so as calculated before,

$$\begin{aligned}
F(t, z, y) &= L(\mathbf{U}^a + \dot{\mathbf{U}}, \chi^a + \dot{\chi}, \mathrm{d}(\chi^a + \dot{\chi}))\mathbf{U}^a|_{(t,z,y)} \\
&= \mathbb{E}L(\mathbf{U}^a + \dot{\mathbf{U}}, \chi^a + \dot{\chi}, \mathrm{d}(\chi^a + \dot{\chi}))\mathbf{U}^a|_{(t,z,-y)} \\
&= \mathbb{E}F(t, z, -y).
\end{aligned}$$

This plus the fact that F is periodic in y-variable with period 2 show proper symmetry of F. So b) in Theorem 4.2 is true.

5.1.3. *Extension to $t < 0$.* We already know that $\mathbf{U}^a$ is defined for all $t \in \mathbb{R}$. Also, by property 3) on $\mathcal{S}_{T,M}$, we may extend $\dot{\mathbf{U}}, \dot{\chi}$ to be 0 for $t < 0$ and the extended functions belong to $H^m((-\infty, T] \times D)$ and $H^{m+1}((-\infty, T] \times I)$ respectively. By property of approximate solutions, assumptions c) and d) in Theorem 4.2 hold.

Also, with such an extension of $\dot{\mathbf{U}}$ and $\dot{\chi}$, by v) in Lemma 3.6, we can extended F, G, g to be zero for $t < 0$ and the extended functions are still in $H^m((-\infty, T] \times D)$ and $H^m((-\infty, T] \times I)$ respectively. This verifies f) and g) in Theorem 4.2.

Next consider assumption h) in Theorem 4.2. Since the initial data of $\dot{\mathbf{V}}, \dot{\psi}$ at $t = 0$ vanishes, we need only check that $\partial_t^j \dot{\mathbf{V}}|_{t=0} = 0$ for $j = 1, \cdots, m-1$ and $\partial_t^j \dot{\psi}|_{t=0} = 0$ for $j = 1, \cdots, m$. The procedure is similar to that of deriving compatibility conditions, by using the fact that $\dot{\mathbf{U}}|_{t\leq 0} = 0$ and $\dot{\chi}|_{t\leq 0} = 0$, and v) in Lemma 3.6 (note $F|_{t=0} = f_0$ by property 3) of $\mathcal{S}_{T,M}$). For example, from the equation, for $t=0$, we get $\mathbb{A}^0(\mathbf{U}^a)\partial_t\dot{\mathbf{V}} = f_0 = 0$, so $\partial_t\dot{\mathbf{V}}|_{t=0}=0$. Then acting ∂_t on the equation and taking $t=0$, by the fact that $\partial_t f_0|_{t=0}$ and $\partial_t\dot{\mathbf{V}}|_{t=0}=0$ we have proved, one gets $\partial_t^2\dot{\mathbf{V}}|_{t=0}$, etc. We note actually this verifies the solution $(\dot{\mathbf{V}},\dot{\psi})$, once it exists, must satisfy property 3) in definition $\mathcal{S}_{T,M}$.

Finally we demonstrate assumption e) in Theorem 4.2. For a fixed M_0 so that $M\le M_0$, we may take

$$K = M_0 + \max\left\{\|\mathbf{U}^a-\underline{\mathbf{U}}\|_{H^{m+1}(\mathbb{R}\times D)}, \|\chi^a\|_{H^{m+1}(\mathbb{R}\times I)}\right\} < \infty.$$

Next, to guarantee $\mathbf{U}^a+\dot{\mathbf{U}}\in\mathscr{W}_\mu$, by iv) in Lemma 3.6 (we will take $T<T_0$ below), it is sufficient that $\left\|\dot{\mathbf{U}}\right\|_{L^\infty(D^T)}\le\mu/3$. Since $\dot{\mathbf{U}}|_{t=0}=0$, for any $0\le t\le T$, we have $|\dot{\mathbf{U}}(t,z,y)|\le\int_0^T|\partial_t\dot{\mathbf{U}}(s,z,y)|\,\mathrm{d}s$, hence

$$\begin{aligned}\left\|\dot{\mathbf{U}}(t)\right\|_{H^{m-1}(D)} &\le \int_0^T\left\|\partial_s\dot{\mathbf{U}}(s)\right\|_{H^{m-1}(D)}\mathrm{d}s \le \sqrt{T}\left(\int_0^T\left\|\partial_t\dot{\mathbf{U}}(t)\right\|^2_{H^{m-1}(D)}\mathrm{d}t\right)^{\frac12}\\ &\le \sqrt{T}\left\|\dot{\mathbf{U}}\right\|_{H^m(D^T)}.\end{aligned}$$

