Harmonic Analysis— Excercises（3）

1. Let $\{u_p\}_{p=1}^{\infty}\subset L^2(\mathbb{R}^n)$ satisfy $\operatorname{supp}\hat{u}_p\subset\{\xi\in\mathbb{R}^n:\frac{1}{C}2^{p-1}\le|\xi|\le C2^{p+1}\}$. Then there holds

$$||\sum_{p=1}^{\infty}u_p||_0^2\le C'\sum_{p=1}^{\infty}||u_p||_0^2.$$

Here C,C' are constants independent of u and p.

2. Using Littlewood-Paley decomposition to show that $H^s(\mathbb{R}^n)$ is embedded into $C^{\alpha}(\mathbb{R}^n)$ provided $s=n/2+\alpha$ and $\alpha>0$ is not an integer.

3. Prove the following classical time-decay estimates for the Cauchy problem of heat equation

$$\begin{cases} u_t-\Delta u=0, & x\in\mathbb{R}^n,\ t>0,\\ u(0,x)=\phi(x), & x\in\mathbb{R}^n. \end{cases}$$

Suppose that $\phi\in L^p(\mathbb{R}^n),\ 1\le p\le\infty$. Then for $t>0$, there hold

$$(1)\ \ ||u(t)||_{L^p(\mathbb{R}^n)}\le||\phi||_{L^p(\mathbb{R}^n)},$$
$$(2)\ \ ||\partial^k u(t)||_{L^q(\mathbb{R}^n)}\le Ct^{-(\frac{n}{2r}+\frac{k}{2})}||\phi||_{L^p(\mathbb{R}^n)}.$$

Here $\frac{1}{r}=\frac{1}{p}-\frac{1}{q}$, $1\le r,q\le\infty$; C is a positive constant depending only on $p,\ q,\ r,\ k$; ∂^k is any $k-$th derivative with respect to x.

Hint: Using the Young inequality of convolutions.

4. Suppose for $f\in L^1(\mathbb{R}^1)$ there holds

$$\int_{\mathbb{R}^1}|f(x+t)-f(x)|\,dx\le|t|^2,\ \ t\in\mathbb{R}^1.$$

Show that $f=0$.

5. (1) Let δ be the Dirac distribution: $\langle\delta,f\rangle=f(0)$. Prove that $\hat{\delta}=1$.

(2) Show that any harmonic function in the whole space $\mathbb{R}^n$ must be a polynomial.