Harmonic Analysis— Excercises（1）

1. (1) For $f \in L^1(T)$, show that $\sigma_N f(\theta) \to f(\theta)$ whenever f is continuous at $\theta \in T$. Here $\sigma_N f(\theta)$ is the N-th Cesàro sum.

(2) Show that trigonometric polynomials are dense in $C(T)$.

2. Prove that if $f \in L^1(\mathbb{R})$ and its Fourier transform $\hat{f}$ both have compact support, then $f = 0$.

3. Prove that for $f \in C^k(T)$, there holds $\hat{f}(n) = O(\frac{1}{n^k})$, hence the Fourier series $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}$ converges absolutely to f if $k \geq 2$. Would you try to define spaces $C^s(T)$ for s a positive real number by this observation?

4. For $\psi \in S(\mathbb{R}^d)$, prove the following Heisenberg uncertainty principle in d-dimensions:

$$(\int_{\mathbb{R}^d} |x|^2 |\psi(x)|^2 dx)(\int_{\mathbb{R}^d} |\xi|^2 |\hat{\psi}(\xi)|^2 d\xi) \geq \frac{d^2}{16\pi^2}.$$

5. Let $u(t,x)$ be a bounded solution to the problem:

$$\begin{cases} u_{tt} + \Delta u = 0, & \text{in } [0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x). \end{cases}$$

Show that there holds

$$||u||_{L^\infty(0,T;L^2(\mathbb{R}^n))} \leq ||u_0||_{L^2(\mathbb{R}^n)}.$$