Then by Sobolev embedding theorem $H^{m-1}(D)\hookrightarrow L^\infty(D)$, there is a constant c_0 independent of T,M so that $\left\|\dot{\mathbf{U}}\right\|_{L^\infty(D^T)}\le c_0\sqrt{T}M$. Similarly we also obtain $\|\dot{\chi},\mathrm{d}\dot{\chi}\|_{L^\infty(I^T)}\le c_0\sqrt{T}M$. So once we choose $M\le M_0\doteq\mu/(3c_0\sqrt{T_0})$, assumption e) holds.

5.1.4. *Applying Theorem 4.2.* We can now apply Theorems 4.1 and 4.2 to (5.1) to conclude there is uniquely one solution $(\dot{\mathbf{V}},\dot{\psi})$ and it has the following properties:

i) it is properly symmetric with $s=m$;
ii) $\dot{\mathbf{V}}\in H^m((-\infty,T]\times D)$, $\dot{\mathbf{V}}|_{z=0}\in H^m((-\infty,T]\times I)$, and $\dot{\psi}\in H^{m+1}((-\infty,T]\times I)$.

So to guarantee that $(\dot{\mathbf{V}},\dot{\psi})\in\mathcal{S}_{T,M}$, we only need verify property 1).

Using estimate in Theorem 4.2 and nonlinear estimates we derived above, there holds

$$\begin{aligned}&\frac1T\left\|\dot{\mathbf{V}}\right\|^2_{H^m(D^T)}+\left\|\dot{\mathbf{V}}|_{z=0}\right\|^2_{H^m(I^T)}+\left\|\dot{\psi}\right\|^2_{H^{m+1}(I^T)} \qquad (5.4)\\ \le\ & c_K\left(T(C_1(M)^2M^2+\|f_0\|^2_{H^m(D^T)})+\|g\|^2_{H^m(I^T)}+C_2(M)^2M^4+\|h_0\|^2_{H^m(I^T)}\right).\end{aligned}$$

Recall here that c_K is a constant depending only on K, and $C_{1,2}(M)$ are constants depending non-decreasingly on M. Denote

$$M_1\doteq\left(\|f_0\|^2_{H^m(D^T)}+\|g\|^2_{H^m(I^T)}+\|h_0\|^2_{H^m(I^T)}\right)^{\frac12}.$$

Then for $0 < T \le T_0' = \min\{T_0,\ T_1\} < 1$ with T_0, T_1 determined in Lemma 3.6 and Theorem 4.2, there holds

$$\begin{aligned}&\left\|\dot{\mathbf{V}}\right\|_{H^m(D^T)} + \left\|\dot{\mathbf{V}}|_{z=0}\right\|_{H^m(I^T)} + \left\|\dot{\psi}\right\|_{H^{m+1}(I^T)}\\ \le\ & \sqrt{3}c_K\left(T(C_1(M_0)^2M^2 + C_2(M_0)^2M^4 + M_1^2\right)^{\frac{1}{2}}.\end{aligned}$$

Now we take $M = 3c_K M_1,\ T = \min\left\{\frac{1}{9c_K^2C_1(M_0)^2},\ T_0'\right\}$ with

$$M_1 \le \min\left\{\frac{M_0}{3c_K},\quad \frac{1}{9c_K^2C_2(M_0)}\right\}, \tag{5.5}$$

we readily get

$$\left\|\dot{\mathbf{V}}\right\|_{H^m(D^T)} + \left\|\dot{\mathbf{V}}|_{z=0}\right\|_{H^m(I^T)} + \left\|\dot{\psi}\right\|_{H^{m+1}(I^T)} \le M.$$

Finally, we need show that (5.5) holds. This follows easily from (3.11), just by taking T further small (depending only on approximate solution and initial-boundary data) !

5.2. **Contraction.** For $(\mathbf{U}^j, \chi^j) \in \mathcal{S}_{T,M},\ j = 1, 2$, denote the corresponding solution of (5.1) be $(\mathbf{V}^j, \psi^j)$, and set $\mathbf{U} = \mathbf{U}^1 - \mathbf{U}^2, \chi = \chi^1 - \chi^2, \mathbf{V} = \mathbf{V}^1 - \mathbf{V}^2, \psi = \psi^1 - \psi^2$. Then $(\mathbf{V}, \psi)$ satisfies the following linear problem

$$\begin{cases} L(\mathbf{U}^a + \mathbf{U}^1, \chi^a + \chi^1, \mathrm{d}(\chi^a + \chi^1))\mathbf{V} = F, & \text{in } D^T,\\ \mathbf{V} = 0, & \text{on } \{t = 0\} \times D,\\ \mathbb{J}\mathrm{d}\psi + \nabla Q(\mathbf{U}^a + \mathbf{U}^1)\mathbf{V} = G, & \text{on } \Sigma_0^T,\\ \psi = 0, & \text{at } \{0\} \times [0, 1],\\ M\mathbf{V} = 0, & \text{on } \Sigma_1^T,\\ M'\mathbf{V} = 0, & \text{on } \Gamma_{0,1}^T, \end{cases} \tag{5.6}$$

where

$$F \doteq \Big(L(\mathbf{U}^a + \mathbf{U}^2, \chi^a + \chi^2, \mathrm{d}(\chi^a + \chi^2)) - L(\mathbf{U}^a + \mathbf{U}^1, \chi^a + \chi^1, \mathrm{d}(\chi^a + \chi^1))\Big) \times (\mathbf{U}^a + \mathbf{V}^2),$$

and $G = G_1 + G_2$, with

$$G_1 \doteq \Big((\nabla Q(\mathbf{U}^a + \mathbf{U}^1)\mathbf{U}^1 - Q(\mathbf{U}^a + \mathbf{U}^1)) - (\nabla Q(\mathbf{U}^a + \mathbf{U}^2)\mathbf{U}^2 - Q(\mathbf{U}^a + \mathbf{U}^2))\Big)$$

and

$$G_2 \doteq \Big(\nabla Q(\mathbf{U}^a + \mathbf{U}^2) - \nabla Q(\mathbf{U}^a + \mathbf{U}^1)\Big)\mathbf{V}^2.$$

By Sobolev embedding $H^3(D^T) \hookrightarrow \mathscr{C}^1(D^T) \subset W^{1,\infty}(D^T)$, $H^4(I^T) \hookrightarrow \mathscr{C}^2(I_T) \subset W^{2,\infty}(I^T)$, and our choice of M above, we can apply Theorem 4.1 to (5.6) to have the inequality

$$\frac{1}{T}\left\|\mathbf{V}\right\|^2_{L^2(D^T)} + \left\|\mathbf{V}|_{z=0}\right\|^2_{L^2(I^T)} + \left\|\psi\right\|^2_{H^1(I^T)} \le c_K(T\left\|F\right\|^2_{L^2(D^T)} + \left\|G\right\|^2_{L^2(I^T)}). \tag{5.7}$$

We need control the right-hand side.

It is rather easy to estimate F. By using the simple fact $\|uv\|_{L^2} \leq \|u\|_{L^\infty} \|v\|_{L^2}$, we have

$$
\begin{aligned}
& \|F\|_{L^2(D^T)} \\
\leq\ & \left\|\mathbf{U}^a + \mathbf{V}^2\right\|_{W^{1,\infty}(D^T)} \Big(\sum_{j=0,2} \left\|\mathbb{A}^j(\mathbf{U}^a + \mathbf{U}^1) - \mathbb{A}^j(\mathbf{U}^a + \mathbf{U}^2)\right\|_{L^2(D^T)} \\
& + \left\|\mathbb{A}^1(\mathbf{U}^a + \mathbf{U}^1, \chi^a + \chi^1, \mathrm{d}(\chi^a + \chi^1)) - \mathbb{A}^1(\mathbf{U}^a + \mathbf{U}^2, \chi^a + \chi^2, \mathrm{d}(\chi^a + \chi^2))\right\|_{L^2(D^T)} \Big) \\
\leq\ & c(\|\mathbf{U}^a\|_{H^m(D^T)} + M) c_K (\|\mathbf{U}\|_{L^2(D^T)} + \|\chi\|_{H^1(I^T)}) \\
\leq\ & c_K (\|\mathbf{U}\|_{L^2(D^T)} + \|\chi\|_{H^1(I^T)})
\end{aligned}
$$

with the help of mean value theorem and Sobolev embedding theorem. We may treat similarly the second term in G:

$$
\begin{aligned}
\|G_2\|_{L^2(I^T)} &= \left\| \Big(\nabla Q(\mathbf{U}^a + \mathbf{U}^2) - \nabla Q(\mathbf{U}^a + \mathbf{U}^1)\Big) \mathbf{V}^2 \right\|_{L^2(I^T)} \\
&\leq \left\|\mathbf{V}^2\right\|_{L^\infty(I^T)} c_K \|\mathbf{U}\|_{L^2(I^T)} \\
&\leq c_K M \left\|\mathbf{U}|_{z=0}\right\|_{L^2(I^T)}.
\end{aligned}
$$

For G_1, we can write it as a sum of two terms $G_1 = G_{11} + G_{12}$, with

$$
G_{11} \doteq \Big(\nabla Q(\mathbf{U}^a + \mathbf{U}^1) - \nabla Q(\mathbf{U}^a + \mathbf{U}^2)\Big) \mathbf{U}^2,
$$

and

$$
\begin{aligned}
G_{12} &\doteq \Big(Q(\mathbf{U}^a + \mathbf{U}^2) - Q(\mathbf{U}^a + \mathbf{U}^1)\Big) + \nabla Q(\mathbf{U}^a + \mathbf{U}^1)(\mathbf{U}^1 - \mathbf{U}^2) \\
&= (\nabla Q(\mathbf{U}^a + \mathbf{U}^1) - \nabla Q(\mathbf{U}^a))\mathbf{U} - (\nabla Q(\mathbf{U}^a + \theta \mathbf{U}^1 + (1-\theta)\mathbf{U}^2) \\
&\quad - \nabla Q(\mathbf{U}^a))\mathbf{U}, \quad \theta \in [0,1].
\end{aligned}
$$

Similar as before, we see

$$
\|G_{11}\|_{L^2(I^T)} \leq c_K \left\|\mathbf{U}^2\right\|_{L^\infty(I^T)} \|\mathbf{U}\|_{L^2(I^T)} \leq c_K M \left\|\mathbf{U}|_{z=0}\right\|_{L^2(I^T)}
$$

and

$$
\|G_{12}\|_{L^2(I^T)} \leq C_K M \left\|\mathbf{U}|_{z=0}\right\|_{L^2(I^T)}.
$$

Substituting the above estimates of F and G into (5.7), recall that $T < 1$, we then have

$$
\begin{aligned}
& \|\mathbf{V}\|_{L^2(D^T)} + \left\|\mathbf{V}|_{z=0}\right\|_{L^2(I^T)} + \|\psi\|_{H^1(I^T)} \\
\leq\ & \sqrt{3} c_K (\sqrt{T}(\|\mathbf{U}\|_{L^2(D^T)} + \|\chi\|_{H^1(I^T)}) + M \left\|\mathbf{U}|_{z=0}\right\|_{L^2(I^T)}).
\end{aligned}
$$

If we choose T, hence M further small so that $T \leq \frac{1}{3}\left(\frac{1}{2c_K}\right)^2, M \leq \frac{1}{2\sqrt{3}c_K}$, then

$$
\|\mathbf{V}\|_{L^2(D^T)} + \left\|\mathbf{V}|_{z=0}\right\|_{L^2(I^T)} + \|\psi\|_{H^1(I^T)} \leq \frac{1}{2}(\|\mathbf{U}\|_{L^2(D^T)} + \|\chi\|_{H^1(I^T)} + \left\|\mathbf{U}|_{z=0}\right\|_{L^2(I^T)}).
$$

So the nonlinear mapping $\mathfrak{N}$ actually contracts in $\mathcal{S}_{T,M}$ with the above $L^2(D^T) \times H^1(I^T)$ topology.

5.3. **Solution of nonlinear problem.**

Lemma 5.1. *$\mathcal{B} = \mathcal{S}_{T,M}$ is a bounded closed convex set in $X = H^m(D^T;\mathbb{R}^8) \times H^{m+1}(I^T)$.*

Proof. Boundedness and convexity is simple. Suppose now $(\mathbf{U}_k, \chi_k) \in \mathcal{B}$ so that $\mathbf{U}_k \to \mathbf{U}$ in $H^m(D^T)$ and $\chi_k \to \chi$ in $H^{m+1}(I^T)$ as $k \to \infty$. To prove $\mathcal{B}$ is close, we only need show $\mathbf{U}|_{z=0} \in H^m(I^T)$ and $\|\mathbf{U}|_{z=0}\|_{H^m(I^T)} \le \liminf_{k\to\infty} \|\mathbf{U}_k|_{z=0}\|_{H^m(I^T)}$. By trace theorem, we see $\mathbf{U}_k|_{z=0} \to \mathbf{U}|_{z=0}$ in $H^{m-\frac{1}{2}}(I^T)$. While since $\{\mathbf{U}_k|_{z=0}\}$ is bounded in $H^m(I^T)$, there is a subsequence $\mathbf{U}_{k_j}|_{z=0}$ converges weakly to a $\mathbf{V}$ in $H^m(I^T)$, and by lower semi-continuous of norm with respect to weak convergence,

$$\|\mathbf{V}\|_{H^m(I^T)} \le \liminf_{k\to\infty} \|\mathbf{U}_k|_{z=0}\|_{H^m(I^T)}\,.$$

Uniqueness of limit in the sense of distribution then implies $\mathbf{U}|_{z=0} = \mathbf{V}$. □

We define

$$d((\mathbf{U},\chi),(\mathbf{V},\psi)) \doteq \|\mathbf{U}-\mathbf{V}\|_{L^2(D^T)} + \|(\mathbf{U}-\mathbf{V})|_{z=0}\|_{L^2(I^T)} + \|\chi-\psi\|_{H^1(I^T)}$$

for any $(\mathbf{U},\chi)$ and $(\mathbf{V},\psi) \in \mathcal{B}$, which is a metric on $\mathcal{B}$.

Lemma 5.2. *$\mathcal{B}$ is complete under the metric d.*

Proof. Suppose $\{(\mathbf{U}_k,\chi_k)\}_{k=1}^\infty$ is a Cauchy sequence in $(\mathcal{B}, d)$. Then there are $\mathbf{U} \in L^2(D^T), \mathbf{W} \in L^2(I^T)$ and $\chi \in H^1(I^T)$, so that $\mathbf{U}_k \to \mathbf{U}$ in $L^2(D^T)$, $\mathbf{U}_k|_{z=0} \to \mathbf{W}$ in $L^2(I^T)$, $\chi_k \to \chi$ in $H^1(I^T)$, as $k \to \infty$. We only need show $(\mathbf{U},\chi) \in \mathcal{B}$ and $\mathbf{W} = \mathbf{U}|_{z=0}$.

Note $\{(\mathbf{U}_k,\chi_k)\}_{k=1}^\infty$ is also bounded in X, so there is a subsequence $\mathbf{U}_{k_j} \to \mathbf{U}'$ weakly in $H^m(D^T)$, and $\chi_{k_j} \to \chi'$ weakly in $H^{m+1}(I^T)$. Moreover, since closed and convex subset of a Banach space is weakly closed, by Lemma 5.1, we infer $(\mathbf{U}',\chi')$ still belongs to $\mathcal{B}$. Then by uniqueness of limit in the sense of distribution, we conclude $(\mathbf{U},\chi) = (\mathbf{U}',\chi') \in \mathcal{B}$, and the subsequence $\{k_j\}$ may be taken as the original sequence $\{k\}$.

Now note both $\mathbf{U}_k, \mathbf{U}$ are bounded in $H^m(D^T)$ and $\|\mathbf{U}_k - \mathbf{U}\|_{L^2(D^T)} \to 0$, then by interpolation inequality of Sobolev spaces, $\mathbf{U}_k \to \mathbf{U}$ in $H^1(D^T)$, hence by trace theorem, $\mathbf{U}_k|_{z=0} \to \mathbf{U}|_{z=0}$ in $L^2(I^T)$. This proves $\mathbf{W} = \mathbf{U}|_{z=0}$. □

We have shown $\mathfrak{N}$ is a contract mapping on $(\mathcal{B}, d)$. So by Banach fixed-point theorem, $\mathfrak{N}$ has uniquely one fixed-point $(\mathbf{V},\chi)$ in $\mathcal{B}$. Obviously $(\mathbf{V}+\mathbf{U}^a, \chi+\chi^a)$ solve Problem (N). Furthermore, by our construction, $\mathbf{V}+\mathbf{U}^a$ takes values in $\mathscr{W}_\mu$ and $(\chi+\chi^a, \mathrm{d}(\chi+\chi^a))$ takes value in $\mathscr{V}_\mu$. For μ chosen small, we infer that the flow is still supersonic ahead of shock-front, subsonic behind of it, and the transform $\Psi_\pm$ used to fixed shock-front is actually a homeomorphism, for $t \in [0, \bar{T}]$, as required. This finishes the proof of Theorem 2.1.

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